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THE ORDERABILITY OF IDEMPOTENT SEMIGROUPS

by Toru SAITÔ

The orderability condition for idempotent semigroups are studied by some authors : M.-L. DUBREIL-JACOTIN [3] for idempotent semigroups with identity, T. MERLIER [4] for finite idempotent semigroups, and T. SAITÔ [6] for commutative idempotent semigroups. The purpose of this note is to give orderability conditions for general idempotent semigroups. The detailed version [7] will appear elsewhere.

The terminologies of CLIFFORD and PRESTON [2] are used throughout. Let E be a semilattice with respect to a partial order $\preccurlyeq$. E is called a tree semilattice, if, for every $\alpha \in E$, the set $\{\xi \in E \; ; \; \xi \preccurlyeq \alpha\}$ is a simply ordered set. Let α be an element of a tree semilattice E . We define a binary relation $\sim$ on the set $\overline{U}_\alpha = \{\xi \in E \; ; \; \alpha < \xi\}$ by

$$\text{for } \xi , \eta \in \overline{U}_\alpha , \quad \xi \sim \eta \text{ if and only if } \alpha < \xi\eta .$$

Then $\sim$ is an equivalence relation on $\overline{U}_\alpha$. Each $\sim$-equivalence class is called a branch at α .

Let S be an idempotent semigroup. Then S is a semilattice of rectangular bands $\{D_\alpha \; ; \; \alpha \in S^*\}$, and the decomposition of S into $\{D_\alpha \; ; \; \alpha \in S^*\}$ coincides with the decomposition of S into $\mathcal{D}$-classes. The semilattice S^* is called the associated semilattice of S .

Let S be an idempotent semigroup such that the associated semilattice S^* is a tree semilattice and let D be a $\mathcal{D}$-class of S . Let $\mathfrak{B}$ be a branch of S^* at $D \in S^*$. Then, the subset $B = \{x \in S \; ; \; D_x \in \mathfrak{B}\}$ of S is called the component-branch at D associated with the branch $\mathfrak{B}$.

If a $\mathcal{D}$-class D of an idempotent semigroup S consists of one $\mathcal{L}$-class, then D is called of L-type, while if D consists of one $\mathcal{R}$-class, then D is called of R-type.

By an ordered semigroup S , we mean a semigroup S with a simple order $\leqslant$ satisfying the condition

$$\text{for } x , y , z \in S , \quad x \leqslant y \text{ implies } xz \leqslant yz \text{ and } zx \leqslant zy .$$

A semigroup S is called orderable if there exists a simple order $\leqslant$ on S such that the system $S(. , \leqslant)$ is an ordered semigroup.

Here we refer to some preliminary lemmas :

LEMMA 1 ([5], theorem 3). - *The associated semilattice* S^* *of an ordered idem-*

potent semigroup S is a tree semilattice.

LEMMA 2 ([5], theorem 1). - In an ordered idempotent semigroup S , each $\mathcal{D}$-class consists of either one $\mathcal{L}$-class or one $\mathcal{R}$-class.

LEMMA 3 [1]. - Let S be a set with a ternary relation β satisfying the conditions :

(a) $(x, y, z)\beta$ implies $(z, y, x)\beta$;

(b) $(x, y, x)\beta$ implies $x = y$;

(c) $(x, y, z)\beta$, $(y, z, u)\beta$ and $y \neq z$ imply $(x, y, u)\beta$;

(d) For every $x, y, z \in S$, either $(x, y, z)\beta$ or $(y, z, x)\beta$ or $(z, x, y)\beta$;

(e) $(x, y, z)\beta$ and $(x, z, u)\beta$ imply $(y, z, u)\beta$.

Then, there exists a simple order $\leqslant$ on S such that $(x, y, z)\beta$ if and only if either $x \leqslant y \leqslant z$ or $z \leqslant y \leqslant x$.

I

THEOREM A. - An idempotent semigroup S is orderable if and only if it satisfies the following conditions :

(A) The associated semilattice S^* of S is a tree semilattice ;

(B) Each $\mathcal{D}$-class of S consists of either one $\mathcal{L}$-class or one $\mathcal{R}$-class ;

(C) If D is a $\mathcal{D}$-class of S and $a \in S$ such that $D \prec D_a$ in the associated semilattice S^* , then either aD or Da consists of at most two elements of S ;

(D) If D is a $\mathcal{D}$-class of S of L-type [R-type] and if $a, b, x \in S$ such that $D \prec D_{ab}$ in the associated semilattice S^* and $x \in D$, then $ax=bx$ [$xa=xb$];

(E) If D is a $\mathcal{D}$-class of S of L-type [R-type] and if $a, b, x \in S$ such that $a, b \notin D$, $ab \in D$, x D and $ab \neq ax$ [$ba \neq xa$] , then $ba=bx$ [$ab=xb$];

(F) If D is a $\mathcal{D}$-class of S of L-type [R-type] and if $a, b, c \in S$ such that $a, b, c \notin D$, $ab \in D$, $ab = ac$ and $bc = ba$ [$ba = ca$ and $cb = ab$] , then $ca \neq cb$ [$ac \neq bc$] ;

(G) If D is a $\mathcal{D}$-class of S of L-type [R-type] and if $a, b, x \in S$ such that $a, b \notin D$, $ab \in D$, $x \in D$ and $ax = bx$ [$xa = xb$] , then $x=ab$ [$x=ba$] .

Here we give only a brief survey of the proof of the "if" part by steps.

Let S be an idempotent semigroup satisfying the conditions given in the theorem and let D be a $\mathcal{D}$-class of S . We denote by $\{B_\lambda ; \lambda \in \Lambda\}$ the set of all component-branches at D .

(1) <u>If</u> $B_\lambda \neq B_\mu$, <u>then</u> $B_\lambda B_\mu$ <u>is a one-element subset of</u> S .

We define the ternary relation β on $\{B_\lambda ; \lambda \in \Lambda\}$ by :

If D is of L-type [R-type], then $(B_\lambda , B_\mu , B_\nu)\beta$ if and only if either $B_\lambda = B_\mu$ or $B_\mu = B_\nu$ or $B_\lambda \neq B_\mu$, $B_\mu \neq B_\nu$ and $B_\mu B_\lambda \neq B_\mu B_\nu$ [$B_\lambda B_\mu \neq B_\nu B_\mu$] .

(2) β <u>satisfies the conditions in lemma 3</u>.

For each $\lambda \in \Lambda$, we define a subset L_λ of D as follows :

(i) The case when Λ contains at least two elements.

(ia) If there exists $\mu \in \Lambda$ such that $B_\mu < B_\lambda$ and if D is of L-type [R-type] then put

$$L_\lambda = \{x \in D ; B_\lambda x = B_\lambda B_\mu\} \quad [L_\lambda = \{x \in D ; xB_\lambda = B_\mu B_\lambda\}] .$$

(ib) If there exists $\nu \in \Lambda$ such that $B_\lambda < B_\nu$ and if D is of L-type [R-type], then put

$$L_\lambda = \{x \in D ; B_\lambda x \neq B_\lambda B_\nu\} \quad [L_\lambda = \{x \in D ; xB_\lambda \neq B_\nu B_\lambda\}] .$$

(ii) The case when Λ consists of one and only one element λ . We take $x_0 \in D$ arbitrarily and fix it. Then put

$$L_\lambda = \{x \in D ; B_\lambda x = B_\lambda x_0\} \quad [L_\lambda = \{x \in D ; xB_\lambda = x_0 B_\lambda\}] .$$

Further, we define the binary relation γ on D by

$$x \gamma y \text{ if and only if } \lambda \in \Lambda \text{ and } y \in L_\lambda \text{ implies } x \in L_\lambda .$$

(3) γ <u>is a reflexive and transitive relation on</u> D . <u>Moreover, for each pair of elements</u> x <u>and</u> y <u>of</u> D , <u>we have either</u> $x \gamma y$ <u>or</u> $y \gamma x$.

Hence, if we define

$$x \,\delta\, y \overset{\text{def}}{\Longleftrightarrow} x \gamma y \text{ and } y \gamma x ,$$

then δ is an equivalence relation on D and the quotient set D/δ is a simply ordered set with respect to the relation $\leq$ defined by :

$$\text{for } K_1 , K_2 \in D/\delta , \quad K_1 \leq K_2 \overset{\text{def}}{\Longleftrightarrow} x \gamma y \text{ for some } x \in K , \text{ and } y \in K_2 .$$

We denote the quotient set D/δ by $\mathfrak{K}_D$ and call an element of $\mathfrak{K}_D$ a <u>component</u> of D .

(4) <u>Let</u> D <u>be of</u> <u>L-type</u> [R-<u>type</u>].

(a) <u>If</u> $L_\lambda \neq \square$, <u>then</u> $B_\lambda L_\lambda$ [$L_\lambda B_\lambda$] <u>consists of one and only one element</u> ℓ_λ <u>of</u> D .

(b) <u>If</u> $D \setminus L_\lambda \neq \square$, <u>then</u> $B_\lambda(D \setminus L_\lambda)$ [$(D \setminus L_\lambda)B_\lambda$] <u>consists of one and only one element</u> u_λ <u>of</u> D .

The element ℓ_λ is called the lower distinguished element of D corresponding to λ and the element u_λ is called the upper distinguished element of D corresponding to λ.

(5) (a) $\ell_\lambda \curlyvee \ell_\mu$ if and only if $B_\lambda \leqslant B_\mu$,

(b) $u_\lambda \curlyvee u_\mu$ if and only if $B_\lambda \leqslant B_\mu$.

(6) If $\ell_\lambda = u_\mu$, then the component K containing the element ℓ_λ consists of one and only one element.

Hence, by the well-ordering principle, we can take a simple order in such a way that, if K contains a lower distinguished element ℓ_λ, then ℓ_λ is the greatest element of K and, if K contains an upper distinguished element u_μ, then u_μ is the least element of K. Now, we define the simple order on D as the ordinal sum of these simply ordered components.

Finally, we define, for $x, y \in S$, $x < y$ if and only if either one of the following conditions is satisfied :

(a) $D_{xy} < D_x$, $D_{xy} < D_y$, B_λ is the component-branch at D_{xy} containing x, B_μ is the component-branch at D_{xy} containing y, and $B_\lambda < B_\mu$;

(b) $D_x = D_{xy} < D_y$, B_λ is the component-branch at D_{xy} containing y, and $x \leqslant \ell_\lambda$ in D_{xy};

(c) $D_x > D_{xy} = D_y$, B_μ is the component-branch at D_{xy} containing x, and $u_\mu \leqslant y$ in D_{xy};

(d) $D_x = D_y = D_{xy}$ and $x < y$ in D_{xy}.

(7) The relation $<$ on S defines a simple order which is compatible with the semigroup operation.

II

Let S be an idempotent semigroup.

We divide the condition (D) into following three conditions :

(D1) If D is a $\mathcal{D}$-class of S of L-type [R-type] and if $a, b \in S$, $ab=ba=b$ and $D < D_{ab}$, then $ax = bx$ [$xa = xb$] for every $x \in D$;

(D2) If D is a $\mathcal{D}$-class of S of L-type [R-type] and if $a, b \in S$, $a \mathcal{L} b$ [$a \mathcal{R} b$], and $D < D_{ab}$, then $ax = bx$ [$xa = xb$] for every $x \in D$;

(D3) If D is a $\mathcal{D}$-class of S of L-type [R-type] and if $a, b \in S$, $a \mathcal{R} b$ [$a \mathcal{L} b$], and $D < D_{ab}$, then $ax = bx$ [$xa = xb$] for every $x \in D$.

Now, we have :

(a) S satisfies condition (B) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to the semigroup S_1 :

S_1 :

	x	y	z	u
x	x	z	z	x
y	u	y	y	u
z	x	z	z	x
u	u	y	y	u

(b) S satisfies condition (D1) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to either one of the two semigroups :

S_2 :

	a	b	x	y
a	a	b	x	y
b	b	b	y	y
x	x	x	x	x
y	y	y	y	y

S_2^* :

	a	b	x	y
a	a	b	x	y
b	b	b	x	y
x	x	y	x	x
y	y	y	x	y

(c) Let S satisfy (B) and (D1). Then, S satisfies condition (A) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to the semigroup S_3 :

S_3 :

	a	b	c	x
a	a	b	c	x
b	b	b	x	x
c	c	x	c	x
x	x	x	x	x

(d) Let S satisfy (B). Then, S satisfies condition (C) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to either one of the two semigroups :

S_4 :

	a	x	y	z
a	a	x	y	z
x	x	x	x	x
y	y	y	y	y
z	z	z	z	z

S_4^* :

	a	x	y	z
a	a	x	y	z
x	x	x	y	z
y	y	x	y	z
z	z	x	y	z

(e) S satisfies condition (D2) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to either one of the two semigroups :

S_5 :

	a	b	x	y
a	a	a	x	x
b	b	b	y	y
x	x	x	x	x
y	y	y	y	y

S_5^* :

	a	b	x	y
a	a	b	x	y
b	a	b	x	y
x	x	y	x	y
y	x	y	x	y

(f) S satisties condition (G) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to either one of the two semigroups :

S_6 :

	a	b	x	y
a	a	y	y	y
b	y	b	y	y
x	x	x	x	x
y	y	y	y	y

S_6^* :

	a	b	x	y
a	a	y	x	y
b	y	b	x	y
x	y	y	x	y
y	y	y	x	y

(g) Let S satisfy (G). Then, S satisfies condition (F) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to either one of the three semigroups :

S_7 :

	a	b	c	x
a	a	x	x	x
b	x	b	x	x
c	x	x	c	x
x	x	x	x	x

S_8 :

	a	b	c	x	y	z
a	a	x	x	x	x	x
b	y	b	y	y	y	y
c	z	z	c	z	z	z
x	x	x	x	x	x	x
y	y	y	y	y	y	y
z	z	z	z	z	z	z

S_8^* :

	a	b	c	x	y	z
a	a	y	z	x	y	z
b	x	b	z	x	y	z
c	x	y	c	x	y	z
x	x	y	z	x	y	z
y	x	y	z	x	y	z
z	x	y	z	x	y	z

(h) S satisfies condition (D3) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to either one of the two semigroups :

S_9 :

	a	b	x	y	z
a	a	b	x	y	x
b	a	b	x	y	y
x	x	x	x	x	x
y	y	y	y	y	y
z	z	z	z	z	z

S_9^* :

	a	b	x	y	z
a	a	a	x	y	z
b	b	b	x	y	z
x	x	x	x	y	z
y	y	y	x	y	z
z	x	y	x	y	z

(i) S satisfies condition (E) $\Longleftrightarrow$ S does not contain a subsemigroup isomorphic to either one of the four semigroups :

S_{10} :

	a	b	x	y	z	u
a	a	u	y	y	u	u
b	u	b	z	u	z	u
x	x	x	x	x	x	x
y	y	y	y	y	y	y
z	z	z	z	z	z	z
u	u	u	u	u	u	u

S_{10}^* :

	a	b	x	y	z	u
a	a	u	x	y	z	u
b	u	b	x	y	z	u
x	y	z	x	y	z	u
y	y	u	x	y	z	u
z	u	z	x	y	z	u
u	u	u	x	y	z	u

S_{11} :

	a	b	x	y	z	u	v
a	a	u	y	y	u	u	u
b	v	b	z	v	z	v	v
x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z
u	u	u	u	u	u	u	u
v	v	v	v	v	v	v	v

S^*_{11} :

	a	b	x	y	z	u	v
a	a	v	x	y	z	u	v
b	u	b	x	y	z	u	v
x	y	z	x	y	z	u	v
y	y	v	x	y	z	u	v
z	u	z	x	y	z	u	v
u	u	v	x	y	z	u	v
v	u	v	x	y	z	u	v

THEOREM B. - An idempotent semigroup S is orderable if and only if it does not contain a subsemigroup isomorphic to either one of semigroups S_1 - S^*_{11} given above.

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