Séminaire Équations aux dérivées partielles - École Polytechnique

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Séminaire Équations aux dérivées partielles (Polytechnique) (1977-1978), exp. nº 17, p. 1-7
<http://www.numdam.org/item?id=SEDP_1977-1978____A18_0>

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FREE SYSTEMS OF VECTOR FIELDS

(d'après L. Hörmander et A. Melin)

par A. MELIN

Exposé n° XVII 23 Mai 1978

Let $X_1,\dots,X_n$ be C^∞ vector fields near a point o of a manifold M and assume that the commutators have maximal rank at that point. In their paper [2] Rothschild and Stein have shown how such systems of vector fields can be made free by the introduction of auxiliary variables (the "lifting theorem"). They also examined to what extent free systems of vector fields could be approximated by corresponding free systems of left invariant vector fields on appropriate nilpotent Lie groups (the "approximation theorem"). I shall here present a joint work with L. Hörmander in which an elementary proof of these two theorems are given [1]. At the end of the talk I shall recall briefly how the approximation theorem may be used.

Let us use the following notations : If $I = (i_1,\dots,i_k)$ is a sequence with $k = |I|$ integers between 1 and n we put

$$X_I = X_{i_1}\dots X_{i_k} \quad ; \quad X_{[I]} = \operatorname{ad} X_{i_1} \dots \operatorname{ad} X_{i_{k-1}} X_{i_k} \quad ,$$

we can define $(\operatorname{ad} X)_{[I]}$ in a similar way and it follows then from Jacobi's identity that $(\operatorname{ad} X)_{[I]} = \operatorname{ad} X_{[I]}$. It is clear that there are constants A_{IJ} with $A_{IJ} = 0$ if $|I| \neq |J|$ and $A_{IJ} = \delta_{IJ}$ if $|I| = |J| = 1$, so that

$$X_{[I]} = \Sigma A_{IJ} X_J \quad . \tag{1}$$

<u>Definition 1</u> : We say that the system $\{X_1,\dots,X_n\}$ is free of order s at o if

$$\sum_{|I| \le s} a_I X_{[I]}(o) = 0 \Rightarrow \sum_{|I| \le s} a_I A_{IJ} = 0 \quad \text{for all } J \ .$$

<u>Proposition 2</u> : $\{X_1,\dots,X_n\}$ is free of order s at o if and only if for arbitrary C_I, $|I| \le s$, one can find $u \in C^\infty$ such that

$$X_I u(o) = C_I \quad ; \quad |I| \le s \ . \tag{2}$$

<u>Proof</u> : We shall show that (2) is always solvable if the system is free to order s. (The other direction is trivial). If u is a solution of (2) then by (1) we see that

$$(3) \qquad X_{[I_1]}\cdots X_{[I_\nu]}\, u(o) = \Sigma\, A_{I_1J_1}\cdots A_{I_\nu J_\nu} C_{J_1\cdots J_\nu} \quad \text{if } |I_1|+\cdots+|I_\nu| \le s.$$

We shall make induction over those integers j for which there is a u satisfying (3) when $\nu \le j$. When $j = s$ we then get a solution of (2) and when $j = 1$ we have just to show that there exists a linear form du(o) on the tangent space at o which sends $X_{[I]}(o)$ to $\Sigma A_{IJ}C_J$ when $|I| \le s$. This is however clear from Definition 1 since $\sum_{|I|\le s} a_I X_{[I]}(o) = 0$ implies that $\Sigma\, a_I\, A_{IJ} C_J = 0$.

In the following we may assume that (3) holds when $\nu < j$ for some function u_0. Then the same equations are satisfied by $u = u_0 + v$ if v vanishes to order j at o and the condition for (3) to hold when $\nu = j$ then takes the form

$$(4) \qquad p(X_{[I_1]},\ldots,X_{[I_j]}) = \Sigma\, A_{I_1J_1}\cdots A_{I_jJ_j}\, C_{J_1\cdots J_j} - X_{[I_1]}\cdots X_{[I_j]}\, u_0(o)\ ; \quad |I_1|+\cdots+|I_j| \le s\ .$$

Here $p = v^{(j)}(o)$ may be any symmetric j-linear form. Note that the right hand side of (4) is symmetric in the indices $I_1,\ldots,I_j$ for a commutator of $X_{[I']}$ and $X_{[I'']}$ is a sum of commutators of length $\le |I'| + |I''|$.

Using the fact that the system is free to order s we see that a relation among the $X_{[I_1]}(o)$ in the left hand side of (4) gives rises to the same kind of relation in the right hand side. The same argument applies to the other variables and this shows that one can find p so that (4) is satisfied.

THE LIFTING THEOREM

Assume that $\{X_1,\dots,X_n\}$ is free of order s-1 but not of order s for some s. This means that we can find a_I such that

$$\sum_{|I|\leq s} a_I\, X_{[I]}(o) = 0 \quad ; \quad \sum_{|I|\leq s} a_I\, A_{IJ} \neq 0 \qquad \text{for some } J\ .$$

Set $\tilde{X}_k = X_k + u_k(x,t)\,\partial/\partial t$, $t \in \mathbb{R}$, $x \in M$. If we write $K = (K',K'')$ with $|K''| = 1$ then we find that

$$\tilde{X}_{[K]}(o) = X_{[K]}(o) + \sum_{|L|=|K|} A_{KL}\, X_{L'}\, u_{L''}(o)\,\partial/\partial t\ .$$

This shows that

$$\sum_{|I|\leq s} a_I\, X_{[I]}(o) = \sum_{|I|\leq s}\ \sum_{|L|=|I|} a_I A_{IL}\, X_{L'}\, u_{L''}(o)\,\partial/\partial t\ .$$

Since the coefficient $X_{L'}u_{L''}(o)$ may be any number C_L we see that the number of linearly independent $\tilde{X}_{[I]}(o)$, $|I| \leq s$, is increased and some commutator has a non-vanishing component in the direction of $\partial/\partial t$. Repeating this arguments in a finite number of steps we get the following result :

Theorem 3 : Let $X_1,\dots,X_n$ be vector fields near $o \in M$ such that the commutators of length $\leq r$ span the tangent space at 0. Then one can find p and $u_{kj} \in C^\infty(M\times \mathbb{R}^p)$ so that

$$\tilde{X}_k = X_k + \sum_{j=1}^{p} u_{kj}(x,t)\,\partial/\partial t_j$$

form a system free of order r with the commutators of length $\leq r$ spanning the tangent space.

THE APPROXIMATION THEOREM

We assume from now on that $\{X_1,\dots,X_n\}$ is free to order r and that the commutators up to that length span the tangent space. Let B be a maximal subset of sequences I with $|I| \leq r$ so that

$$\sum_{I \in B} a_I A_{IJ} = 0 \quad \text{for all } J$$

implies that $a_I = 0$ when $|I| \le r$. Then $\{X_{[I]} \,;\, I \in B\}$ form a basis for the tangent space near o and by the map

$$\mathbb{R}^B \ni \{u_I\} \longrightarrow \exp \Sigma\, u_I X_{[I]} \cdot 0$$

we then introduce local coordinates near $o \in M$. These "normal" coordinates are characterized by the equation

$$\sum_B u_I e_I = \sum_B u_I X_{[I]} \tag{5}$$

if $e_I = \partial/\partial u_I$. Then we can also introduce the local groups $\{\delta_t\} = \{\delta_t^B\}$, $t \in \mathbb{R}_+$, of dilations described by $\delta_t\{u_I\} = \{t^{|I|} u_I\}$. The weight of a smooth function f is then the largest integer ρ for which $\delta_t^* f = f \circ \delta_t$ is $\mathcal{O}(t^\rho)$. This notion of weight can also be carried over to vector fields.

<u>Theorem 4</u> :
1) If $\{u_I\}_{I \in B}$ are normal coordinates for $\{X_1,\dots,X_n\}$ then X_i has weight -1.
2) If $\{u_I\}_{I \in B}$ are normal coordinates for the system $\{Y_1,\dots,Y_n\}$ also, then $X_i - Y_i$ has weight ≥ 0.

<u>Remark 5</u> : We see that the vector fields are modulo fields of non negative weight completely determined from their normal coordinates. It is also easy to see that the weight associated to a function or vector field is independent of the choice of B and does only depend on $\{X_1,\dots,X_n\}$ modulo fields of non-negative weight.

<u>Sketch of proof for Theorem 4</u> : We concentrate on the first part since the ideas are essentially the same in the second part. Let F_s^q and V_s^q be the set of C^∞-functions f and vector fields v so that it is true that f and v have weight $\ge$ s as long as we only consider their Taylor expansions up to order q in the set of normal coordinates. Then we have relations

between these spaces

(6)
$$F^q_s F^q_t \subset F^q_{s+t} \quad , \quad F^q_s V^q_t \subset V^q_{s+t}$$

$$V^{q-1}_s(F^q_t) \subset F^{q-1}_{s+t} \quad ; \quad [V^q_s, V^q_t] \subset V^{q-1}_{s+t} \quad .$$

Also we have some better information when we know that one of the factors above wanishes at the origin .

We shall prove inductively over q that

(7)
$$X_{[J]} \in V^q_{-[J]} \quad ,$$

and since all fields have weight $\geq - r$ and since the fields are free to that order we can assume that $J \in B$. Since $X_{[J]}(o) = e_J$ it is clear than that (7) holds when $q = 0$ and we have then to show that (7) implies that

(8)
$$W = \operatorname{ad} X_{[J]} e_I \in V^q_{-(|I|+|J|)} \cdot$$

Multiplying (5) by $\operatorname{ad} e_I$ we find that

(9)
$$e_I = X_{[I]} + \sum_{K \in B} u_K \operatorname{ad} e_I X_{[K]} \quad .$$

If we multiply this equation by $\operatorname{ad} X_{[J]}$ we get

$$W = \operatorname{ad} X_{[J]} X_{[I]} + \sum_{K \in B} X_{[J]}(u_K) \operatorname{ad} e_I X_{[K]}$$

$$+ \sum_{K \in B} u_K \operatorname{ad} X_{[J]} \operatorname{ad} e_I X_{[K]} \quad .$$

The first term in the right hand side belongs to $V^q_{-(|I|+|J|)}$ and using (6) we can replace $X_{[J]}$ by e_J in the second term if we make our computations modulo this space. Since $\operatorname{ad} e_I X_{[K]} = -\operatorname{ad} X_{[K]} e_I$ we see that

(10) $$2W \equiv -\sum_{K\in B} u_K \operatorname{ad} X_{[J]} \operatorname{ad} X_{[K]}\, e_I \quad .$$

By Jacobi's identity and (6) the orders between ad $X_{[J]}$ and $X_{[K]}$ may be interchanged and this shows that

(11) $$2W \equiv -\sum_{K\in B} u_K \operatorname{ad} X_{[K]} . W \equiv$$

$$-\sum_{K\in B} \operatorname{ad}(u_K X_{[K]})\, W - \sum_{K\in B} W(u_K)\, X_{[K]} \quad .$$

Now we use (5) and may replace $\operatorname{ad}(u_K X_{[K]})$ by $u_K \operatorname{ad} e_K$ if we add $-W(u_K)\, e_K$ in the last sum of (11). This shows if we apply (6) that

(12) $$(2+\Sigma\, u_K \operatorname{ad} e_K)\, W \equiv 0 \quad \text{modulo } V^q_{-(|I|+|J|)} \quad .$$

Since the operator in the left hand side is injective in the space considered we obtain (8) .

Let me finally recall the main reason for having the approximation theorem. Suppose that $P = P(X_1,\ldots,X_n)$ is a non-commutative polynomial in $X_1,\ldots,X_n$ satisfying the assumptions of the approximation theorem. Of course, it is then no restriction to assume that the fields live near the unit e of an appropriate nilpotent Lie group G whose Lie algebra is generated by fields $Y_1,\ldots,Y_n$ whose commutators to order r satisfy the same relations as the $X_{[I]}$, $|I|\le r$, at e. To construct a right parametrix for P now means to solve an equation $P\,E(x,y) = \delta_e(x^{-1}y)$ near e in $G\times G$ where P acts on the first variable. Composing this equation with the map

$$\theta : (\exp u\,Y, y) \to (\exp uX.y, y),$$

where $uY = \sum_B u_I Y_{[I]}$ we get instead an equation for $\tilde{E} = E\circ\theta$:

$$\tilde{P}\tilde{E} = P(\tilde{X})\tilde{E} = \delta_e(x) \quad .$$

Keeping y fixed we see that $\Sigma u_I Y_I = \Sigma u_I \widetilde{X}_{[I]}$ if $\widetilde{X}_i$ are the transformed operators. It follows then from the approximation theorem that we can write

$$\widetilde{X}_i = Y_i + R(x,y)$$

where $R(x,y)$ is a vector field in the variable x depending smoothly on y and of weight ≥ 0 (w.r.t. $\{Y_1,\dots,Y_n\}$). A natural "Ansatz" to construct E is therefore to define $\widetilde{E}$ as some perturbation of the fundamental solution of $P(Y_1,\dots,Y_n)$ if this exists.

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[2] L. P. Rothschild and E. M. Stein : Hypoelliptic differential operators and nilpotent groups. Acta Math. 137 (1976), 247-320.
