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## **Diffraction by convex bodies**

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DIFFRACTION BY CONVEX BODIES

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In the spectral theory of the laplacian in exterior domains "distorted plane waves" are fundamental. For the exterior domain  $\mathbf{R}^n \setminus K$ , where  $K$  is a compact set with smooth boundary  $\partial K$ , one defines the distorted plane wave  $\phi(x, \omega, k)$  for the Dirichlet problem as follows :

- i)  $(\Delta + k^2)\phi = 0$  on  $\mathbf{R}^n \setminus K$ ,
- ii)  $\phi = 0$  on  $\partial K$  (Dirichlet condition),
- iii)  $\phi = e^{-ikx \cdot \omega} + v$ , where as  $|x| \rightarrow \infty$   
 $v = |x|^{\frac{1-n}{2}} e^{-ik|x|} (f(\frac{x}{|x|}) + O(\frac{1}{|x|}))$  (Sommerfeld condition).

For a proof of the existence and uniqueness of  $\phi$  satisfying i) - iii) one may consult [11].

This seminar deals with an approximate construction of  $\phi(x, \omega, k)$  in the case that  $K$  is strictly convex - in the sense that the normal curvatures of  $\partial K$  are everywhere strictly positive. The construction is asymptotic to order  $k^{-N}$  for any given  $N$  as  $k$  tends to  $\infty$ , and it permits the explicit asymptotic expansion of two quantities of interest in scattering theory, the scattering phase  $s(k)$  and the forward diffraction peak  $a(\theta, \theta, k)$ . These can be expressed in terms of  $\phi(x, \omega, k)$  as follows :

$$\frac{ds}{dk} = \frac{1}{8\pi^2} \left(\frac{k}{2\pi}\right)^{n-3} \int_{|\omega|=1} d\omega \int_{\partial K} \left| \frac{\partial \phi}{\partial \nu} \right|^2 (x \cdot \nu) dS$$

$$a(\theta, \omega, k) = \frac{i}{4\pi} \left(\frac{k}{2\pi}\right)^{n-2} \int_{\partial K} e^{ik\theta \cdot x} \frac{\partial \phi}{\partial \nu} dS ,$$

where  $\nu$  is the unit normal to  $\partial K$  pointing into  $\mathbf{R}^n \setminus K$ . By substituting the approximations for  $\phi$  into these formulas one can conclude that  $ds/dk$  and  $a(\theta, \theta, k)$  have complete asymptotic expansions of the form

$$\sim \sum k^{-n_i} (a_i + b_i \log k), \quad n_i \searrow -\infty ,$$

and compute the first few terms :

$$(1) \quad \frac{1}{2\pi} \frac{ds}{dk} = \frac{n(4\pi)}{\Gamma(1+\frac{n}{2})} k^{n-1} V(K) - \frac{(n-1)(4\pi)}{\Gamma(1+\frac{n-1}{2})} k^{n-2} A(\partial K) + O(k^{n-3})$$

$$(2) \quad a(\theta, \theta, k) = \left(\frac{k}{2\pi}\right)^{n-1} A(\theta) + c_n k^{n-\frac{5}{3}} \int_{\Gamma} K^{-\frac{1}{3}}(\theta) dS + \dots$$

Here  $V(K)$  is the volume of  $K$ ,  $A(\partial K)$  is volume of  $\partial K$ ,  $A(\theta)$  is the volume of the projection of  $K$  onto  $x \cdot \theta = 0$ ,  $\Gamma$  is the boundary of this projection,  $dS$  is the volume form on  $\Gamma$ , and  $K(\theta)$  is the normal curvature in direction  $\theta$  on the pre-image of  $\Gamma$  in  $K$ . The constant  $c_n$  is the finite part of a definite integral of Airy functions and depends only on  $n$ .

The constructions given here follows those of Ludwig [3] very closely but make use of improvements made possible by Melrose's proof of the symplectic equivalence of glancing hypersurfaces [7]. For a discussion of (1) one may see [5]. The expansion (2) was derived when  $K$  is a sphere by Rubinow and Wu [10], and conjectured for convex bodies by Keller and Rubinow [2]. The leading term was derived rigorously by Majda and Taylor [6]. The complete asymptotic expansion is due to R. Melrose [9]. The method of [9] is different from that used here and appears to be more powerful as it yields the same results for the Neumann problem. Still more refined results on  $a(\theta, \omega, k)$ -which permit a uniform expansion near  $\theta = \omega$ -have been obtained by Melrose and M. E. Taylor. The construction given here seems sufficiently intuitive -at least to the author - that it may serve as a prologue to the results of Melrose and Melrose-Taylor.

### Localization

Using the standard construction of geometric optics one can decompose  $e^{-ikx \cdot \omega}$  into a sum of terms  $u_e$ , where

$$u_e = e^{-ikx \cdot \omega} \left( a_0 + \frac{a_1}{k} + \dots + \frac{a_M}{k^M} \right)$$

such that

$$i) \quad (\Delta + k^2)u_e = O(k^{-N})$$

ii) the projections of the supports of  $u_e$  onto  $x \cdot \omega = 0$  can be made subordinate to any given cover of  $x \cdot \omega = 0$ .

The strategy here will be, given  $u_e$  to construct a  $u_s$  satisfying

- i)  $(\Delta + k^2)u_s = O(k^{-N})$
- ii)  $u_s = -u_e$  on  $\partial K$
- iii)  $u_s$  satisfies the Sommerfeld condition.

Actually one has only to construct  $u_s$  on a neighborhood of  $\partial K$  in  $\mathbf{R}^n \setminus K$  satisfying i) and ii) with wave fronts -or more precisely "frequency set" (see [1]) - over points near  $\partial K$  but strictly inside  $\mathbf{R}^n \setminus K$  directed toward  $\partial K$ . Then  $u_s$  can be extended to satisfy the Sommerfeld condition by the outgoing Green's function for the laplacian on  $\mathbf{R}^n$  (see [4], pp.521-3).

If the projection of the support of  $u_e$  on  $x \cdot \omega$  does not intersect  $\Gamma$ , the construction of  $u_s$  is a standard application of geometric optics. Hence from here on we consider only  $u_e$  whose support projects onto a neighborhood -which we may take as small as we wish - of a point on  $\Gamma$ .

#### The Ludwig-Melrose construction

The idea here is to find a representation of  $u_e$  in the form

$$(3) \quad u_e = \int_{\mathbf{R}^{n-1}} e^{ik\theta} (a \text{Ai}(-k^{2/3}\rho) + b \text{Ai}'(-k^{2/3}\rho)) d\xi + O(k^{-N})$$

where the integrand is an asymptotic solution to  $(\Delta + k^2)w = 0$  uniformly in  $\xi$ , and one has additionally

$$(4) \quad \text{i) } \rho = \xi_1 \text{ and } \frac{\partial \rho}{\partial \nu} > 0 \text{ on } \partial K ,$$

$$(5) \quad \text{ii) } b = 0 \text{ on } \partial K.$$

The function  $\text{Ai}$  is the standard Airy function

$$\text{Ai}(s) = \int_{-\infty}^{\infty} e^{i(\beta s + \frac{\beta^3}{3})} d\beta ,$$

and  $a$  and  $b$  have the form

$$a = \sum_{i=0}^R a_i(x, \xi) k^{-i + \frac{n}{2} - \frac{1}{3}}, \quad b = \sum_{i=0}^R b_i(x, \xi) k^{-i + \frac{n}{2} - \frac{2}{3}}.$$

Once we have (3) - (5) the function  $u_s$  will be given by

$$(6) \quad u_s = - \int_{\mathbf{R}^{n-1}} e^{ik\theta} (aA(-k^{2/3}\rho) + bA'(-k^{2/3}\rho)) \frac{A_i(-k^{2/3}\xi_1)}{A(-k^{2/3}\xi_1)} d\xi,$$

where  $A(s) = \text{Ai}(e^{-\frac{2\pi i}{3}} s)$ . Note that, since  $A$  satisfies Airy's differential equation, the integrand is automatically an asymptotic solution to  $(\Delta + k^n)w = 0$  in  $\mathbf{R}^n \setminus K$  - since  $A(s)$  is exponentially increasing as  $s \rightarrow +\infty$ , we use the fact  $\rho > \xi$  in  $\mathbf{R}^n \setminus K$  here. The choice of  $A$  is made so that the frequency set of  $u_s$  is directed toward  $\partial K$  from  $\mathbf{R}^n \setminus K$ .

As we mentioned earlier the constructions here are strictly local. We assume that we are given  $x_0 \in \partial K$  with  $\omega \cdot \nu(x_0) = 0$  and, writing  $\xi = (\xi_1, \xi')$ , a  $\xi'_0$  such that  $\nabla\theta(x_0, 0, \xi'_0) = -\omega$ . All the assertions (of existence etc...) in the constructions that follow are to be qualified by "for  $(x, \xi)$  in a neighborhood of  $(x_0, 0, \xi'_0)$ " - even though this will always be omitted. Just how small the support of  $u_e$  must be is only determined at the end of the construction.

The representation (3) with conditions (4), (5) is the delicate part of the construction. One first determines  $\theta$  and  $\rho$  and then  $a$  and  $b$ . In order that the integrand in (3) be an asymptotic solution to  $(\Delta + k^2)w = 0$ ,  $\theta$  and  $\rho$  must satisfy the "eichonal" equations :

$$(7) \quad \text{a) } |\nabla_x \theta|^2 + \rho |\nabla_x \rho|^2 = 1$$

$$\text{b) } \nabla_x \rho \cdot \nabla_x \theta = 0$$

on  $\rho \geq 0$ . These equations are solved by choosing a smooth family of strictly convex surfaces  $S_\xi$  with  $S_\xi = \partial K$  when  $\xi_1 = 0$ , and defining  $\rho(x, \xi) = 0$  on  $S_\xi$ . Note that, since we want  $\nabla_x \rho \neq 0$ , this implies  $|\nabla_x \theta(x, \xi)|^2 = 1$  on  $S_\xi$  and  $\nabla_x \theta(x, \xi)$  is tangent to  $S_\xi$ . Thus we must choose  $\theta$  on  $S_\xi$  to be a solution of the surface eichonal on  $S_\xi$ . With these choices 7a) and 7b) determine  $\theta$  and  $\rho$  uniquely for  $x$  outside  $S_\xi$ , i.e. in the region where we will have  $\rho \geq 0$ . The condition (4) implies and, modulo a change of variables in  $\xi$ , is equivalent to the following geometric

condition on  $S_\xi$  and  $\theta \uparrow S_\xi$  : if the straight line through  $x_0 \in S_{\xi_0}$  with direction  $\nabla_x \theta(x_0, \xi_0)$  hits  $\partial K$  at  $x'$ , then the reflection of this line in  $\partial K$  is, for some  $x_1 \in S_{\xi_0}$ , the line through  $x_1$  with direction  $\nabla_x \xi(x_1, \xi_0)$ .

In [3] the surfaces  $S_\xi$  were only chosen so that (4) held up to an error which was  $O(\xi_1^N)$  for all  $N$ . However, it is a direct consequence of [7] (the derivation is given in [8] that  $S_\xi$  and  $\theta \uparrow S_\xi$  can be chosen so that (4) holds exactly. Then one completes the construction by extending  $\theta$  and  $\rho$  as  $C^\infty$  functions in the complement of  $\rho \geq 0$ , maintaining (4).

If we replace  $A_i$  and  $A_i'$  by their integral representations, (3) becomes

$$u_e = \int e^{ik(\theta - \beta\rho + \frac{\beta^3}{3})} (a + ik^{1/3}\beta b) d\xi d\beta .$$

Note that, writing  $\xi = (\xi_1, \xi')$ , if  $\det \frac{\partial^2 \theta}{\partial \xi' \partial \xi'} \neq 0$ , then this integral can be expanded by the method of stationary phase. If the result of this expansion agrees with  $u_e$ , then we must have

$$(8) \quad -x \cdot \omega = \phi(x) \equiv (\theta - \beta\rho + \frac{\beta^3}{3}) \uparrow \xi = \xi(x), \beta = \beta(x),$$

where  $\xi(x)$  and  $\beta(x)$  are defined by

$$\theta_\xi - \beta\rho_\xi = 0 \quad \text{and} \quad -\rho + \beta^2 = 0$$

However, since  $\phi$  is automatically a solution of the standard eichonal ( $|\nabla \phi|^2 = 1$ ) it suffices to have (8) hold for  $x$  on a surface transverse to  $\omega$ . The eichonal equations (7) and condition (4) remain valid if we replace  $\theta$  by  $\theta + \psi(\xi)$ , and we must exploit this freedom to obtain  $\det \frac{\partial^2 \theta}{\partial \xi' \partial \xi'} \neq 0$  and (8) .

Introducing local coordinates  $(z, y)$  where  $z = 0$  on  $\partial K$ ,  $\partial f / \partial z = \partial f / \partial \nu$  on  $\partial K$  and  $y_1 = x \cdot \omega$  on  $\partial K$ , we can assume  $S_\xi$  is given by  $z = \alpha(y, \xi)$ . Writing  $y = (y_1, y')$ , it is a consequence of the constructions in [7] that  $S_\xi$  and  $\theta \uparrow S_\xi$  can be chosen so that  $\det \frac{\partial^2 \theta}{\partial y \partial \xi} \neq 0$  (this is used in [8]) and  $\det \frac{\partial^2 \theta}{\partial y' \partial y'} \neq 0$ . We let  $\beta$  denote  $x \cdot \omega$  written as a function of  $(z, y)$ .



To achieve 8) we begin by solving  $(\theta_z, \theta_y) = (\beta_z, \beta_y)$  on  $z = \alpha(y, \xi)$  for  $y = y(\xi)$ . This is over determined, but since  $|\nabla_x \theta| = |\nabla_x(x \cdot \omega)| = 1$  when  $z = \alpha(y, \xi)$  it suffices to solve  $(\theta_z, \theta_{y'}) = (\beta_z, \beta_{y'})$  on  $z = \alpha(y, \xi)$ . To check the hypothesis of the implicit function theorem, we set  $\xi_1 = 0$  (so that  $\alpha = \theta_z = 0$ ) and compute

$$\begin{pmatrix} \frac{\partial^2(\theta - \beta)}{\partial z \partial y_1} & \frac{\partial^2(\theta - \beta)}{\partial z \partial y'} \\ \frac{\partial^2(\theta - \beta)}{\partial y' \partial y_1} & \frac{\partial^2(\theta - \beta)}{\partial y' \partial y'} \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 \beta}{\partial z \partial y_1} & -\frac{\partial^2 \beta}{\partial z \partial y'} \\ \frac{\partial^2 \theta}{\partial y' \partial y_1} & \frac{\partial^2 \theta}{\partial y' \partial y'} \end{pmatrix}$$

Since  $\partial^2 \beta / \partial z \partial y_1$  is nonzero by the strict convexity of  $\partial K$ , and we may assume  $\partial^2 \theta / \partial y' \partial y_1$  vanishes at the base point, we conclude that

$(\theta_z, \theta_y) = (\beta_z, \beta_y)$  can be solved for  $y(\xi)$  on  $z = \alpha(y, \xi)$ .

Now we defined  $\psi$  by the requirement

$$(9) \quad \theta_\xi(\alpha(y(\xi), \xi), y(\xi), \xi) + \psi_\xi = 0 \quad .$$

To check that

$$\theta_{\xi\xi} + \theta_{\xi z} \alpha_y y_\xi + \theta_{\xi y} y_\xi$$

is symmetric, we note that

$$\theta_{z\xi} + \theta_{zz} \alpha_y y_\xi + \theta_{zy} y_\xi = \beta_{zz} \alpha_y y_\xi + \beta_{zy} y_\xi$$

$$\theta_{y\xi} + \theta_{yz} \alpha_y y_\xi + \theta_{yy} y_\xi = \beta_{yz} \alpha_y y_\xi + \beta_{yy} y_\xi \quad .$$

Since (9) determines  $\psi$  up to an additive constant, we complete the construction of  $\psi$  by choosing this constant so that  $\theta + \psi = -x \cdot \omega$  at the base point. Further work along exactly the same lines shows that  $\det \partial^2(\theta + \psi) / \partial \xi' \partial \xi' \neq 0$  (This uses  $\det \frac{\partial^2 \theta}{\partial y \partial \xi} \neq 0$ ) and that

$z = \alpha(y(\xi), \xi)$ ,  $y = y(\xi)$  defines a surface transverse to  $\omega$ . Then it follows that (8) holds when  $\theta$  is replaced by  $\theta + \psi$ .

We will not discuss the construction of the amplitudes  $a(x, \xi, k)$  and  $b(x, \xi, k)$  here. In [3]  $a$  and  $b$  are constructed so that, given the preceding construction of  $\theta$  and  $\rho$ , (3) holds and in place of (5) one has  $b = O(\xi_1^N)$  for any  $N$  on  $\partial K$ . The modifications needed to improve this to (5), i.e.  $b = 0$  on  $\partial K$  are substantially simpler than those that were used in obtaining (4) - no use of [7] is involved. Actually, imposing (5) for all  $x \in \partial K$  (or even the weaker condition  $b = O(\xi_1^N)$ ) would make it impossible to keep the intersection of the support of  $a$  and  $b$  with  $\partial K$  strictly inside the set where  $\theta$  and  $\rho$  are defined. This is a turn would prevent us from making the integrand in (3) an asymptotic solution to  $(\Delta + k^2)w = 0$  on a neighborhood of  $\partial K$  in  $\mathbf{R}^n \setminus K$ . However, we only impose (5) for  $(x, \xi)$  in a small neighborhood of the base point. Provided the projection of the support of  $u_e$  is made sufficiently small, one still has  $u_s(x, k) + u_e(x, k) = O(k^{-N})$  for  $x \in \partial K$  in this case.

#### The representation of $\partial\Phi/\partial\nu$

Away from the intersection of  $\partial K$  with the pre-image of  $\Gamma$  the expansion of  $\partial\Phi/\partial\nu$  is easy to compute from geometric optics; the leading term is

$$\frac{\partial\Phi}{\partial\nu} = \begin{cases} -2ik\omega \cdot \nu e^{-ikx \cdot \omega} & \text{if } \omega \cdot \nu < 0 \\ 0 & \text{if } \omega \cdot \nu > 0 \end{cases}$$

(the "Kirchhoff approximation"). The next term is  $O(1)$  and it does not contribute to the second term in (1) and (2).

In a neighborhood of a point  $x_0 \in \partial K$  where  $\omega \cdot \nu(x_0) = 0$ , i.e. a point that projects to  $\Gamma$ , one can combine (3)-(6) to get

$$\frac{\partial\Phi}{\partial\nu} = \int_{\mathbf{R}^{n-1}} e^{ik\theta} \left( -k^{2/3} \frac{\partial\rho}{\partial\nu} a + \frac{\partial b}{\partial\nu} \right) F(-k^{2/3} \xi_1) d\xi + O(k^{-N})$$

where  $F(x) = Ai'(x) - \frac{A'(x)}{A(x)} Ai(x)$ . Expanding by stationary phase in the variable  $\xi'$ , this can be further simplified to a representation in the form

$$(10) \quad \frac{\partial\Phi}{\partial\nu}(x, k) = \int_{\mathbf{R}} e^{ik\check{\theta}(x, \xi_1)} G(x, \xi_1, k) F(-k^{2/3} \xi_1) d\xi_1 + O(k^{-N})$$

where  $G$  has the form

$$G = \sum_{i=0}^M G_i(x, \xi_1) k^{\frac{1}{6} + i}$$

Substituting (10) and the analogous expression (derived from (3)),

$$e^{-ikx \cdot \omega} = \int_{\mathbf{R}} e^{ik \tilde{\theta}(x, \xi_1)} H(x, \xi_1, k) \text{Ai}(-k^{2/3} \xi_1) d\xi_1 + O(k^N)$$

into the integral formulas for  $ds/dk$  and  $a(\omega, \omega, k)$  one derives (1) and (2). The crucial advantage here of (10) over the formulas that could be obtained from [3] is that one can eliminate the oscillatory  $e^{ik\tilde{\theta}}$  factors by an integration in an  $x$ -variable without disturbing the Airy functions. At the final stage in the derivation of (1) and (2) one must expand integrals of the form

$$\int_{\mathbf{R}} H(s) G(-k^{2/3} s) ds$$

where  $G$  is a polynomial in  $A'/A$ ,  $\bar{A}'/\bar{A}$ ,  $\text{Ai}$  and their derivatives, it is here that the  $k^r \log k$  terms seem to appear in the asymptotic expansions. It is known that there are no logarithms (and only integral powers of  $k$ ) in the expansion of  $ds/dk$  (see [5]), but unknown whether logarithms actually appear in the expansion of  $a(\theta, \theta, k)$ .

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