

# SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES – ÉCOLE POLYTECHNIQUE

G. ESKIN

**Well-posedness and propagation of singularities for initial  
boundary value problem for second order hyperbolic equation  
with general boundary condition**

*Séminaire Équations aux dérivées partielles (Polytechnique)* (1979-1980), exp. n° 2,  
p. 1-14

[http://www.numdam.org/item?id=SEDP\\_1979-1980\\_\\_\\_A2\\_0](http://www.numdam.org/item?id=SEDP_1979-1980___A2_0)

© Séminaire Équations aux dérivées partielles (Polytechnique)  
(École Polytechnique), 1979-1980, tous droits réservés.

L'accès aux archives du séminaire Équations aux dérivées partielles (<http://sedp.cedram.org>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

PLATEAU DE PALAISEAU - 91128 PALAISEAU CEDEX

Téléphone : 941.82.00 - Poste N°

Télex : ECOLEX 691 596 F

S E M I N A I R E   G O U L A O U I C - S C H W A R T Z   1 9 7 9 - 1 9 8 0

WELL-POSEDNESS AND PROPAGATION OF SINGULARITIES  
FOR INITIAL BOUNDARY VALUE PROBLEM FOR SECOND  
ORDER HYPERBOLIC EQUATION WITH  
GENERAL BOUNDARY CONDITION.

by G. ESKIN



§1. STATEMENT OF RESULTS

Let  $G$  be a domain in  $\mathbb{R}^n$  with a smooth boundary  $\gamma$  and let  $A(x, \mathcal{D})$  be a second order differential operator in the cylinder  $\Omega = (-\infty, +\infty) \times G \subset \mathbb{R}^{n+1}$ , strictly hyperbolic with respect to  $x_0$ , where  $x = (x_0, x_1, \dots, x_n)$ ,  $x_0 \in (-\infty, +\infty)$ ,  $(x_1, x_2, \dots, x_n) \in G$ ,  $\mathcal{D} = (i \frac{\partial}{\partial x_0}, \dots, i \frac{\partial}{\partial x_n})$ .

Consider the following mixed problem

$$(1.1) \quad A(x, \mathcal{D})u = 0 \quad \text{in } \Omega,$$

$$(1.2) \quad u = 0 \quad \text{for } x_0 < 0, \quad x \in \Omega,$$

$$(1.3) \quad B(x, \mathcal{D})u|_{\Gamma} = h(x') \quad \text{for } x' \in \Gamma = (-\infty, +\infty) \times \gamma,$$

where  $h \in \mathcal{D}'(\Gamma)$ ,  $h = 0$  for  $x_0 < 0$ ,  $u \in \mathcal{D}'(\Omega)$  and  $B(x, \mathcal{D})$  is a differential operator of order  $r$ . We shall find when for a given  $B(x, \mathcal{D})$  the problem (1.1), (1.2), (1.3) is well-posed or ill-posed. Denote by  $G_\delta \approx \Gamma \times [0, \delta)$  the  $\delta$ -neighbourhood of  $\Gamma$  in  $G$ . We shall use in  $G_\delta$  the coordinates  $(x', x_n)$ , where  $x' \in \Gamma$  and  $x_n$  is the distance to  $\Gamma$ ,  $x_n \in [0, \delta)$ . The dual coordinates in  $T^*(G_\delta)$  will be denote by  $(\xi', \xi_n)$  where  $\xi' \in T^*(\Gamma)$  and  $\xi_n \in \mathbb{R}^1$ .

The principal symbol of  $A(x, \mathcal{D})$  has in this system of coordinates the following form

$$(1.4) \quad A^{(0)}(x, \xi) = (\xi_n - \lambda(x, \xi'))^2 - \mu(x, \xi'),$$

where  $\lambda(x, \xi')$ ,  $\mu(x, \xi')$  are real and  $\frac{\partial \mu}{\partial \xi_0} \neq 0$  when  $\mu = 0$  (cf. [3]). As in [3] denote  $N_0 = \{(x', \xi') \in T_0^*(\Gamma), \mu(x', 0, \xi') = 0\}$ ,  $N_\pm = \{(x', \xi') \in T_0^*(\Gamma), \mu(x', 0, \xi') \gtrless 0\}$ .

Set

$$(1.5) \quad b(x', \zeta') = B^{(0)}(x', 0, \zeta', \lambda(x', 0, \zeta') - \sqrt{\mu(x', 0, \zeta')}),$$

where  $B^{(0)}(x', x_n, \xi', \xi_n)$  is the principal symbol of the operator  $B(x, \mathcal{D})$  written in the system of coordinate introduced above,  $\zeta' = (\xi_0 + i\tau, \xi_1, \dots, \xi_{n-1})$ ,  $\tau > 0$ , and  $\sqrt{\mu(x', 0, \zeta')}$  is the branch of the square root such that  $\text{Im} \sqrt{\mu(x', 0, \zeta')} > 0$  for  $(x', \xi') \in T_0^*(\Gamma)$ ,  $\tau > 0$ .

It is known (see [12] ) that the condition

$$(1.6) \quad b(x', \zeta') \neq 0 \text{ for } \forall \tau > 0, (x', \xi') \in T_0^*(\Gamma)$$

is necessary for the well-posedness of the initial-boundary problem (1.1), (1.2), (1.3).

We shall see below that one needs additional assumptions on  $b(x', \zeta')$  for real  $\zeta' = \xi'$ . It is well-known (see, for example, [13] ) that microlocally on  $N_+ \cup N_-$  the problem (1.1), (1.2), (1.3) can be reduced to the solution of a pseudo-differential equation

$$(1.7) \quad b(x', \vartheta') u_0 = h,$$

where  $u_0 = u|_{\Gamma}$ .

We shall assume that  $b(x', \vartheta')$  is an operator of principal type in  $N_+ \cup N_-$ ; more exactly we assume that

$$(1.8) \quad \frac{\partial b(\hat{x}, \hat{\xi})}{\partial \xi_0} \neq 0 \text{ if } (\hat{x}, \hat{\xi}) \in N_+ \cup N_- \text{ and } b(\hat{x}, \hat{\xi}) = 0.$$

Therefore  $b(x', \xi')$  has the following form in the neighbourhood of  $(\hat{x}, \hat{\xi})$  :

$$(1.9) \quad b(x', \xi') = (\xi_0 + b_1(x', \xi')) b^{(1)}(x', \xi'),$$

where  $b^{(1)}(x', \xi') \neq 0$ . The main problem is to find the conditions on  $b(x', \xi')$  for  $(x', \xi') \in N_0$ .

We shall assume additionally that

$$(1.10) \quad \frac{\partial B^{(0)}(\hat{x}, 0, \hat{\xi}, \lambda(\hat{x}, 0, \hat{\xi}))}{\partial \xi_1} \neq 0 \text{ for all } (\hat{x}, \hat{\xi}) \in N_0 \text{ such that } b(\hat{x}, \hat{\xi}) = 0.$$

Then in a small neighbourhood of such point  $(\hat{x}, \hat{\xi}) \in N_0$  we have

$$(1.11) \quad B^{(0)}(x', 0, \xi', \xi_n) = B^{(2)}(x', \xi', \xi_n) (\xi_n + b_2(x', \xi')),$$

where  $B^{(2)}(x', \xi', \xi_n) \neq 0$ . Therefore in this neighbourhood we have

$$(1.12) \quad b(x', \zeta') = -B^{(2)}(x', \zeta', \lambda(x', 0, \zeta') - \sqrt{\mu(x', 0, \zeta')}) (\sqrt{\mu(x', 0, \zeta')} + \lambda_1(x', \zeta'))$$

where

$$(1.13) \quad \lambda_1(x', \zeta') = -(\lambda(x', 0, \zeta') + b_2(x', \zeta')).$$

The conditions of the well-posedness of the initial-boundary problem depend on the geometry of the boundary. We shall consider only the cases when the boundary is strictly convex or concave with respect to the null-bicharacteristics of  $A(x, \mathcal{D})$ . The boundary  $\Gamma$  is called strictly convex for  $(x', \xi') \in N_0$  if for  $x_n = 0$  and  $\mu(x', 0, \xi') = 0$  we have (cf. [3]) :

$$(1.14) \quad \{\xi_n - \lambda(x', x_n, \xi'), \mu(x', x_n, \xi')\} < 0,$$

where  $\{\xi_n - \lambda, \mu\}$  is the Poisson bracket.

The boundary  $\Gamma$  is called strictly concave for  $(x', \xi') \in N_0$  if (1.14) with the opposite sign holds.

In the case of the strictly concave boundary there are few restrictions for the well-posedness of the initial-boundary problem.

Theorem 1.1 : Let the boundary  $\Gamma$  be strictly concave for all  $(x', \xi')$  and let the assumptions (1.6), (1.8) and (1.10) hold. Then for each  $h \in H_{s+r-1}^\tau(\Gamma)$  there exists a unique solution  $u \in H_s^\tau(\Omega)$  of the problem (1.1), (1.2) (1.3) when  $\tau$  is sufficiently large.

Here  $H_s^\tau(\Omega)$  is a Sobolev's space with a finite norm

$\|u\|_{s,\tau}^2 = \|e^{-\tau x} u\|_s^2 = \|\Lambda_\tau^{2s} u\|_0^2$ ,  $\Lambda_\tau^{2s}$  is a pseudo-differential operator with a symbol  $(|\xi|^2 + \tau^2)^s$ ,  $u = 0$  for  $x_0 < 0$ ,  $h = 0$  for  $x_0 < 0$ . For the convex boundary we shall consider separately three cases.

It follows from (1.12) that if  $(\hat{x}, \hat{\xi}) \in N_0$  and  $b(\hat{x}, \hat{\xi}) = 0$  then also  $\lambda_1(\hat{x}, \hat{\xi}) = 0$ . Suppose that if  $(\hat{x}, \hat{\xi}) \in N_0$  and  $b(\hat{x}, \hat{\xi}) = 0$  then there exists a conic neighbourhood  $U_0$  of  $(\hat{x}, \hat{\xi})$  and a constant  $C_0$  such that

$$(1.15) \quad \frac{-\text{Re } \lambda_1(x', \xi')}{\ln \frac{1}{|\text{Re } \lambda_1(x', \xi')|}} \text{sgn } \frac{\partial \mu(\hat{x}, \hat{\xi})}{\partial \xi_0} < C_0 (\text{Im } \lambda_1(x', \xi'))^2$$

for any  $(x', \xi') \in U_0 \cap N_0$  and  $|\xi'| = 1$ .

Theorem 1.2 : Let the boundary  $\Gamma$  be strictly convex and the conditions (1.6), (1.8) (1.10) be satisfied. Suppose that for any  $(\hat{x}, \hat{\xi}) \in N_0$  such that  $b(\hat{x}, \hat{\xi}) = 0$  the condition (1.15) holds. Then for each  $h \in H_{s+r-1}(\Gamma_T)$  there exists a unique solution  $u \in H_s(\Omega_T)$  of the problem (1.1), (1.2), (1.3) where  $T > 0$  is sufficiently small,  $\Omega_T = (-\infty, T) \times G$ ,  $\Gamma = (-\infty, T) \times \gamma$  and  $H_s(\Omega_T)$  is a Sobolev's space on  $\Omega_T$ ,  $u = 0$  for  $x_0 < 0$ ,  $h = 0$  for  $x_0 < 0$ .

Remark 1.1 : If  $\text{Im } \lambda_1 \equiv 0$  then the condition (1.15) means that  $\lambda_1(x', \xi') \geq 0$  in  $U_0 \cap N_0$ , if  $\frac{\partial \mu(\hat{x}, \hat{\xi})}{\partial \xi_0} > 0$  and  $\lambda_1(x', \xi') \leq 0$  in  $U_0 \cap N_0$  if  $\frac{\partial \mu(\hat{x}, \hat{\xi})}{\partial \xi_0} < 0$ . We note that  $\frac{\partial \mu}{\partial \xi_0} \neq 0$  when  $\mu = 0$  since the symbol  $A^{(0)}(x, \xi)$  is hyperbolic with respect to  $\xi_0$ .

Now we shall consider the case when the condition (1.15) is not satisfied.

We suppose that there exists a point  $(\hat{x}, \hat{\xi}) \in N_0$  with the following properties :

$$(1.16) \quad \begin{aligned} &\lambda_1(\hat{x}, \hat{\xi}) = 0 \text{ and there is a sequence } (\hat{x}_m, \hat{\xi}_m) \in N_0, |\hat{\xi}_m| = 1 \text{ such} \\ &\text{that } (\hat{x}_m, \hat{\xi}_m) \rightarrow (\hat{x}, \hat{\xi}) \text{ and } \text{Re } \lambda_1(\hat{x}_m, \hat{\xi}_m) < 0 \text{ if } \frac{\partial \mu(\hat{x}, \hat{\xi})}{\partial \xi_0} > 0 \\ &\text{and } \text{Re } \lambda_1(\hat{x}_m, \hat{\xi}_m) > 0 \text{ if } \frac{\partial \mu(\hat{x}, \hat{\xi})}{\partial \xi_0} < 0. \end{aligned}$$

We assume also that

$$(1.17) \quad \frac{(\text{Im } \lambda_1(\hat{x}_m, \hat{\xi}_m))^2 \ln |\text{Re } \lambda_1(\hat{x}_m, \hat{\xi}_m)|}{\text{Re } \lambda_1(\hat{x}_m, \hat{\xi}_m)} \rightarrow \infty \text{ when } m \rightarrow \infty$$

and

$$(1.18) \quad | \{ \lambda_1(x', \xi'), \mu(x', 0, \xi') \} | < C |\text{Re } \lambda_1(x', \xi')|^{\delta_1} \text{ for } (x', \xi') = (\hat{x}_m, \hat{\xi}_m)$$

where  $\delta_1$  is an arbitrary positive constant and  $C, \delta_1$  are independent on  $m$ .

Theorem 1.3 : Suppose that there exists  $(\hat{x}, \hat{\xi}) \in N_0$  such that (1.16), (1.17), (1.18) hold and that the boundary  $\Gamma$  is strictly convex at the point  $(\hat{x}, \hat{\xi})$ . Then the initial-boundary problem (1.1) (1.2) (1.3) is ill-posed in the space of distributions. More precisely, there exists  $h(x') \in C_0^\infty(\Gamma)$ ,  $h(x') = 0$  for

$x_0 < 0$  such that there is no  $u(x) \in \mathcal{D}'(\Omega_T)$  which is a solution of the problem (1.1), (1.2), (1.3) for  $x_0 < T$ .

Here  $T > \hat{x}_0$  where  $\hat{x} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{n-1})$ ,  $(\hat{x}, \hat{\xi})$  is the point satisfying (1.14), (1.16), (1.17), (1.18) and  $\mathcal{D}'(\Omega_T)$  is the space of distributions on  $\Omega_T$ .

Remark 1.2 : It is obvious that (1.17) is satisfied, for example, when  $\text{Im } \lambda_1(x', \xi') \equiv 0$ . We note that (1.18) implies

$$(1.18') \quad \{\lambda_1(x', \xi'), \mu(x', 0, \xi')\} = 0 \quad \text{for } (x', \xi') = (\hat{x}, \hat{\xi}).$$

One can show that when  $\text{Im } \lambda_1(x', \xi') \equiv 0$  and (1.16), (1.18') hold then there exists a sequence  $(\tilde{x}_m, \tilde{\xi}_m) \in N_0$  such that  $(\tilde{x}_m, \tilde{\xi}_m) \rightarrow (\hat{x}, \hat{\xi})$ ,  $|\tilde{\xi}_m| = 1$ ,

$$\lambda_1(\tilde{x}_m, \tilde{\xi}_m) \cdot \text{sgn} \frac{\partial \mu(\hat{x}, \hat{\xi})}{\partial \xi} < 0 \text{ and (1.18) holds for } (x', \xi') = (\tilde{x}_m, \tilde{\xi}_m) \text{ with } \delta_1 = \frac{1}{2}.$$

Theorem 1.4 : Let the boundary  $\Gamma$  be strictly convex and the assumptions (1.6), (1.8) and (1.10) hold. Suppose that for each  $(\hat{x}, \hat{\xi}) \in N_0$  such that  $b(\hat{x}, \hat{\xi}) = 0$  and (1.15) is not satisfied we have

$$(1.19) \quad \{\lambda_1(x', \xi'), \mu(x', 0, \xi')\} \neq 0 \quad \text{for } (x', \xi') = (\hat{x}, \hat{\xi})$$

Then for each  $h \in H_{s+r-1+m}(\Gamma_T)$ ,  $h = 0$  for  $x_0 < 0$ , there exists a unique solution  $u \in H_s(\Omega_T)$  of the problem (1.1), (1.2), (1.3) where  $T$  is sufficiently small and

$$(1.20) \quad m > \max \frac{1}{3} \frac{-\{\xi_n - \lambda_1, \mu\}}{|\{\lambda_1, \mu\}|}$$

where the maximum in (1.20) is taken over all  $(\hat{x}, \hat{\xi}) \in N_0$  such that  $b(\hat{x}, \hat{\xi}) = 0$  and (1.15) is not satisfied .

Remark 1.3 : Part of the results containing in the Theorems 1.1 and 1.3 was proved by M. Ikawa (see [7], [8], [9]). The Theorem 1.2 has an intersection with results of M. Miyatake [11], L. Gårding [5] and R. Melrose and J. Sjöstrand [10].

Remark 1.4 : We have described also the singularities of the solution of the problem (1.1), (1.2), (1.3) under the assumptions of the Theorems 1.1 and 1.4 and additional assumptions that  $b_1(x', \xi')$  and  $b_2(x', \xi')$  are real (see (1.9) and (1.11)).



Remark 1.5 : Roughly speaking the phenomenon of ill-posedness of the problem (1.1) (1.2), (1.3) under the assumptions of the Theorem 1.3 can be explained in a following way :

When (1.16) holds and  $\lambda_1(x', \xi')$  is real there is a propagation of singularities along the boundary (the boundary waves). These singularities are overgoing multiple reflections the number of which tends to the infinity for any time interval containing  $\hat{x}_0$  when the reflecting points approach  $(\hat{x}, \hat{\xi})$ . Therefore under the assumptions of the Theorem 1.3 the Green function of the problem (1.1), (1.2), (1.3) has a singularity of an infinite order, i.e. it is not a distribution. Indeed it can be shown for the model initial boundary problem (see the section 3) that under the assumptions of the Theorem 1.3 the problem (1.1), (1.2), (1.3) is well-posed in the space of ultra-distributions, i.e. functionals over  $C^\infty$ -functions belonging to some Gevrey classes. More precisely such ultradistributions have a finite order of singularities outside of  $N_0$  which increases when we approach  $N_0$  and which becomes infinite only on  $N_0$ .

It follows from the Theorem 1.4 that when (1.18') don't take place there is still a well-posedness of the problem (1.1), (1.2), (1.3). We note that when  $\{\mu, \lambda_1\} \rightarrow 0$  then  $m \rightarrow \infty$  (see (1.20)) in accordance with the Theorem 1.3.

## §2. EXAMPLES

We give now two simple examples which illustrate the Theorems 1.1, 1.2, 1.3, 1.4.

Example 2.1 : Let  $G$  be a strictly convex bounded domain in  $\mathbb{R}^2$  and  $A(x, \partial) \equiv \square = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$  is the wave operator. Consider the following initial boundary

problem

$$(2.1) \quad \square u = 0 \quad \text{for } x \in \Omega$$

$$(2.2) \quad u = 0 \quad \text{for } x_0 < 0, \quad x \in \Omega$$

$$(2.3) \quad \frac{\partial u}{\partial n} - a(x_0) \frac{\partial u}{\partial x_0} \Big|_{\Gamma} = h,$$

where  $\frac{\partial u}{\partial n}$  is the normal derivative to  $\partial G$  and  $a(x_0)$  is a real  $C^\infty$  function.

The condition (1.6) gives

$$(2.4) \quad a(x_0) > -1.$$

If  $a(x_0) \geq 0$  then all assumptions of the Theorem 1.2 are satisfied.

If all zeros of  $a(x_0)$  are of the first order and (2.4) holds then according to the Theorem 1.4 the problem (2.1), (2.2), (2.3) is well-posed.

If  $x_0^{(0)}$  is a zero of  $a(x_0)$  and  $a'(x_0)$  such that there exists a sequence  $\{x_0^{(n)}\}$  where  $a(x_0^{(n)}) < 0$  and  $x_0^{(n)} \rightarrow x_0^{(0)}$  then it follows from the Theorem 1.3 that the problem (2.1), (2.2), (2.3) is ill-posed.

We note that if  $\Omega = (-\infty, +\infty) \times \mathcal{G}$  where  $\mathcal{G}$  is the exterior of the domain  $G$  then according to the Theorem 1.1 the problem (2.1), (2.2), (2.3) is well-posed under the only assumption (2.4).

Example 2.2 : Let  $G$  be the same domain as in the example 2.1. Consider the following boundary operator

$$(2.5) \quad \frac{\partial u}{\partial n} + b(s) \frac{\partial u}{\partial s} \Big|_{\Gamma} = h,$$

where  $\frac{\partial}{\partial n}$  is the normal derivative to  $\partial G$ ,  $\frac{\partial}{\partial s}$  is the tangential derivative and  $b(s)$  is a real  $C^\infty$ -function.

It follows from the Theorem 1.4 that the problem (2.1), (2.2), (2.5) is well-posed if  $b(s)$  has no roots on  $\partial G$  of the multiplicity greater than one. If there is a point  $s_0$  such that  $b(s_0) = b'(s_0) = 0$  and  $b(s) \neq 0$  then the Theorem 1.3 gives that the problem (2.1), (2.2), (2.5) is ill-posed.

The exterior problem (2.1), (2.2) (2.5) in the domain  $\Omega = (-\infty, +\infty) \times \mathcal{G}$  is well-posed for any real  $b(s) \in C^\infty$  (see the Theorem 1.1) and [7]).

### §3. A MODEL FOR THE THEOREM 1.3

To clarify the results of the section 1 we shall consider some model equations with model boundary conditions. A good model for the initial-boundary problem in the case of convex domain is the following problem in  $\mathbb{R}_+^{n+1}$  :

$$(3.1) \quad a_-(x_n, \mathcal{D})u = 0, \quad x \in \mathbb{R}_+^{n+1},$$

$$(3.2) \quad u = 0 \quad \text{for } x_0 < 0, \quad x \in \mathbb{R}_+^{n+1},$$

$$(3.3) \quad B(\mathcal{D})u \Big|_{x_n=0} = h(x'),$$

where  $h(x')$  is a tempered distribution,  $h = 0$  for  $x_0 < 0$ ,

$$a_-(x_n, \xi) = \xi_n^2 - (\xi_0 - x_n |\xi''|) |\xi''|,$$

$B(\xi) = \xi_n - B_1(\xi'')$ ,  $B_1(\xi'') \in C^\infty$  for  $\xi'' \neq 0$  and  $B_1(\xi'')$  is a homogenous function of degree 1.

The initial-boundary Dirichlet problem for the equation (3.1) in  $\mathbb{R}_+^{n+1}$  was considered in [4].

As it was stated in [4] the unique solution of (3.1) in the class of tempered distributions which satisfies the initial data (3.2) and the Dirichlet condition  $u|_{x_n=0} = v$ ,  $v = 0$  for  $x_0 < 0$  is given by the following formula

$$(3.4) \quad h(x', x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \frac{A_0\left(\frac{\xi_0 + i\tau_0}{|\xi''|^{1/3}} - x_n |\xi''|^{2/3}\right) e^{-i(x'', \xi'') - i(\xi_0 + i\tau_0)x_0} \tilde{v}(\xi_0 + i\tau_0, \xi'') d\xi_0 d\xi''}{A_0\left(\frac{\xi_0 + i\tau_0}{|\xi''|^{1/3}}\right) |\xi''|}$$

where  $\tau_0 > 0$  is arbitrary and  $A_0(z)$  is the Airy function with the following properties

$$(3.5) \quad A_0(t) \approx \frac{C}{|t|^{1/4}} e^{-\frac{2}{3}|t|^{3/2}} \quad \text{for } t \rightarrow -\infty$$

$$A_0(t) \approx \frac{C}{t^{1/4}} \sin\left(\frac{2}{3}t^{3/2} + \frac{\pi}{4}\right) \quad \text{for } t \rightarrow +\infty.$$

To solve the problem (3.1), (3.2), (3.3) it is enough to find  $v(x')$  such that

$$(3.6) \quad b_{\tau_0}(\mathcal{D})v = h,$$

where  $b_{\tau_0}(\mathcal{D})$  is a pseudodifferential operator with a symbol

$$(3.7) \quad b_{\tau_0}(\xi_0, \xi'') = b(\xi_0 + i\tau_0, \xi'') = -i|\xi''|^{2/3} \frac{A_0'((\xi_0 + i\tau_0)|\xi''|^{-1/3})}{A_0((\xi_0 + i\tau_0)|\xi''|^{-1/3})} - B_1(\xi'').$$

The necessary and sufficient condition for the existence of the solution  $v(x')$  of the equation (3.6) which is a tempered distribution in  $(-\infty, T) \times \mathbb{R}^{n-1}$  for any finite  $T$  is the following: there exist  $C_1, C_2, C_3, N$  such that

$$(3.8) \quad |b(\xi_0 + i\tau, \xi'')| \geq C_1 (1 + |\xi'| + \tau)^{-N}$$

for any  $\tau > C_2 \ln |\xi'| + C_3$ .

Suppose that there exists  $\omega_0''$ ,  $|\omega_0''| = 1$  such that  $B_1(\omega_0'') = 0$  and suppose that there is a sequence  $\omega_m''$ ,  $|\omega_m''| = 1$  such that

$$(3.9) \quad \operatorname{Re} B_1(\omega_m'') < 0, \quad \omega_m'' \rightarrow \omega_0''$$

and

$$(3.10) \quad (\operatorname{Im} B_1(\omega_m''))^2 < \frac{1}{N} \frac{|\operatorname{Re} B_1(\omega_m'')|}{|\ln |\operatorname{Re} B_1(\omega_m'')||} \quad \text{for } m > m_N,$$

where  $N > 0$  is arbitrary,  $m_N$  is depending on  $N$ .

Let  $K_0$  be an arbitrary zero of the Airy function  $A_0(Z)$ , i.e.

$$(3.11) \quad A_0(K_0) = 0$$

Take a sequence  $|\xi_m''| \rightarrow \infty$  such that

$$(3.12) \quad \frac{1}{|\xi_m''|^{1/3}} \frac{1}{B_1(\omega_m'')} \rightarrow 0 \quad \text{for } m \rightarrow \infty.$$

We note that (3.10) implies  $\operatorname{Re} \frac{1}{B_1(\omega_m'')} \rightarrow -\infty$  when  $m \rightarrow \infty$ .

For example, take  $|\xi_m''|^{1/3} = |\operatorname{Re} B_1(\omega_m'')|^{1+\varepsilon}$ ,  $\varepsilon > 0$ . It is easy to show that there exists a root  $\xi_0^{(m)} + i\tau_m$  of the equation

$$\frac{A_0((\xi_0 + i\tau) |\xi_m''|^{-1/3})}{A_0'((\xi_0 + i\tau) |\xi_m''|^{-1/3})} = - \frac{i}{|\xi_m''|^{1/3} B_1(\omega_m'')}$$

which has the following asymptotic behaviour

$$\xi_0^{(m)} + i\tau_m = K_0 |\xi_m''|^{-1/3} - \frac{i}{B_1(\omega_m'')} + \dots$$

Therefore we have

$$(3.13) \quad b(\xi_0^{(m)} + i\tau_m, |\xi_m''| \omega_m'') = 0$$

where

$$(3.14) \quad \tau_m \approx -\operatorname{Re} \frac{1}{B_1(\omega_m)} \geq CN \ln |\xi''| ,$$

$N$  is arbitrary,  $m > m_N$ .

Since the condition (3.8) is not satisfied we obtain that the problem (3.1), (3.2), (3.3) is ill-posed.

Remark 3.1 : Solution of the problem (3.1), (3.2), (3.3) in the space of ultradistributions.

We shall show that the solution of the ill-posed problem (3.1), (3.2) (3.3) exists in some class of ultradistributions. The general initial boundary problems for hyperbolic equations in the spaces of ultradistributions was studied by J. Chazarain [2] and R. Beals [1], but their results are not precise. We note that the Cauchy problem for partial differential equations with constant coefficients in the spaces of ultradistributions was studied at first by G. E. Shilov and his students (see [6]). Denote by  $\tau_0(\xi')$  the following function :

$$\begin{aligned} \tau_0(\xi') &= \frac{C}{\sqrt{\alpha}} \ln(1+|\xi''|) \quad \text{for } \alpha = \frac{\xi_0}{|\xi''|} > \frac{C_1}{|\xi''|^{2/3}} \\ \tau_0(\xi') &= C_2 |\xi''|^{1/3} \quad \text{for } |\alpha| < \frac{C_1}{|\xi''|^{2/3}} \\ \tau_0(\xi') &= 0 \quad \text{for } \alpha < -\frac{C_1}{|\xi''|^{2/3}} . \end{aligned}$$

One can show assuming that the condition (1.6) holds that

$$(3.15) \quad |b(\xi_0 + i\tau, \xi'')| \geq C \quad \text{for } \tau \geq \tau_0(\xi') .$$

Let  $\varphi(x_0, x'') \in C_0^\infty(\mathbb{R}_+^n)$  and  $\varphi(x_0, x'')$  belongs to the Gevrey class  $G_{3-\varepsilon}$  of order  $3 - \varepsilon$  in  $x''$ . Then

$$(3.16) \quad |\tilde{\varphi}(\xi_0 + i\tau, \xi'')| \leq C_N (1 + |\xi'| + \tau)^{-N} e^{-N|\xi''|^{1/3}} \quad \text{for any } N.$$

Let  $h(x_0, x'')$  be an arbitrary tempered distribution,  $h = 0$  for  $x_0 < 0$ . We define  $v(x')$  by the following formula

$$v = \frac{1}{(2\pi)^n} \int_L \frac{\tilde{h}(z, \xi'')}{b(z, \xi'')} e^{-ix_0 z - i(x'', \xi'')} dz d\xi'' ,$$

where  $L$  is a curve  $z(\xi_0) = \xi_0 + i\tau(\xi_0)$  in the complex plane  $\mathbb{C}$  such that  $C |\xi''|^{1/3} \geq \tau(\xi_0) \geq \tau_0(\xi')$ .

One can see that  $v(x')$  is a ultradistribution belonging to  $G'_{3-\varepsilon}$  and  $v(x')$  is a solution of the equation (3.6). We note that  $v(x)$  is a ultradistribution microlocally only for  $|\xi_0| < \varepsilon |\xi''|$ . Let  $\chi$  be a pseudodifferential

operator with a symbol  $\chi\left(\frac{\xi_0}{|\xi''|}\right)$  where  $\chi(t) \in C^\infty(\mathbb{R}^1)$ ,  $\chi(t) = 0$  for  $|t| < \varepsilon$ . Since

$$e^{x_0 \tau_0(\xi')} < e^{\frac{x_0}{\sqrt{\varepsilon}} \ln(1+|\xi''|)} = (1+|\xi''|)^{\frac{x_0}{\sqrt{\varepsilon}}} \quad \text{for } \alpha = \frac{\xi_0}{|\xi''|} > \varepsilon$$

we have that  $\chi(\frac{\xi_0}{|\xi''|})v$

belongs to a Sobolev's space of the order  $\frac{T}{\sqrt{\varepsilon}}$  when  $x_0 < T$ .

§4. A MODEL FOR THE THEOREM 1.4

In this section we consider the equation (3.1) in  $\mathbb{R}_+^{n+1}$  with the initial data (3.2) and with the following boundary condition

$$(4.1) \quad i \frac{\partial u}{\partial x_n} + \gamma(x_0 - a) \Lambda'' u \Big|_{x_n=0} = h(x'),$$

where  $h$  is a tempered distribution,  $h = 0$  for  $x_0 < 0$ ,  $\Lambda''$  is a pseudodifferential operator with the symbol  $|\xi''|$ ,  $a > 0$  and  $\gamma \neq 0$ . We note that the condition (1.6) gives that  $\text{Re } \gamma \cdot \text{Im } \gamma < 0$ .

As in §3 (see (3.6)) we reduce the solution of the problem (3.1), (3.2), (4.1) to the solution of the following equation in  $\mathbb{R}^n$  :

$$(4.2) \quad N^\tau v_\tau + \gamma(x_0 - a) \Lambda'' v_\tau = h_\tau$$

where  $v_\tau = e^{-x_0 \tau} v$ ,  $h_\tau = e^{-x_0 \tau} h$ ,  $v(x') = u(x', 0)$  and  $N^\tau$  is the Neumann operator, i.e.

$$(4.3) \quad N^\tau v_\tau = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{-i(x', \xi')} (-i|\xi''|^{2/3}) \frac{A'_0((\xi_0 + i\tau)|\xi''|^{-1/3})}{A_0((\xi_0 + i\tau)|\xi''|^{-1/3})} \tilde{v}_\tau(\xi') d\xi' .$$

If we take the Fourier transform in (4.2) we obtain an ordinary differential equation of the first order :

$$(4.4) \quad N^\tau \tilde{v}(\xi_0 + i\tau) - i\gamma|\xi''| \frac{d\tilde{v}(\xi_0 + i\tau, \xi'')}{d\xi_0} - \gamma a |\xi''| \tilde{v} = \tilde{h}(\xi_0 + i\tau, \xi''),$$

where

$$(4.5) \quad N^\tau = (-i|\xi''|^{2/3}) \frac{A'_0(z)}{A_0(z)}, \quad z = (\xi_0 + i\tau)|\xi''|^{-1/3}.$$

The general solution of the homogeneous equation (4.4), i.e. with  $h = 0$ , is given by the formula

$$(4.6) \quad w(\xi_0 + i\tau, \xi'') = C(\xi'') A_0^{-\frac{1}{\gamma}} ((\xi_0 + i\tau) |\xi''|^{-1/3}) e^{ia(\xi_0 + i\tau)},$$

where we take a branch of  $Z^{-\frac{1}{\gamma}}$  which is positive for the positive  $Z$ .

By using (4.6) one can show that the equation (4.4) has a unique solution which is a tempered distribution and so that the problem (3.1), (3.2), (4.1) is well-posed. The following indirect way to solve the equation (4.2) is more useful : in the region  $\xi_0 < 0$  the symbol  $N^\tau$  is a symbol of class  $S_{1/3,0}$  and so that there is no difficulties to solve the equation (4.2) microlocally in this region. Let now  $\xi_0 > 0$ . Denote by  $\chi_1(t)$  a  $C^\infty$ -function such that  $\chi_1(t) = 1$  for  $t > 1$ ,  $\chi_1(t) = 0$  for  $t < 0$ . Let  $\chi_1(\xi_0 |\xi''|^{-1/3})$  be a pseudodifferential operator with a symbol  $\chi_1(\xi_0 |\xi''|^{-1/3})$ .

Let  $B\chi_1$  be the following operator

$$(4.7) \quad B\chi_1 v = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} B(Z) \chi_1(\xi_0 |\xi''|^{-1/3}) \tilde{v}(\xi') e^{-i(x', \xi')} d\xi',$$

where  $Z = (\xi_0 + i\tau) |\xi''|^{-1/3}$ ,  $B(Z) = \frac{A_1(Z)}{A(Z)}$ ,  $A_1(Z) = \overline{A(\bar{Z})}$  and  $A(Z)$  is the Airy function with the following asymptotic behaviour for  $Z \rightarrow +\infty$  (see, for example [4])

$$(4.8) \quad A(Z) \approx C Z^{-1/4} e^{-2/3 iZ^{3/2}}$$

We note that  $A_0(Z) = A(Z) - A_1(Z)$ .

The operator  $B\chi_1$  is a Fourier integral operator with the phase function  $\varphi(x', \zeta') = (x', \zeta') - \frac{4}{3} \zeta_0^{3/2} |\zeta''|^{-1/2}$ . It was indicated in [4] that the phase function  $\varphi(x', \zeta')$  for  $\zeta_0 > 0$  is the generating function for the canonical transformation  $(y', \zeta') \rightarrow (x', \xi')$  such that the image  $(x'_0, \xi'_0)$  of a point  $(y'_0, \zeta'_0)$ ,  $\zeta'_0 > 0$  is the endpoint of an outgoing bicharacteristics in  $\mathbb{R}_+^{n+1}$  of the operator (3.1) which starts at the point  $(y'_0, \zeta'_0)$ .

It is easy to verify that the following identity holds

$$(4.9) \quad N^\tau \chi_1 v + \gamma (X_0 - a) \Lambda'' \chi_1 v = (I - B)^{-1/\gamma} L_0 (I - B)^{1/\gamma} \chi_1 v$$

where  $L_0$  is a pseudodifferential operator with a symbol  $-i|\xi''|^{2/3} \frac{A'(Z)}{A(Z)} + \gamma(X_0 - a)\Lambda''$ ,  $Z = (\xi_0 + i\tau) |\xi''|^{-1/3}$ .

We note that an operator of the form  $L_0$  arises also when one consider the case of the concave boundary. It is not difficult to prove the existence of the inverse of  $L_0$  and therefore to find the inverse of (4.9).

It was shown in [4] that

$$(4.10) \quad 1 - |B(Z)| > \frac{C\tau}{|\xi''|} (1 + \sqrt{\alpha} |\xi''|^{1/3}) \quad \text{for } \alpha = \frac{\xi_0}{|\xi''|} > 0$$

One can obtain from (4.10) an a priori estimate for the solution of the equation (4.2) with a loss of  $\frac{1}{3} \frac{1}{\gamma} + \frac{2}{3}$  derivatives.

The identity (4.9) allows also to describe the wave front set of the solution  $v(x')$ .

---

#### REFERENCES

- [1] R. Beals : Mixed boundary value problems for non strict hyperbolic equations, Bull. AMS (1972) pp.520-521.
- [2] J. Chazarain : Problèmes de Cauchy abstraits et application à quelques problèmes mixtes, J. Funct. Analysis v.7 (1971), pp.386-446.
- [3] G. Eskin : Parametrix and propagation of singularities for the interior mixed hyperbolic problem, Journ. d'Analyse Math. v.32 (1977) pp.17-62.
- [4] G. Eskin : Propagation of singularities for the interior mixed hyperbolic problem, Sem. Goulaouic-Schwartz 1976-1977, exposé n° XII.
- [5] L. Gårding : Le problème de la dérivée oblique pour l'équation des ondes, C. R. Acad. Sc. Paris t.285, 1977, pp.773-775, ET C. R. Acad. Sc. Paris t.286 (1978) p.1199.
- [6] I. M. Gelfand, G. E. Shilov : Generalized functions v.3, Fizmatgiz Moscow, 1958.
- [7] M. Ikawa : Problèmes mixtes pour l'équation des ondes II, Pibl. R.I.M.S. Kyoto Univ. v.13 n° 1 (1977), 61-106.
- [8] M. Ikawa : On the mixed problems for the wave equation in an interior domain, Comm. in P. D. E. 3(3), 249-295 (1978).
- [9] M. Ikawa : Preprint (1979).
- [10] R. Melrose, J. Sjostrand : Propagation of singularities near the boundary II (in preparation).



- [11] S. Miyataké : A sharp form of the existence theorem for hyperbolic mixed problems of second order, J. Math. Kyoto Univ. 17 (2), 1977, p.199-223.
- [12] K. Kajitani : A necessary condition for the well-posed hyperbolic mixed problem with variable coefficients, J. Math. Kyoto University 14 (1974), 231-242.
- [13] L. Nirenberg : Lectures on Linear P. D. E., Regional conference series in Math., n° 17, Providence, R. I., 1973.

\*\*\*  
\*\*