Séminaire Équations aux dérivées partielles - École Polytechnique

M. Kuranishi

On a construction of the normal Cartan connections for CR-structures

Séminaire Équations aux dérivées partielles (Polytechnique) (1982-1983), exp. nº 17, p. 1-8

<http://www.numdam.org/item?id=SEDP_1982-1983____A17_0>

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

91128 PALAISEAU CEDEX - FRANCE

Tél. (6) 941.82.00 - Poste N°
Télex : ECOLEX 691 596 F

S E M I N A I R E   G O U L A O U I C - M E Y E R - S C H W A R T Z   1 9 8 2 - 1 9 8 3

ON A CONSTRUCTION OF THE NORMAL CARTAN CONNECTIONS FOR CR-STRUCTURES

par M. KURANISHI

Exposé n° XVII

22 mars 1983

We are here concerned with the equivalence problem of systems of partial differential equations. To state the problem more precisely, let E (resp. U) be a vector space over $\mathbb{R}$ (resp. an open neighborhood of a point p in $\mathbb{R}^m$) and denote by (Σ, E, U, p) a system of partial differential equations Σ for an unknown E-valued function on U. p is considered as a reference point so that we may shrink U whenever it is necessary to do so. We now fix (Σ, E, U, p), and ask for an arbitrary (Σ', E', U', p') to find a constructive procedure to check if the latter is obtained from the former by a change of variables $E' \times U' \to E \times U$ over $(U', p') \to (U, p)$, or a more general change like a canonical transformation.

The problem is obviously very important. However, the answer is known only for very special cases :

1) when $E = \mathbb{R}$ and $\Sigma : \partial u/\partial x^j = 0$ $(j = 1, \ldots, p)$, it is the classical theorem of Hamilton -Jacobi-Lie ;

2) in the case $E = \mathbb{C}$ (considered as $\mathbb{R}^2$), $U = \mathbb{C}^n$, and Σ is the Cauchy-Riemann equation, it is the Newlander-Nirenberg Theorem [4] . In these cases the procedure to check the equivalence is surprising simple : we only have to check rank conditions and closedness under the bracket operation.

It is desirable to have an answer in the case when Σ has constant coefficients or more generally when U is homogeneous under an action of a Lie group G and Σ is invariant under the action of G. However, very little seems to be known in these cases, except the Frobenius theorem for completely integrable systems. When Σ is the defining equation of a local Lie group, the third fundamental theorem of Lie is the answer .
In the case of infinite pseudo-groups of Lie instead of local Lie groups, or more generally for Lie equations, a great deal of work has been done recently. In particular in the transitive case and in the real analytic category, the situation seems to be fairly well understood. As for the study on the relation between C^∞ category and formal category in the above context, we may mention a monumental work of H.Goldschmidt and D.C. Spencer [3].

A problem which falls in the same category is the equivalence problem of Riemannian geometry : to see if a given Riemann manifold is locally isometric to a euclidean space we check the vanishing of its curvature.

We have a similar answer for conformal classes of Riemannian manifolds. Note that, in the above cases 1) and 2), nothing like the curvature makes its apparance . We may say that the curvature is always zero in these cases.

Today we consider the case of the tangential Cauchy Riemann equation. This is the case $E = \mathbb{C}$ and

(1) $U \subset \mathbb{C}^n$

is a real hypersurface of codimension 1 . To define Σ we may regard it as a system of partial differential equations with complex coefficients. Denoting by v (resp. $z = (z^1, \ldots, z^n)$) a general complex valued function on U (resp. a general element in $\mathbb{C}^n$), we definie Σ by

(2) $\bar{Z}\, v = 0$

for any $\bar{Z} = a^j\, \partial/\partial_{\bar{z}^j}$ which is tangential to U. The set of such $\bar{Z}$ is a subbundle, say E, of the complex tangent bundle $\mathbb{C}TU$. Thus, instead of the system of partial differential equations Σ,we may speak of the subbundle E. Obviously E satisfies the following conditions :

1) $E \cap \bar{E} = \{0\}$ and the fiber dimension over $\mathbb{C}$ of the quotient bundle

$\mathbb{C}T\,U/(E \oplus \bar{E})$ is 1 ;

2) if X and Y are sections of E, so is $[X,Y]$

We call any E satisfying the above two conditions a CR-structure. Such an E has an invariant which is absent in the above cases 1) and 2). Namely, pick a supplementary real vector field S on U so that $\mathbb{C}\,T\,U = \mathbb{C}\,S \oplus E \oplus \bar{E}$. Write for sections X, Y of E.

(3) $[X, \bar{Y}] \equiv i C\, (X, Y)\, S \mod E \oplus \bar{E}.$

see easily that $C(X, X)$ is a hermitian quadratic form defined on fibers of E. It is called the Levi-form of E. C may depend on S. However, it is defined up to multiplications by non-vanishing real valued functions. We assume that C is non-degenerate. The simplest such E is the case when U is the a non-degenerate real quadric in $\mathbb{C}^n$. In this case, we call it flat. Our problem is to find a constructive procedure to decide whether a given CR-structure with non-degenerate Levi form is locly isomorphic to a flat CR-structure.

We wish to point out here an analogy with the conformal hermitian structures. In this case, a conformal class of hermitian metrics is given on each tangent vector space. In our case, it is given only on E. However, we can exploit the analogy. In fact pursuing this analogy, E. Cartan introduced the curvature for CR-structures with non-degenerate Levi-form (in the case dim U = 3) and showed that the vanishing of the curvature is equivalent to its flatness [1]. In this respect, "pseudo-conformal structure" used by E. Cartan is more indicative than the modern "CR-structure".

In the language of differential geometry the curvature is obtained by constructing a Cartan connection. However, in our case there is no unique Cartan connection. We have to select one by imposing conditions for its curvature. The unique Cartan connection thus obtained is called normal. The construction of the normal Cartan connection for general dimension was first achieved by N. Tanaka [5]. Independently, S.S. Chern and J.K. Moser gave a different construction [2]. We wish to outline here another approach.

Let us first recall the definition of Cartan connections. We fix a Lie group G and its closed subgroup H. Denote by $\mathfrak{g}$ (resp. by $\mathfrak{h}$) the Lie algebra of G (resp. of H). A Cartan connection in (G, H) of a manifold M is a principal bundle P with structure group H together with a $\mathfrak{g}$ - valued Paffian ω form on P satisfying the following conditions :

(i) for each $X \in P$, ω induces an isomorphism of $T_x P$ with $\mathfrak{g}$;

(ii) if we identify a fiber of P with H by a chart of P, ω restricted to H is the left invariant Maurer-Cartan form ;

(iii) if R_h denotes the right action of H on P, $R_h^* \omega = Ad(h^{-1})\omega$.

In our case, we consider the following G : we denote by Q the quadric in $\mathbb{C}^n$ given by

$$\text{(4)} \qquad \operatorname{Im} z^n = \frac{1}{2} \sum_{j,R=1}^{n-1} h_{j\bar{k}} z^j \bar{z}^k$$

where (h_{jk}) is a non-singular hermitian matrix. The CR-structure on Q extends to the one-point compactification $\hat{Q}$ of q. G is the component of the automorphism group of the CR-structure $E_{\hat{Q}}$ on $\hat{Q}$.

H is the isotropy group at the origin O. If we write elements in $\mathbb{C}^{n-1}$ $(\zeta^0, \zeta^1 z^1, \ldots, z^{-1})$, G is isomorphic to the quotient by the center of the subgroup of $G\ell(n+1\ \mathbb{C}^{n-1})$ leaving the hermitian form

(5) $i(\bar{\zeta}^0 \zeta^1 - \zeta^0 \bar{\zeta}^1) + h_{j\bar{k}}\ z^i\ \bar{z}^k$ invariant. Note that $G \ni g \mapsto g(0) \in \hat{Q}$ induces an isomorphism of G/H onto Q. Hence it induces $\mathfrak{g}/\mathfrak{h} \to T_0 Q = T_0 Q$. Thus we have an isomorphism :

(6) $(\mathfrak{g}/\mathfrak{h}) \otimes \mathbb{C} \to \mathbb{C}\ T_0 Q$

The fiber of the CR-structure E_Q over 0 is a vector subspace of $\mathbb{C}T_0 Q$.

Let E be a CR-structure over a manifold M. We say that a Cartan connection (P, ω) of M in (G, H) is a Cartan connection for E, if for each $X \in P$ the inverse image of E_Q in $\mathbb{C}\ T_0 Q$ under the map

(7) $\mathbb{C}\ T_x P \to \mathfrak{g} \otimes \mathbb{C} \to (\mathfrak{g}/\mathfrak{h}) \otimes \mathbb{C} \to \mathbb{C}\ T_0 Q$

(where arrows are ω, the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, the map (6)) projects into E. Thus E is actually given by (P, ω).

To give an idea of our construction, let us outline our construction in the case of Riemannian manifold of dimension n. In this case the group G is the group of euclidean motions of $\mathbb{R}^n$ and H is its isotropy group at the origin. In the case of the flat Riemannian manifold $\mathbb{R}^n$, P is by definition equal to G and the projection $P \to \mathbb{R}^n$ is the projection $G \to G/H$ where G/H is identified with the bundle of orthonormal coframes of $\mathbb{R}^n$ by $G \ni g \to ge$, where e is the standard cobase of $\mathbb{R}^n$. ω is by definition the left invariant Mauer-Cartan form. Thus Cartan connection is defined for $\mathbb{R}^n$ which we denote by $(P_{\mathbb{R}^n}, \omega_{\mathbb{R}^n})$. For a Riemannian manifold M, we take P to be the bundle of orthonormal coframes of M. We wish to transplant $(P_{\mathbb{R}^n}, \omega_{\mathbb{R}^n})$ to P. Namely, we take for each $p \in M$ an osculating chart

$$i : (M', p) \quad (\mathbb{R}^n, 0)$$

where M' is an open neighborhood of p. It is a chart such that the Riemannian metric of M and the metric induced by the map from $\mathbb{R}^n$ agree at p up to order 2. We check easily that such chart exists. Then $i^* P_{\mathbb{R}^n}$ and P are sub-bundles of the bundle of coframes of M. Thanks to the osculating property of i, $i^* P_{\mathbb{R}^n}$ and P agree at all points over p and they also have the same tangent space.

Therefore $(i^* w_{\mathbb{R}} n) : T_X P \to \mathcal{G}$ is well defined for any $X \in P$ over p. We then show the $(i^* \omega_{\mathbb{R}} n)_X$ is independent of the choice of osculating i, which we define to be $(\omega)_X$. Since $(\omega)_X$ is now defined uniquely for any X over P, we have $\omega : TP \to \mathcal{G}$. This our Cartan connection for M.

In the case of CR-structures with non-degenerate Levi-form that we use the automorphism G of $\hat{Q}$ (where $(h_{j\bar{k}})$ in (4) is taken to be equivalent to the Levi-form) instead of the group of euclidean motions. Then the above construction makes sense once we find correct notions which replace the notions of coframes, orthonormal coframes, and of osculating charts. To define these notions we fix some notations. Let V be a C^∞ vector bundle over M. Then the vector space $C^\infty(M,V)$ of C^∞ sections of V over M is a module over the ring of C^∞ functions on M. Let I be a submodule of $C^\infty(M,V)$. we set

$$V/I = C^\infty(M,V)/I.$$

Thus V/I is a module over $C^\infty(M)$. For a point p in M we denote by $\mathfrak{m}_p$ the ideal of the point p in M. Then $\mathfrak{m}_p^\ell C^\infty(M,V)$ is a submodule of $C^\infty(M,V)$. We also set

$$V/\mathfrak{m}_p^\ell = V/\mathfrak{m}_p^\ell C^\infty(M,V)$$

$V/\mathfrak{m}_p^\ell$ is also a finite dimensional vector space. V/I has the obvious functorial properties.

Let Q be the quadric defined by (4). We have the CR-structure E_Q over Q. We set

$$SQ = TQ/E_Q^\circ \quad , \quad E_Q^\circ = (E_Q + \bar{E}_Q) \cap TQ.$$

SQ is a vector bundle over Q. By a hyper-coframe of a manifold M at p, we mean a pair of linear maps (a,b) where

$$a : T_0 Q \to T_p^* M \quad \text{is an isomorphism}$$

$$b : S^*Q / \mathfrak{m}_0^2 \to T^*M/\mathfrak{m}_p^2$$

such that there is a map $(M,P) \to (Q,0)$ which induces a and b.

The above replaces coframes. To define the notion which plays the role of orthonormal coframes, we set

$$SM = TM/E^\circ \ , \ E^\circ = (E+\bar{E}) \cap TM$$

for a CR-structure E over M. Since SM is a quotient of TM,

$$S^*M \subseteq T^*M,$$

and hence

$$S^*M/\mathfrak{m}_p^2 \subset T^*M/\mathfrak{m}_p^2.$$

In order to make our bundle connected we also introduce orientations on the vector bundles SQ and SM in such a way it is consistent with the equivalence of the Levi-form of E_Q and E. We say a hyper-coframe (a,b) is a CR-hyper-coframe if

1°) $a \otimes \mathbb{C}$ sends $(\mathbb{C}\, T_0 Q/E_Q)^*$ to $(\mathbb{C}\, T_p M/E)^*$

2°) a preserves the orientation

3°) the image of b is in $S^*M/\mathfrak{m}_p^2$

The set of hyper-coframes (resp. of CR-hyper-coframes) forms a bundle HM (resp. P^E) over M. P^E is the bundle on which we construct the normal Cartan connection.

A diffeomorphism $h : M \to N$ obviously induces

$$[h] \text{ from } H^N \text{ to } H^M.$$

A diffeomorphism $f : (Q',0) \to (M',p)$, where Q' (resp. M') is a neighborhood of 0 (resp. of P) in Q (resp. in M) is called admissible at P if (when considered as P^{E_Q}, $[h]P^E \subseteq H^Q$)

1°) $(P^{E_Q})_0 = [h]\,(P^E)_p$, where $(\ )_p$ denotes the fiber over p ;

2°) $T_X P^{E_Q} = T_X([h](P^E))$ for all $X \in P^{E_Q}$ over O.

To see the nature of an admissible map, note that SQ (resp. SM) is defined by

$$\theta_Q = 0 \quad (\text{resp. } \theta = 0)$$

where θ_Q (resp. θ) is a real Pfaffian form on Q (resp. on M). Here we shrunk M if necessary. Then we can pick complex Pfaffian forms $\omega_Q^1,\dots,\omega_Q^{n-1}$ on Q (resp. $\omega^1,\dots,\omega^{n-1}$ on M) such that E_Q (resp. E) is defined by the equations :

$$\theta_Q = \omega_Q^1 = \dots = \omega_Q^{n-1} = 0 \quad (\text{resp. } \theta = \omega^1 = \dots = \omega^{n-1} = 0).$$

Obviously, $\theta_Q, \omega_Q^1,\dots,\omega_Q^{n-1}, \overline{\omega_Q^1},\dots,\overline{\omega_Q^{n-1}}$ form a base of complex Pfaffian forms on Q. Similarly for θ, ω^j, $\overline{\omega^j}$.

Hence for $f : (Q,0) \to (M,p)$ we can set

$$f^*\theta = H^{00}\theta_Q + \sum_k (H^{0k}\omega_Q^k + \overline{H^{0k}\omega_Q^k})$$
$$f^*\omega^j = H^{j0}\theta_Q + \sum_k (H^{j\bar{k}}\overline{\omega_Q^k} + H^{jk}\omega_q^k) \quad .$$

When θ, θ_Q are chosen properly (depending on the orientation), we have the following proposition : f is admissible at p if and only if

$$H^{00}(0) > 0, \quad H^{0k} \equiv 0 \bmod \mathfrak{m}_0^3, \quad H^{jk} \equiv 0 \bmod \mathfrak{m}_0^2.$$

Note that Q, being defined by (4), has a global chart $(z^1,\dots,z^{n-1}, x)$ where $x = \mathfrak{Re}\, z^n$. Hence a function F on Q is considered as $F(z^1,\dots,z^{n-1}, x)$. We give a weight to a homogenous polynomial $F(z^1,\dots,z^{n-1},x)$ by assigning the weight 1 (resp. 2) to $z^1,\dots, z^{n-1}$ (resp. to x). We say F is of type (s,t) when it is of that type in $(z^1,\dots z^{n-1})$. We define an operator $\Box$ by

$$\Box F = h^{j\bar{k}} \, \partial^2 F/\partial z^j \, \partial \bar{z}^k \quad , \quad (h^{j\bar{k}}) = (h_{j\bar{k}})^{-1}$$

We also set

$$Z^{\bar{k}} = \partial/\partial \bar{z}^k - \frac{i}{2} h_{j\bar{k}} z^j \, \partial/\partial x$$

We say that $f : (Q,0) \to (M,p)$, admissible at p, is normal at p if the following conditions are satified :

(N_0) the weight 3 part of H^{0k} is of type (2,1) ;

(N_1) $H^{\alpha k} \equiv 0, \ \sum_k Z^{\bar{k}} H^{\alpha k} \equiv 0 \bmod \mathfrak{m}_0^3$, $\alpha=0,\dots,n-1$;

(N_2) $\Box H^{\alpha k} \equiv 0, \ \Box \sum_k Z^{\bar{k}} H^{\alpha k} \equiv 0 \bmod \mathfrak{m}_0^2$, $\alpha=0,\dots,n-1$;

(N_3) $\Box^2 H^{0k} \equiv 0, \ \Box^2 \sum_k Z^{\bar{k}} H^{0k} \equiv 0 \bmod \mathfrak{m}_0$.

By using a normal admissible map instead of the osculating chart we can define a unique Cartan connection.

$$\omega_E : P^E \to \mathfrak{g}$$

This is our definition of the normal Cartan connection for the CR-structure E with a non-degenerate Levi-form. Now the curvature Ω_E of E is defined by

$$\Omega_E = d\,\omega_E + \frac{1}{2} [\omega_E, \omega_E]$$

where the bracket is defined in terms of the bracket in the Lie algebra $\mathfrak{g}$.

BIBLIOGRAPHIE

[1] Cartan, E., Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes I. Ann. Math. Pura Appl. (4) 11 (1932) 17-90 ; II, Ann. Scuola Norm. Sup. Pisa (2) 1 (1932) 333-354.

[2] Chern, S.S and Moser, J.K., Real Hypersurfaces in complex manifolds. Acta Math. 113 (1974) 219-271.

[3] Goldschmidt, H. and Spencer, D.C., On the non-linear cohomology of Lie equations, I and II. Acta Math. 136 (1976), 103-239 ; III and LV. J. Dif. Geo. 13 (1978), 409-526.

[4] Newlander, A and Nirenberg, L. Complex coordinates in almost-complex manifolds. Ann. of Math. 65 (1957) 391-404.

[5] Tanaka, N., On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables, J. Math. Soc. Japan 14 (1962), 397-429 ; On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japanese J. Math. vol. 2, No 1 (1976).