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CRITERIA FOR HYPOELLIPTICITY

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Introduction. In order to explain the motivation of this study, we begin with the following simple example. Let L_0 be a differential operator in R^3 of the form

$$L_0 = D_x^2 + \phi(x)^2 D_y^2 + D_t^2 , \quad D_x = -i\partial_x , \ldots, \ldots,$$

where $\phi \in C^\infty$, $\phi(0) = 0$, $\phi(x) > 0$ ($x \neq 0$), $\phi(x) = \phi(-x)$ and ϕ is non-decreasing in $[0,\infty)$. It was proved by Kusuoka-Strook [K-S] that L_0 is hypelliptic in R^3 if and only if ϕ satisfies

$$\lim_{x \to 0} |x||\log \phi(x)| = 0 .$$

When $\phi = \exp(-1/|x|^\sigma)$, $\sigma > 0$, this condition means $\sigma < 1$. This result was obtained by using the Malliavin calculus, which is a theory of stochastic differential equations.

The main motivation of this study is to prove the above result by means of the microlocal analysis in the theory of partial differential equations. In the next section we give a new sufficient condition for hypoellipticity and discuss its necessity.

1. Main results

Let $P = p(x,D_x)$ be a differential operator of order $m \geq 1$ with coefficients in $C^\infty(R_x^n)$, that is,

$$p(x,D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha , \quad a_\alpha(x) \in C^\infty(R_x^n),$$

where for multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_x^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $D_j = -i\partial_{x_j}$.

We say that P is *hypoelliptic* (C^∞*-hypoelliptic*) *in* R^n if for any $u \in \mathcal{D}'(R^n)$ and for any open set Ω of R^n , $Pu \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$. Let Λ and $\log \Lambda$ be pseudodifferential operators with symbols $\langle\xi\rangle$ and $\log \langle\xi\rangle$, respectively, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. We write $p^{(\alpha)}_{(\beta)}(x,\xi) = \partial^\alpha_\xi D^\beta_x p(x,\xi)$ for multi-indices α and β. We set $\|u\|_s = \|\Lambda^s u\|$ for real s and $u \in C_0^\infty(R^n)$, where $\|\cdot\|$ denotes the usual L^2 norm.

Theorem 1. *Assume that for any* $\varepsilon > 0$ *and any compact set* K *of* R^n *there exists a constant* $C_{\varepsilon,K}$ *such that for any* $u \in C_0^\infty(K)$

(1) $$\|(\log \Lambda)^m u\| \le \varepsilon\|Pu\| + C_{\varepsilon,K}\|u\| ,$$

(2) $$\sum_{0<|\alpha+\beta|<m} \|(\log \Lambda)^{|\alpha+\beta|} P^{(\alpha)}_{(\beta)} u\|_{-|\beta|} \le \varepsilon\|Pu\| + C_{\varepsilon,K}\|u\| ,$$

where $P^{(\alpha)}_{(\beta)} = p^{(\alpha)}_{(\beta)}(x, D_x)$. *Then* P *is hypoelliptic in* R^n. *Furthermore we have*

(3) $$\mathrm{WF}\, Pv = \mathrm{WF}\, v \quad \text{for any} \quad v \in \mathcal{D}'(R^n).$$

Corollary 1. *Let* P *be a differential operator of second order with* C^∞*-coefficients, that is,*

$$P = \sum_{j,k} a_{jk}(x) D_j D_k + \sum_j i b_j(x) D_j + c(x) .$$

We assume that

(*) $$\begin{cases} a_{jk} \text{ and } b_j \text{ are real valued,} \\ \sum a_{jk}(x)\xi_j\xi_k \ge 0 \quad \text{for all } (x,\xi) \in R^{2n} \end{cases}$$

If for any $\varepsilon > 0$ *and any compact set* K *of* R^n *the estimate*

$$(4) \qquad \|(\log \Lambda)^2 u\| \le \varepsilon\|Pu\| + C_{\varepsilon,K}\|u\|, \qquad u \in C_0^\infty(K)$$

holds with a constant $C_{\varepsilon,K}$ *then we have (3).*

Corollary 2. *Let* P *be the same as in Corollary 1. If for any* $\varepsilon > 0$ *and any compact set* K *of* R^n *the estimate*

$$(5) \qquad \|(\log \Lambda)u\|^2 \le \varepsilon \operatorname{Re}(Pu, u) + C_{\varepsilon,K}\|u\|^2, \quad u \in C_0^\infty(K)$$

holds with a constant $C_{\varepsilon,K}$ *then we have (3).*

Proof of Corollaries. The estimate (2) with $m = 1$ follows from (*) and (4) (cf. Oleinik-Radkevich [O-R]). (4) easily follows by substituting $(\log \Lambda)u$ into (5). Q.E.D.

We discuss the necessity of hypoellipticity for second order differential operators given in Corollary 1. The estimate (4) is not always necessary for hypoellipticity. We have a counter example given by Fediĭ [Fd], $\mathcal{A}_0 \equiv D_x^2 + \exp(-1/|x|^\sigma)D_y^2$, $\sigma > 0$. Indeed, this example does not satisfy (4) when $\sigma \ge 1$, while it is proved by [Fd] that $\mathcal{A}_0$ is hypoelliptic for any $\sigma > 0$. The fact that $\mathcal{A}_0$ with $\sigma \ge 1$ does not satisfy (4) is easily seen if we consider an eigenvalue problem

$$\begin{cases} (-d^2/dx^2 + \exp(-1/|x|^\sigma\eta^2))v = \lambda v \\ v(-1) = v(1) = 0, \qquad v \in C^\infty((-1,1)). \end{cases}$$

The min-max principle shows that the minimal eigenvalue $\lambda_0(\eta)$ satisfies

$$0 < \lambda_0(\eta) \le C_0(\log |\eta|)^{2/\sigma} \quad \text{(see [M}_2\text{])},$$

which contradicts the estimate (4).

However, the estimate (4) is necessary to be hypoelliptic for a class of differential operators, (for example, $D_t^2 + \mathcal{A}_0$). The result concerning the necessity of (4) can be discussed for some class of operators of higher order. Let m be even positive integer and let P_0 be a differential operator of the form

$$(6) \qquad P_0 = D_t^m + \mathcal{A}(x, D_x) \quad \text{in} \quad R_t \times R_x^n ,$$

where $\mathcal{A}(x, D_x)$ is a differential operator of order m with C^∞-coefficients. We assume that $\mathcal{A}(x, D_x)$ is formally self-adjoint in R_x^n and bounded from below, that is, there exists a real c_0 such that $(\mathcal{A}(x, D_x)u, u) \ge c_0\|u\|^2$ for $u \in C_0^\infty(R_x^n)$.

Theorem 2. *Let* P_0 *be the above operator. Assume that* P_0 *is hypoelliptic in* $R_t \times R_x^n$. *Then for any* $x_0 \in R_x^n$ *there exists a neighborhood* ω *of* x_0 *such that for any* $\varepsilon > 0$ *the estimate*

$$(7) \qquad \|(\log \Lambda)^{m/2}u\|^2 \le \varepsilon \operatorname{Re}(P_0 u, u) + C_\varepsilon\|u\|^2, \quad u \in C_0^\infty(R_t \times \omega)$$

holds with a constant C_ε. *Here* Λ , *of course, denotes* $\langle D_t, D_x\rangle = (1+D_t^2+|D_x|^2)^{1/2}$.

Remark 1. When $m = 2$ the estimate (4) follows from (7). In fact, for any compact set K of $R_t \times R_x^n$, let K' be the projection of K to R_x^n and take the partition of unity $\sum \varphi_j^2(x) = 1$ over K'. Since $\operatorname{Re}([P_0, \varphi_j]u, \varphi_j u)$ is majorated by a constant times of $\|u\|^2$, we have (7) for $u \in C_0^\infty(R_t \times K')$, which implies (5) and hence (4).

Remark 2. Almost the same result as Theorem 2 was obtained independently by [Hs_1]. Proof of Theorem 2 is performed by the similar method as in Métivier [Me], who studied nonanalytic hypoellipticity for operators of the same form as (6) (cf. [B-G]).

As an application of Theorems we shall consider the hypoellipticity of degenerate elliptic opertors of the following form;

(8) $$P_1 = D_t^{2\ell} + D_x^{2\ell} + g(x)D_y^{2\ell} \quad \text{in} \quad R^3,$$

where $\ell = 1, 2, \dots$ and $g(x)$ is C^∞ function such that $g(x) > 0$ $(x \neq 0)$ and $g(0) = 0$. When $\ell \geq 2$ we assume that for any $j > 0$

(9) $$|D_x^j g(x)| \leq C_j g(x)^{1-\delta j} \quad \text{in a neighborhood of} \quad x = 0 ,$$

where δ is a number satisfying

(10) $$0 < \delta < 1/2\ell^2 .$$

It is clear that a function $\exp(-1/|x|^\sigma)$, $\sigma > 0$, satisfies (9) for any $\delta > 0$. We refer [M_1] about an example of g which does not satisfy (9) for any $0 < \delta < 1/2$.

Proposition 1. *Let P_1 be the above operator. If $g(x)$ satisfies*

(11) $$\lim_{x \to 0} |x||\log g(x)| = 0$$

then P_1 is hypoelliptic in R^3. Assume in addition that $xg'(x) \geq 0$, that is, g is monotone in R^+ and R^-, respectively. Then the condition (11) is also necessary for P_1 to be hypoelliptic in R^3.

The result of Kusuoka-Stroook stated in Introduction is the case $\ell = 1$ of Proposition 1.

Unfortunately, when $\ell \geq 2$ we can not apply directly Theorem 1 to the proof of Proposition 1 , because it is quite hard to check the hypothesis (2) for P_1 , more precisely, to show

$$\|(\log \Lambda)\, D_x^{2\ell-1}u\| \leq \varepsilon\|P_1u\| + C_{\varepsilon,K}\|u\|, \quad u \in C_0^\infty(K).$$

So we need the following amelioration of Theorem 1 under an additional assumption.

Theorem 3. *Assume that the principal symbol of* $p_m(x,\xi)$ *of* P *satisfies*

(12) $$p_m(x,\xi) \neq 0 \quad for \quad x' \neq 0, \quad where \quad x = (x', x'').$$

Then the conclusion (3) of Theorem 1 still holds even if the estimate (2) is replaced by

(13) $$\sum_{\substack{0<|\alpha+\beta|<m \\ \alpha=(0,\alpha'')}} \|(\log \Lambda)^{|\alpha+\beta|} P_{(\beta)}^{(\alpha)} u\|_{-|\beta|}$$

$$\leq \varepsilon\|Pu\| + C_{\varepsilon,K}\|u\| \quad , \quad u \in C_0^\infty(K).$$

The hypoelliptic operator $\mathscr{A}_0$ given by [Fd] with $\sigma \geq 1$ is not covered by Corollary 2 (nor 1) as stated in the preceding. To cover this exceptional example we give another criterion of hypoellipticity.

Theorem 4. *Assume that the principal symbol of* P *satisfies (12). If for any compact set* K *of* R^n *there exist a* $\kappa_0 > 0$ *and a constant* C_K *such that*

(14) $$\|u\| + \sum_{\substack{0<|\alpha+\beta|\le m \\ \alpha=(0,\alpha'')}} \|P^{(\alpha)}_{(\beta)}u\|_{\kappa_0-|\beta|}$$

$$\le C_K(\|Pu\| + \|u\|_{-1}), \quad u \in C_0^\infty(K),$$

then we have (3).

2. Proofs

Concerning of the detail proofs of Theorem 1,3,4 and Theorem 2 we refer [M_6] (or [M_5]) and [M_5], respectively. Here we only explain the idea.

First, we review the following:

$$\mathcal{A}_k = D_x^2 + x^{2k}D_y^2$$, *analytic hypoelliptic*, see [Ma],

$$L_k = D_t^2 + D_x^2 + x^{2k}D_y^2$$, *nonanalytic hypoelliptic*,

see [B-G] when $k = 1$, see [Me] when $k \ge 2$.

Here the definition of analytic hypoellipticity is that of C^∞-hypoellipticity with replaced " $C^\infty(\Omega)$ " by "*real analytic in* Ω ". It was stated above that

$$\mathcal{A}_0 = D_x^2 + \exp(-1/|x|^\sigma)D_y^2 , \quad \sigma > 0 , \quad C^\infty\text{-hypoelliptic}.$$

$$L_0 = D_t^2 + D_x^2 + \exp(-1/|x|^\sigma)D_y^2 , \quad \sigma \ge 1 , \quad \text{non } C^\infty\text{-hypoelliptic}.$$

The above comparaison between analiticity and C^∞-smootheness leads us to the following characterization of C_0^∞-functions:

$$u \in C_0^\infty \quad \longleftrightarrow \quad u \in \mathcal{E}', \ \|\Lambda^k u\| < \infty \quad \textit{for any} \quad k > 0$$

$$\longleftrightarrow \quad u \in \mathcal{E}' , \ \|\exp(k \log \Lambda)u\| < \infty \quad \textit{for any} \quad k > 0$$

$$\longleftrightarrow \quad u \in \mathcal{S}' \ , \ \|(k \log \Lambda)^N u\| \le N! \ , \ N = 0, 1, 2, \ldots$$
$$\text{for any} \quad k > 0.$$

If we note that C_0^∞ -functions are functions satisfying Cauchy's estimate with D_x replaced by $k \log \Lambda$ for any $k > 0$ then the proof of Theorem 1 is not particular in the usual microlocal analysis. (More precisely, the proof of Theorem 1 is performed by using the microlocalization arguments as in [Hr_2].) As stated in the remark after Theorem 2, the proof of Theorem 2 is easily done if $\log \Lambda$ replaces Λ in the arguments of Métivier [Me].

We remark that, essentially following the above idea, Hoshiro [Hs_2] has recently given another proof of Theorem 1, by using the microlocal energy method studied in Mizohata [Mi].

We mention the sketch of the proof of Proposition 1. For the brevity, we only consider the case of $\ell = 1$ and $g(x) = \exp(-1/|x|^\sigma)$, $\sigma > 0$. The necessity of (11) is proved by Theorem 2. In fact, if (11) does not hold, then (7) contradicts the estimation of the minimal eigenvalue $\lambda_0(\eta)$ considered in the paragragh after Corollary 2. For the proof of the sufficiency, we use Corollary 2. Note that

$$\text{(15)} \qquad \mathrm{Re}\langle P_1 u, u\rangle = \|D_t u\|^2 + \|D_x u\|^2 + \langle \exp(-1/|x|^\sigma) D_y^2 u, u\rangle .$$

If $\check{u}(t, x, \eta)$ is the Fourier transform of $u \in C_0^\infty$ with respect to y and if $\mathrm{supp}\, \check{u} \subset \{ |x| \le (\log|\eta|)^{-1/\sigma} \} \equiv \Omega_0$ then it follows from Poincaré's inequality that

$$\text{(16)} \qquad \|(\log |\eta|)^{1/\sigma} \check{u}\|^2 \le C\|D_x \check{u}\|^2 .$$

On the region $\Omega_0^c = \{ |x| \ge (\log |\eta|)^{-1/\sigma} \}$, the symbol $\sigma(P_1)$ of P_1 satisfies

$$\sigma(P_1) \geq \tau^2 + \xi^2 + |\eta|.$$

Since P_1 is semi-elliptic on Ω_0^c , the estimate (16) together with (15) gives (5). By the similar way, we can prove the hypoellipticity of an operator $D_t^2 + D_x^2 + ig(x)D_y$ if $g(x)$ satisfies (11) and $g(x) > 0$ ($x \neq 0$).

We remark that Theorem 1 is also applicable to more degenerate elliptic operators of second order. For example, let P_2 be a differential operator of the form

$$P_2 = D_x^2 + x^{2\ell}D_y^2 + f(y)D_z^2 , \qquad \ell = 1, 2, 3, \ldots,$$

where $f(y) = g(y)\sin^2(1/|y|)$ and $g(y)$ is C^∞ function such that $g(y) > 0$ ($y \neq 0$) and $g^{(j)}(0) = 0$ for any j. Then P_2 is hypoelliptic if g satisfies

(17) $$\lim_{y \to 0} |y|^{1/(\ell+1)}|\log g(y)| = 0 \quad , \text{ (see } [M_7] \text{).}$$

In order to check (5) for P_2, we need a smaller cutting of the cotangent space $T^*(R_x^n)$ than the one $\Omega_0 \cup \Omega_0^c$ in the preceding paragragh.

From Hörmander's classical theorem in [Hr_1] and its sharp version given in [R-S] and [F-P] we have

(18) $$\begin{aligned} &\mathrm{Re}(P_2u, u) + \|u\|^2 \\ &\geq \|D_xu\|^2 + \|x^\ell D_yu\|^2 + (f(y)D_z^2u, u) + \|u\|^2 \\ &\geq C(\| |D_y|^{1/(\ell+1)}\check{u}\|^2 + (f(y)\xi^2\check{u}, \check{u})) , \quad u \in C_0^\infty , \end{aligned}$$

where $\check{u}(\cdot, \xi)$ denote the Fourier transform of $u(\cdot, z)$. Set $M = |\xi|$

and $V(y) = f(y)M^2$. Then, in view of (18), for the proof of (5) it suffices to show that for any integer $k > 0$ there exists a $M_k > 0$ such that for a $c > 0$

$$\text{(19)} \qquad \| |D_y|^{1/(\ell+1)} u\|^2 + (V(y)u, u) \geq c(k \log M)^2 \|u\|^2 \quad , \quad u \in C_0^\infty(R^1) \ , \quad \text{if } M \geq M_k \ .$$

To derive this we need the following theorem (cf. Theorem B in $[M_7]$). Let λ be $0 < \lambda \leq 1$. We consider a symbol of the form

$$a(x, \xi) = |\xi|^{2\lambda} + V(x) \ , \ x \in R^n,$$

where $V(x)$ is a non-negative measurable function. Following Fefferman-Phong [F-P], [Ff], we consider a set $\mathcal{B}$ of boxes

$$\text{(20)} \quad B \equiv \{ (x, \xi) \in R^{2n} \ ; \ |x_j - x_{0j}| \leq \delta/2 \ , \ |\xi_j - \xi_{0j}| \leq \delta^{-1}/2 \ \}$$

for all $(x_0, \xi_0) \in R^{2n}$ and all $\delta > 0$. Clearly, the volume of $B \in \mathcal{B}$ is equal to 1 .

Theorem 5. *Assume that there exists a* $c > 0$ *such that for any* $B \in \mathcal{B}$

$$\text{(21)} \qquad m(\{ (x,\xi) \in B \ ; \ a(x, \xi) \geq R > 0 \}) \geq c \ ,$$

where m(.) *denotes Lebeague measure. Then we have*

$$\text{(22)} \qquad \| |D_x|^\lambda u\|^2 + (V(x)u, u) \geq c' R\|u\|^2 \ , \quad u \in C_0^\infty(R^n),$$

where $c' > 0$ *depends only on* c *and* n.

Set $a(y,\eta) = |\eta|^{2/(\ell+1)} + f(y)M^2$ and $R = (k \log M)^2$. Then, it is not difficult to check (21) by using (17). So, by Theorem 5 we obtain (19) and hence (5) for P_2.

Finally, we remark that we can prove the hypoellipticity of P_2 by Theorem 5 and Corollary 2 even if $f(y)$ degenerates in a Cantor set with measure 0 (see [M_7]).

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