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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

SYMPLECTIC TOPOLOGY
AND HAMILTONIAN DYNAMICS

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Consider in $\mathbf{R}^{2n}$ the linear operator J with matrix:

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathcal{L}(\mathbf{R}^{2n})$$

It defines a two-form ω by:

$$\omega(x,y) := (Jx, y)$$

$(\mathbf{R}^{2n}, \omega)$ is the **standard symplectic space.**

A linear map $M \in \mathcal{L}(\mathbf{R}^{2n})$ will be called symplectic if it preserves ω ; that is:

$$\omega(Mx, My) = \omega(x,y) \quad \forall (x,y)$$

This leads us to the characterization:

$$M^* J M = M$$

Let $\Omega \subset \mathbf{R}^{2n}$ be an open subset. A nonlinear map $\varphi \in C^1(\Omega; \mathbf{R}^{2n})$ will be called symplectic if its derivative $\varphi'(x)$ is symplectic for every $x \in \Omega$. Traditionaly, such maps were called canonical. Note the requirement that φ be at least C^1.

Symplectic geometry starts with the simplest possible question: given two open subsets $\mathcal{U}$ and $\mathcal{V}$ in $\mathbf{R}^{2N}$, is it possible to send $\mathcal{U}$ into $\mathcal{V}$ by a symplectic transformation ? In other words, does there exist a symplectic φ such that $\varphi(\mathcal{U}) \subset \mathcal{V}$?

A necessary condition has long been known. Since ω^n is the standard measure on $\mathbf{R}^{2n}$, symplectic transformations must preserve volumes (Liouville's theorem). So, if $\mathcal{U}$ can be sent into $\mathcal{V}$ by a symplectic transformation, we must have:

$$\text{vol}(\mathcal{U}) \leq \text{vol}(\mathcal{V})$$

For $n = 1$, this condition is almost sufficient. For $n > 1$ however, this is very for from being the case. Gromov [G] startled the mathematical world by proving:

Theorem 1.— *Consider in* $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$
the unit ball

$$B := \{(p,q) | \sum_{i=1}^{n} (p_i^2 + q_i^2) < 1\}$$

the vertical cylinder

$$C_1 := \{(p,q) | p_1^2 + q_1^2 < 1\}$$

and assume rB *can be sent into* RC_1 *by a symplectic transformation. Then:*

$$r \leq R \quad \blacksquare$$

Here $x = (p,q)$, so the vertical cylinder C_1 is to be distinguished from horizontal cylinders such as:

$$C^1 := \{(p,q) | p_1^2 + p_2^2 < 1\}$$

Note that $\mathrm{vol}(B) < \infty$ while $\mathrm{vol}(C) = \infty$, so that, if one relied on volume considerations, one would have concluded that rB can always be sent into RC.

To understand Gromov's result, and more like it, we introduce a definition.

Définition 2. A **symplectic capacity** is a map $c : \mathcal{P}(\mathbf{R}^{2n}) \to [0,\infty) \cup \{+\infty\}$ with the following properties:

conformal invariance: if $\varphi \in C^1(\mathbf{R}^{2n}, \mathbf{R}^{2n})$ and $a > 0$ are such that $\varphi^*\omega = a\omega$, then

$$c(\varphi(A)) = a\ c(A) \quad \forall A \subset \mathbf{R}^{2n}$$

monotonicity if $A \subset B \subset \mathbf{R}^{2n}$, then

$$c(A) \leq c(B)$$

scaling

$$c(B) = \pi = c \subset (C_1) \quad \blacksquare$$

Once we have a symplectic capacity we can prove Gromov's theorem:

Proof Assume there is $\varphi \in C^1(B, \mathbf{R}^{2n})$ which is symplectic and $\varphi(rB) \subset RC_1$. It is well-known that, for any $\varepsilon > 0$, there is a $\tilde{\varphi} \in C^1(\mathbf{R}^{2n}, \mathbf{R}^{2n})$ which coincides with φ on $(1-\varepsilon)rB$. So henceforth we assume that φ is defined on all of $\mathbf{R}^{2n}$. We have $\varphi \in C^1(\mathbf{R}^{2n}, \mathbf{R}^{2n})$ and $\varphi(rB) \subset RC_1$. Then

$$\begin{aligned} c(\varphi(rB)) &\leq c(RC_1) && \text{(monotonicity)} \\ c(rB) &\leq c(RC_1) && \text{(symplectic invariance)} \\ r^2 c(B) &\leq R^2 \subset C(C_1) && \text{(conformal invariance)} \\ r^2\pi &\leq R^2\pi && \text{scaling} \\ r^2 &\leq R^2 && \text{as desired} \quad \blacksquare \end{aligned}$$

Of course the main problem is to show that symplectic capacities exist at all. The firstone to do so was Gromov [G] who defined "symplectic width" using holomorphic disks. His definition makes sense in any symplectic manifold. In [EH], we give an existence and representation theorem for a symplectic capacity in $\mathbf{R}^{2n}$.

Theorem 3.— *There exists a symplectic capacity c with the following property. Let $\mathcal{U} \subset \mathbf{R}^{2n}$ be a bounded open set, such that its boundary $\partial\mathcal{U}$ is a C^1 hypersurface of contact type. Then ∂U carries a closed C^1 curve γ such that*

(1) $$-J\,\mathring{\gamma}(t) \quad \text{is normal to} \quad \partial\mathcal{U} \quad \text{at} \quad \gamma(t)$$

(2) $$c(\mathcal{U}) = \oint (\gamma, -J\,\mathring{\gamma})dt. \quad \blacksquare$$

To say that $\partial\mathcal{U}$ has contact type means that the restriction of ω to $\partial\mathcal{U}$ has a primitive Ω such that $\Omega \wedge (\omega)^{n-1}$ is a volume form on $\partial\mathcal{U}$. This will be the case if for instance $\mathcal{U}$ is star-shaped with respect to some point.

Conditions (1) and (2) do not depend on the time parametrization of γ. If for instance we choose a non-vanishing continuous section $n(x)$ of the normal bundle, we can rewrite (1) as follows:

$$\mathring{\gamma} = Jn(\gamma)$$

and this equation defines a flow on $\partial\mathcal{U}$ if $n(x)$ is locally Lipschitz. This is the Hamiltonian flow naturally associated with $\partial\mathcal{U}$. Theorem 3 then asserts that the capacity of $\mathcal{U}$ is equal to the action integral along some particular closed trajectory of the Hamiltonian flow.

Note that this particular trajectory may be run around several times ; that is, the right hand side of formula (2) is defined up to multiplication by an integer.

Let us try the representation formula on B. The Hamiltonian flow on ∂B is well-known ; all its trajectories are closed and the action along them is π. So we get

$$c(B) = \oint (\gamma, -J\mathring{\gamma})dt = k\pi$$

for some integer $k \geq 1$. Direct arguments show that $c(B) < 2\pi$ so $k = 1$, and we have proved half of the scaling formula.

Now for C_1. The Hamiltonian flow on ∂C_1 also has only closed trajectories, all of which have action π. We get $c(C_1) = k\pi$, and we show that the integer k must be 1. Hence the scaling formula.

What about the horizontal cylinder C^1? The Hamiltonian flow runs along generatrices and there are no closed trajectories. We find therefore that

$$c(C^1) = \infty.$$

So the capacity is able to distinguish between vertical cylinder ($c(C_1) = \pi$) and horizontal ones ($c(C_1) = \infty$). What is relevant here is clearly the axis of the cylinder, that is the two-planes

$$(p_1, 0, \ldots, 0, q_1, 0, \ldots, 0) \quad \text{for} \quad C_1$$

$$(p_1, p_2, 0, \ldots\ldots\ldots, 0) \quad \text{for} \quad C^1.$$

The second one is isotropic which means that the restriction of ω vanishes. We can exploit this property. Define an ellipsoid to be the set where $q(x) < 1$, for some positive definite quadratic form q.

Proposition 4.— *Assume a linear map $M \in \mathcal{L}(\mathbf{R}^{2n})$ preserves the capacity of ellipsoids.*

$$c(M(E)) = c(E)$$

Then M is symplectic or antisymplectic:

$$M^*\omega = \pm\omega \quad \blacksquare$$

Indeed, such a map will change an isotropic 2-plane into an isotropic 2-plane. Some linear algebra then gives the result. It carries over to the nonlinear case.

Theorem 5.— *Assume a nonlinear map $\varphi \in C^0(B, \mathbf{R}^{2n})$ preserves capacities. If $\varphi'(0)$ exists, then $\varphi'(0)$ is symplectic or antisymplectic.* ∎

This is a remarkable result because it enables us to extend the notion of symplecticity to the C^0 category. It also enables us to prove a C^0-rigidity theorem.

Theorem 6.— *Let $\varphi_n \in C^1(B, \mathbf{R}^{2n})$ be a sequence of C^1 symplectic embeddings converging uniformly to φ. If $\varphi'(0)$ exists, then it is symplectic or antisymplectic.* ∎

In fact, since the φ_n are symplectic they preserve c. Their C^0-limit φ must also preserve c, and by theorem 5 it will be symplectic or antisymplectic at any point of diffrentiability.

As a consequence, we get a celebrated result of Eliashberg and Gromov [G].

Corollary 7.— *Let P be a compact symplectic manifold and φ_n a sequence of symplectic diffeomorphisms, converging uniformly to a diffeomorphism φ. Then φ is symplectic.* ∎

Proofs will be found in [E-H]. The starting point of this investigation is the theorem of Viterbo [V].

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