Séminaire Équations aux dérivées partielles - École Polytechnique

Nicolas Lerner

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Séminaire Équations aux dérivées partielles (Polytechnique) (1991-1992), exp. nº 12, p. 1-22

<http://www.numdam.org/item?id=SEDP_1991-1992____A12_0>

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Séminaire 1991-1992

EQUATIONS AUX DERIVEES PARTIELLES

A NON SOLVABLE OPERATOR SATISFYING CONDITION (ψ)

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Exposé n° XII

18 Février 1992

A non solvable operator satisfying condition (ψ)

by

Nicolas LERNER

Abstract : The main goal of the present paper is to provide an example of a principal type pseudo-differential operator $P = p(x,D_x)$, with symbol p in $S^1_{1,0}$, satisfying condition (ψ), so that the equation $Pu = f$ has no L^2 solution for most f in L^2 .

This is an expanded version of a talk given in the PDE seminar at Ecole Polytechnique in february 1992 ; it contains all the essential arguments necessary to the construction of our counterexample, but could be considered as sketchy at some places. A final version of this article will be submitted soon for publication somewhere.

Contents

1. Introduction

We are interested in local solvability properties for pseudo-differential operators : the operator P is said to be locally solvable if for any smooth f satisfying a finite number of compatibility conditions, there is a distribution u local solution of $P u = f$ (see definition 26.4.1 in [5] for a precise statement). Most of the research in this domain was oriented toward a characterization of local solvability of a principal type pseudo-differential operator in terms of a geometric property of its principal symbol, the so-called condition (ψ). We briefly recall here some of the basic facts related to this problem.

Let P be a pseudo-differential operator of principal type (i.e. the hamiltonian field H_p of the principal symbol p is independent of the Liouville vector field). If the principal symbol is real-valued, a propagation-of-singularities result is true and implies global existence (see theorem 26.1.9 in [5]). When the principal symbol is complex-valued,the situation is much more complicated ; in 1957, Hans Lewy found a principal type differential operator without solution. His example ,

$$\frac{\partial}{\partial \bar{z}} + i z \frac{\partial}{\partial t} \quad \text{in} \quad \mathbb{C}_z \times \mathbb{R}_t ,$$

is a non-academic one as the Cauchy-Riemann operator on the boundary of a strictly pseudo-convex domain. The simpler models $D_t + i\, t^{2k+1} D_x$ were given later on by Mizohata [8]. Local solvability of differential operators is now known to be characterized by condition (P) : the symbol p is said to satisfy condition (P) if the imaginary part Imp does not change sign along the bicharacteristic curves of $\mathbb{R}$ep (see Nirenberg-Treves [10] with an analyticity assumption, Beals-

Fefferman [1] in the general case for local solutions, Hörmander's theorem 26.11.3 in [5] for a semi-global existence result).

In the pseudo-differential case, a (quite natural) extension of condition (P) is condition (ψ): the imaginary part Imp does not change sign from - to + along the oriented bicharacteristic curves of $\mathbb{R}$ep (see definition 26.4.6 in [5]). This condition was proven invariant by multiplication by an elliptic factor in [10] (see also lemma 26.4.10 in [5]). The importance of this geometrical condition was stressed by Nirenberg and Treves [10] who conjectured condition (ψ) is equivalent to local solvability and proved it in a number of cases. The necessity of condition (ψ) for local solvability was established for general pseudo-differential equations after the works of Moyer [9] in two dimensions and Hörmander in the general case (Corollary 26.4.8 in [5]). Moreover, the sufficiency in two dimensions is proved in [6], yielding condition (ψ) as an iff condition for solvability in that case . Hörmander's work on subellipticity (theorem 27.1.11 of [5]) showed that if the symbol p satisfies condition (ψ) and a finite type assumption ((27.1.8) in [5]) then the quantization of $\overline{p}$ is hypoelliptic and thus the operator with symbol p is solvable.

It was shown by Nirenberg and Treves [10] that a solvability problem for a principal type operator is equivalent by localisation and canonical transformation to proving an a priori estimate for a first order pseudo-differential operator

$$\frac{1}{i}\frac{\partial}{\partial t} + i\,Q(t,x,D_x)\ ,$$

where $Q(t,x,D_x)$ is a first-order pseudo-differential operator with real principal symbol q . In that framework, condition (ψ) for the adjoint operator is expressed as

$$(1.1)\qquad q(t,x,\xi) > 0 \ \text{ and } \ s \geq t \qquad \text{imply} \qquad q(s,x,\xi) \geq 0.$$

In this paper, we construct a symbol $q(t,x,\xi)$ in the $S^1_{1,0}$ class, i.e. a smooth function of five real variables such that, for any five-uple of integers $k, \alpha_1, \alpha_2, \beta_1, \beta_2$,

$$(1.2)\qquad \sup_{t,x_1,x_2,\xi_1,\xi_2\in\mathbb{R}^5} |(D_t^k D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_{\xi_1}^{\beta_1} D_{\xi_2}^{\beta_2} q)(t,x_1,x_2,\xi_1,\xi_2)|\ [1+|\xi_1|+|\xi_2|]^{-1+|\beta_1|+|\beta_2|} < +\infty$$

such that q satisfies (1.1) and such that no L^2 estimate can be proved for $\frac{1}{i}\frac{\partial}{\partial t} + i\,q(t,x,D_x)$: we have

$$(1.3)\qquad \inf_{u\in C_0^\infty(\Omega)\,,\ \|u\|_{L^2(\mathbb{R}^3)}=1} \left\| \frac{1}{i}\frac{\partial u}{\partial t} + i\,q(t,x,D_x)u \right\|_{L^2(\mathbb{R}^3)} = 0$$

for any Ω neighborhood of $0_{\mathbb{R}^3}$. We thus prove that the equation $\frac{\partial v}{\partial t} + q(t,x,D_x)v = f$ has no L^2 solution for a general right -hand side f in L^2 although the operator $\partial_t + q(t,x,D_x)$ satisfies condition (ψ). Let's remark here that , first of all, q is not homogeneous though in the most classical class of pseudo-differential operators of order 1, and next, that we are only able to disprove the L^2 solvability (which is quite satisfactory for a first order operator) ; at the present time, the author does not know if these two weaknesses could be easily overcome ; however , we give in the fifth section an example of a bounded sequence of homogeneous symbols of order 1, satisfying condition (1.1), so that no L^2 estimate can hold for this family. At any rate, this counterexample shows that no L^2 estimate can be proved for a pseudo-differential operator $\frac{1}{i}\frac{\partial}{\partial t} + i\, q(t,x,D_x)$ in the $S^1_{1,0}$ class under the sole assumption of condition (1.1).

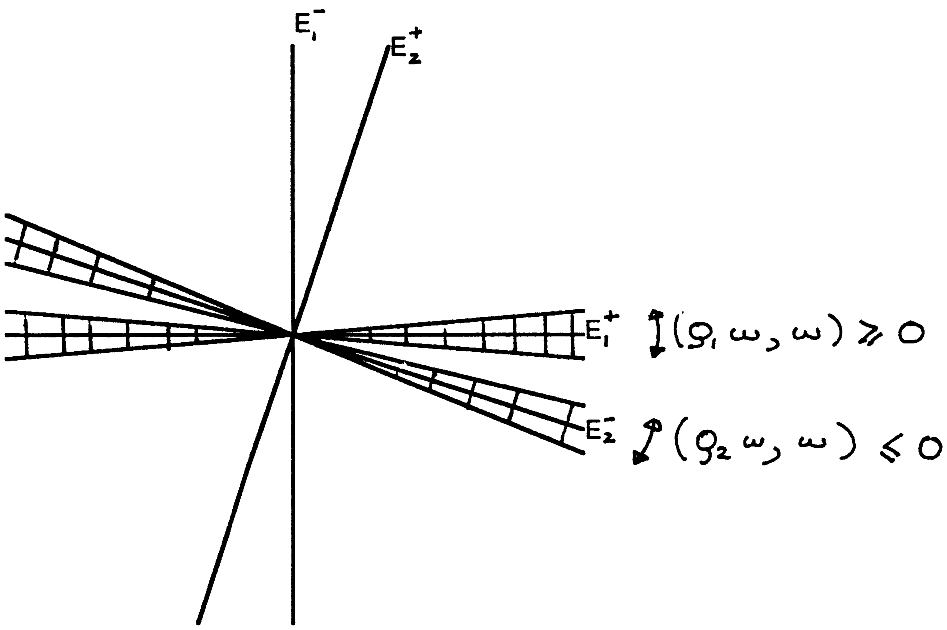
The reader eager for precise statements and proofs may proceed directly to the main body of the text, starting in section 2. However, we wish to devote some space to an informal introduction to the main ideas involved in the construction, as well as outlining the steps of the proof. Let's choose first our notations : we want to discuss the solvability of an evolution operator $\frac{d}{dt} + Q(t)$, where each Q(t) is a selfadjoint operator (unbounded) on a Hilbert space $\mathbb{H}$. This is equivalent to discussing a priori estimates for $\frac{d}{dt} - Q(t) = i\,[\, D_t + i\,Q(t)\,]$. It is quite clear that the above problem is far too general, and so we wish to start our discussion with the simplest non trivial example : instead of dealing with infinite dimensional Hilbert space , let's take $\mathbb{H} = \mathbb{R}^2$, so that Q(t) is a 2 x 2 symmetric matrix, defining a (bounded!) operator on $\mathbb{R}^2$, allowed to depend on large parameters. Since it could be still complicated, let's assume

$$(1.4) \qquad Q(t) = H(-t)\,Q_1 + H(t)\,Q_2 \;,$$

where H is the Heaviside function(characteristic function of $\mathbb{R}^+$), Q_1 and Q_2 (2 x 2) symmetric matrices. There is of course no difficulty solving the equation $\frac{dv}{dt} + Q(t)\,v = f$; however, if we want to get uniform estimates with respect to the size of the coefficients of Q(t), we have to choose carefully our solutions, even in finite dimension . When $Q_1 = Q_2$, the good fundamental solution is given by $H(t)\,E_1^+ \exp -t\,Q_1 - H(-t)\,E_1^- \exp -t\,Q_1$, where E_1^+ and E_1^- are the spectral projections corresponding to the half axes. If we go back to (1.4) with $Q_1 \neq Q_2$, there is a trivial case in which the operator $\frac{d}{dt} + Q(t)$ is uniformly solvable : the monotone increasing situation $Q_1 \leq Q_2$ yielding the estimate $\| D_t u + i\,Q(t)u \|_{L^2} \geq \| D_t u \|_{L^2}$, where L^2 stands for $L^2(\mathbb{R}, \mathbb{H})$.We eventually come to our first point : is it true that solvability for $\frac{d}{dt} + Q(t)$ implies the same property for $\frac{d}{dt} + \alpha(t)\,Q(t)$, where α is a non-negative scalar function ? This question is naturally linked with condition (ψ), since whenever $q(t,x,\xi)$ satisfies (1.1) so does $a(t,x,\xi)\,q(t,x,\xi)$ for a non-negative symbol a. We are thus quite naturally led to discuss the uniform solvability of $\frac{d}{dt} + Q(t)$, with

(1.5) $Q(t) = H(-t)\, Q_1 + H(t - \theta)\, Q_2$, $Q_1 \leq Q_2$, (2 x 2) symmetric matrices , $\theta \geq 0$.

The most remarkable fact for the pair of matrices $Q_1 \leq Q_2$ is the " DRIFT " : the best way to explain it is to look at the following picture :



- figure 1 -

The condition $Q_1 \leq Q_2$ does not prevent E_1^+ and E_2^- to get very close : let's define the drift for the pair Q_1 , Q_2 as the absolute value of the cotangent of the angle between E_1^+ and E_2^-, so that

the drift is zero when $E_2^- \subset E_1^-$ and $E_1^+ \subset E_2^+$,
the drift is infinite when $E_1^+ \cap E_2^-$ is not reduced to zero ,
the drift is unbounded when the distance between the spheres of E_1^+ and E_2^- is zero.

If we consider for instance the following pair of 2x2 symmetric matrices :

(1.6) $$Q_{1,\nu} = \begin{bmatrix} \nu^2 & 0 \\ 0 & -\nu^3 \end{bmatrix} \leq e^{-i\alpha_\nu} \begin{bmatrix} \nu^3 & 0 \\ 0 & -\nu \end{bmatrix} e^{i\alpha_\nu} = Q_{2,\nu}$$

where ν is a large positive parameter, $e^{i\alpha_\nu}$ the rotation of angle α_ν , with $\cos^2\alpha_\nu = 2/\nu$, the drift goes to infinity with ν , since

(1.7) the square of the distance between the spheres of E_1^+ and E_2^- is equivalent to $2/\nu$.

We now claim the non-uniform solvability of the operator $\frac{d}{dt} + Q(t)$,with Q(t) given by (1.5), Q_1 and Q_2 by (1.6). We set up,with ω_1 and ω_2 unit vectors respectively in E_1^+ and E_2^- ,

(1.8)
$$u(t) = \begin{cases} e^{tQ_1}\omega_1 & \text{on } t < 0 \\ \omega_1 + \frac{t}{\theta}(\omega_2 - \omega_1) & \text{on } 0 < t < \theta \\ e^{(t-\theta)Q_2}\omega_2 & \text{on } t > \theta \end{cases} .$$

Let's compute now

(1.9)
$$\| \frac{du}{dt} - Q(t)\, u \|_{L^2}^2 = \int_0^\theta \frac{|\omega_2 - \omega_1|^2}{\theta^2}\, dt = \theta^{-1} |\omega_2 - \omega_1|^2 ,$$

where $|\ |$ stands for the norm on $\mathbb{H}$. On the other hand,

(1.10)
$$\| u \|_{L^2}^2 \geq \int_0^\theta | \omega_1 + \frac{t}{\theta}(\omega_2 - \omega_1) |^2 dt \geq \frac{\theta}{2} - \theta\, |\omega_2 - \omega_1|^2 \geq \frac{\theta}{4}$$

if $|\omega_2 - \omega_1|^2 \leq \frac{1}{4}$, an easily satisfied requirement subsequent to(1.7) . Consequently, using (1.9), (1.10), we get

(1.11)
$$\| u \|_{L^2}^{-2} \| \frac{du}{dt} - Q(t)\, u \|_{L^2}^2 \leq \theta^{-2} |\omega_2 - \omega_1|^2\ 4 .$$

Since ω_1 and ω_2 can be chosen arbitrarily close and independently of the size of the "hole" θ, we get easily a non-solvable operator on $l^2(\mathbb{N})$ by taking direct sums. Note that (1.11) can be satisfied by a compactly supported u since the eigenvalues corresponding to ω_1 and ω_2 are going to infinity with ν, in such a way that there is no difficulty to multiply u by a cut-off function. In the next sections, we 'll say more about the drift of an ordered pair (Q_1 , Q_2) of selfadjoint operators ; it will turn out that the solvability of $\frac{d}{dt} + Q(t)$ will depend very closely upon the drift of the family (Q(t)), and that the solvability of all the operators $\frac{d}{dt} + \alpha(t)Q(t)$, when α is a non-negative scalar function, will require more or less that the drift for the family Q(t) is bounded.

What we've done so far is to get an "abstract" non-solvable operator obtained by change of time-scale from a monotone increasing situation ; the basic device for the construction was the unbounded drift of $Q_1 \leq Q_2$. Since we are interested in pseudo-differential operators, the next question is obviously : is an unbounded drift possible for $Q_1 \leq Q_2$, both of them pseudo-differential ? We 'll see the answer is yes, leading to our counterexample. It is quite interesting to remark that differential operators satisfying condition (P) do not drift, as shown by the Beals-Fefferman reduction : after a non-homogeneous microlocalisation and canonical transformation , their procedure leads to an evolution operator

$$\frac{d}{dt} + Q(t), \qquad \text{with } Q(t) = Q(t,x,D_x) \qquad \text{and } Q(t,x,\xi) = \xi_1\ a(t,x,\xi)\ ,$$

where a is a non-negative symbol of order 0 (in a non-homogeneous class). Then, a Nirenberg-Treves commutator argument gives way to an estimate, after multiplication by the sign of ξ_1 . Quite noticeable too, the fact that 2-dimensional pseudo-differential operators satisfying condition (ψ) do not drift, since the sign function is monotone matrix on operators whose symbols are defined on a lagrangean manifold ; the last remark led the author to a proof of local solvability in two dimensions [6] and for oblique-derivative type operators [7]. Our definition of the Fourier transform is

$$\text{(1.12)} \qquad \hat{u}(\xi) = \int e^{-2i\pi x\xi}\, u(x)\, dx \quad \text{so that } u = \check{\hat{\hat{u}}} \ \text{ with } \check{u}(x) = u(-x)\ .$$

Let's first study the very simple case

$$\text{(1.13)}\ Q_1 = D_{x_1} = \frac{1}{2i\pi}\frac{\partial}{\partial x_1} \leq Q_2 = D_{x_1} + \Lambda x_1^2 = e^{-2i\pi\frac{\Lambda x_1^3}{3}}\, D_{x_1}\, e^{2i\pi\frac{\Lambda x_1^3}{3}}\ ,$$

where Λ is a large positive parameter. Consider ω_1 a unit vector in E_1^+ , i.e.

$$\text{(1.14)} \qquad \omega_1(x) = \int \kappa_1(\xi)\ e^{2i\pi x\xi}\, d\xi\ , \quad 1 = \|\ \kappa_1\ \|_{L^2}\ , \quad \text{supp}\ \kappa_1 \subset \mathbb{R}^+\ ,$$

and ω_2 a unit vector in E_2^- , i.e.

$$\text{(1.15)} \qquad \omega_2(x)\, e^{2i\pi\frac{\Lambda x^3}{3}} = \int \kappa_2(-\xi)\ e^{2i\pi x\xi}\, d\xi\ , \ 1 = \|\ \kappa_2\ \|_{L^2}\ , \ \text{supp}\ \kappa_2 \subset \mathbb{R}^+\ .$$

A convenient way of estimating the drift of the pair (Q_1, Q_2) is to get an upper bound smaller than 1 for $|< \omega_1 , \omega_2 >_{L^2}|$: this quantity is 0 if the pair is not drifting, is 1 if the drift is infinite. We have

$$< \omega_1 , \omega_2 > = \int\int\int \kappa_1(\xi) \, e^{2i\pi x(\xi+\eta)} \, \bar{\kappa}_2(\eta) \, e^{2i\pi \frac{\Lambda x^3}{3}} \, dx \, d\eta \, d\xi \ ,$$

and thus,

$$(1.16) \qquad < \omega_1 , \omega_2 > = \int\int \kappa_1(\Lambda^{1/3}\xi)\Lambda^{1/6} \ \bar{\kappa}_2(\Lambda^{1/3}\eta)\Lambda^{1/6} \, A(\xi+\eta) \, d\eta \, d\xi,$$

where

$$(1.17) \qquad A(\xi) = \int e^{2i\pi \frac{x^3}{3}} \, e^{2i\pi x\xi} \, d\xi \quad \text{is the Airy function.}$$

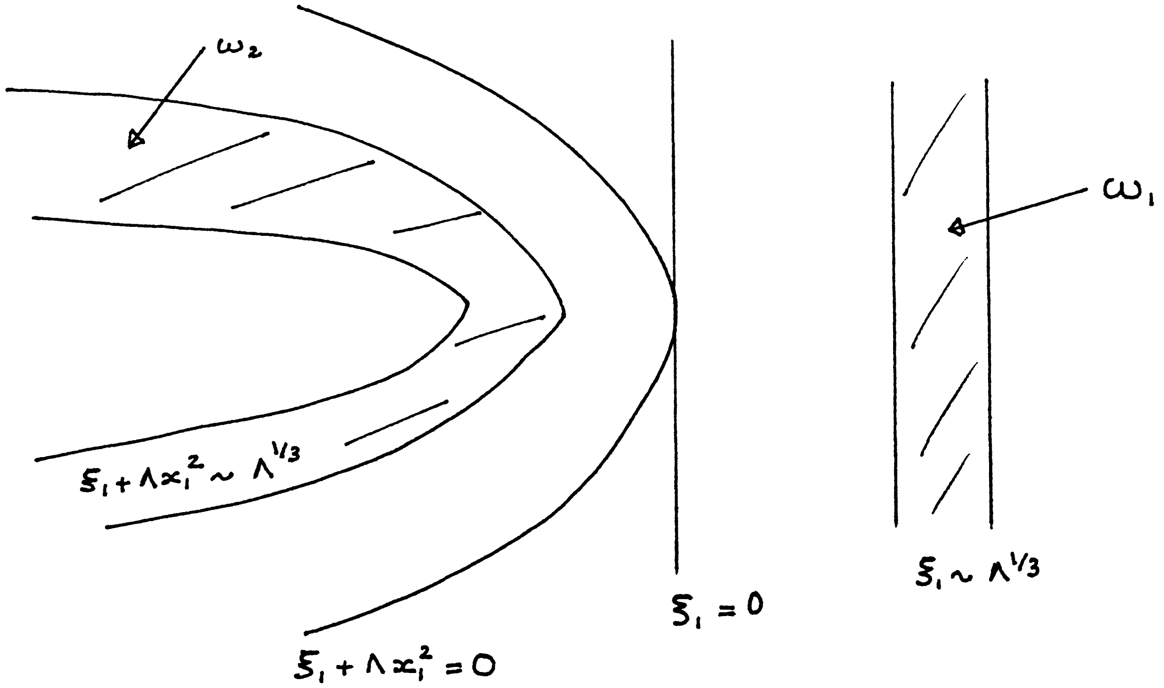
Consequently,

$$(1.18) \qquad \sup_{\omega_1 \in E_1^+, \omega_2 \in E_2^-, \|\omega_1\| = \|\omega_2\| = 1} |< \omega_1 , \omega_2 >_{L^2}| \ > \ 0 \ ,$$

since for $\kappa_1(\xi) = \bar{\kappa}_2(\xi) = \kappa(\xi\Lambda^{-1/3})\Lambda^{-1/6}$ with a non-negative κ, supported in the interval [1,2], $1 = \|\kappa\|_{L^2}$, (1.16) gives

$$(1.19) \qquad < \omega_1 , \omega_2 > = \int\int \kappa(\xi)\kappa(\eta) \, A(\xi+\eta) \, d\eta \, d\xi = \int A(\xi)(\kappa * \kappa)(\xi) d\xi \quad ,$$

the last term is a positive constant, independent of Λ (the Airy function given by (1.15) is positive on $\mathbb{R}^+$) . A picture will be useful for the understanding of these inequalities :



- figure 2 -

this picture in the (x_1,ξ_1) symplectic plane shows that even though ω_1 and ω_2 are "living" in two far away strips, one of which is a curved one, their dot product could be large. If we add one dimension to get an homogeneous version, Λ would be $|\xi_1| + |\xi_2|$, so that the above localisations in the phase space appear as two different second microlocalisations with respect to the hypersurfaces $\{ \xi_1 = 0 \}$ on the one hand, $\{ \xi_1 + |\xi| x_1^2 = 0 \}$ on the other hand (see [2], [3]). These second microlocalisations are somehow incompatible so that the long range interaction between two far away boxes corresponding to two different calculus could be large, as shown by the equality (1.19). However, the pair given by (1.12) has a bounded drift, i.e. the quantity (1.18) is bounded above by a number strictly smaller than 1. This implies the solvability of

$$(1.20) \qquad \frac{d}{dt} + \alpha(t,x,D_x)\left[H(-t)D_{x_1} + H(t) \left(D_{x_1} + x_1^2\sqrt{D_{x_1}^2 + D_{x_2}^2} \right]\right. ,$$

where $\alpha(t,x,\xi)$ is a non-negative symbol of order zero, flat at $t = 0$. Since we are not going to use this result, we leave its proof to the reader with an hint : compute the real parts of
$< D_tu + iQ(t)u , \ i H(t-T)E_2^+u >$, $< D_tu+iQ(t)u , \ -i H(T-t)H(t)E_2^-u >$, for non-negative T,
$< D_tu + iQ(t)u , - i H(T-t)E_1^-u >$, $< D_tu+iQ(t)u , \ i H(t-T)H(-t)E_1^+u >$, for non-positive T ,
use the bounded drift , meaning $E_2^+ + E_1^-$ invertible, and the Nirenberg-Treves commutator argument (see e.g. lemma 26.8.2 in [5]). We now start over our discussion on pseudo-differential operators and study the following case, which turns out to be the generic one, using the microlocalisation procedure of [1] and the Egorov principle of [4] :

$$(1.21) \quad Q_1 = D_{x_1} = \frac{1}{2i\pi}\frac{\partial}{\partial x_1} \leq Q_2 = D_{x_1} + V(x_1) = e^{-2i\pi\phi(x_1)} D_{x_1} e^{2i\pi\phi(x_1)} ,$$

with a non-negative $V = \phi'$. Following the lines of the computations starting at (1.12), the question at hand is to estimate from above

(1.22) $$< \omega_1 , \omega_2 > = \int\int\int \kappa_1(\xi)\ e^{2i\pi x(\xi+\eta)}\ \bar{\kappa}_2(\eta)\, e^{2i\pi\phi(x)}\, dx\, d\eta\, d\xi\ ,$$

with

(1.23) $$1 = \| \kappa_1 \|_{L^2} = \| \kappa_2 \|_{L^2}\ ,\quad \text{supp}\ \kappa_1 \subset \mathbb{R}^+\ ,\ \text{supp}\ \kappa_2 \subset \mathbb{R}^+ .$$

This means estimating from above the $\mathcal{L}(L^2)$norm of the product $\Pi = H(-D_x)\ e^{2i\pi\phi}\ H(D_x)$, where $H(D_x)$ is the Fourier multiplier by the Heaviside function H. If ϕ is $\frac{1}{2}$ H(x), then

(1.24) $$\Pi = i\, H(-D_x)\,[\, -H(x) + H(-x)\,]\, H(D_x) = F\, \Omega\, F,$$

where F is the Fourier transform, and Ω the Hardy operator,whose kernel is $H(\xi)H(\eta)/\pi(\xi+\eta)$ (the norm of Ω is obviously ≤ 1 from (1.24)). It is not difficult to see that the norm of the Hardy operator is exactly 1, as shown in section 4. As a consequence,we get an unbounded drift for the pair (D_{x_1} ; $D_{x_1} + \frac{1}{2}\delta(x_1)$), at least in a formal way ; we will approximate the Dirac mass by a sequence of smooth functions $\frac{1}{2}\nu\, W(\nu\, x_1)$, where W is non-negative with integral 1, and in order to get a symbol , we'll perform this approximation at the frequencies equivalent to 2^ν . Moreover, we shall choose carefully the size and the regularization of the "hole" θ depending on this frequency.

2. Statement of the result

Theorem 2.1

There exists a real valued symbol $q = q(t,x,\xi)$ in the $S^1_{1,0}$ class on $\mathbb{R}_t \times \mathbb{R}^2_x \times \mathbb{R}^2_\xi$, i.e. a smooth function q satisfying (1.2), for which condition (1.1) is fulfilled, such that there is no neighborhood Ω of the origin in $\mathbb{R}_t \times \mathbb{R}^2_x$, so that the equation

$$\frac{\partial v}{\partial t} + q(t,x,D_x)v = f \tag{2.1}$$

has an $L^2_{loc}(\Omega)$ solution v for any f in $\mathcal{D}(\Omega)$. There is a sequence u_k of functions in $C^\infty_0(\mathbb{R}^3)$, with L^2 norm 1 and support $u_k \to \{0\}$ such that

$$\left\| \frac{1}{i}\frac{\partial u_k}{\partial t} + i\, q(t,x,D_x)u_k \right\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{when} \quad k \to +\infty \quad . \tag{2.2}$$

3. The operator Q(t)

We set, for $x \in \mathbb{R}^2$, $\xi \in \mathbb{R}^2$, $t \in \mathbb{R}$,

$$Q(t,x,\xi) = \sum_{\nu=2}^{+\infty} \psi^2(2^{-2\nu}(\xi_1^2 + \xi_2^2))\, \chi^2(\xi_1/\xi_2)\, \alpha_\nu(t)\, q_\nu(t,x_1,\xi_1) \quad , \tag{3.1}$$

so that

$$\psi \in C^\infty_0(\mathbb{R}),\ \operatorname{supp}\psi \subset [2^{-1}, 2],\ \psi = 1 \text{ on } [2^{-1/2}, 2^{1/2}],\ \psi \geq 0, \tag{3.2}$$

$$\chi \in C^\infty_0(\mathbb{R}),\ \operatorname{supp}\chi \subset [-1, +1],\ \chi = 1 \text{ on } [-1/2, 1/2],\ 1 \geq \chi \geq 0, \tag{3.3}$$

$$\alpha_\nu(t) = H(-t)\beta(t) + H(t)\,\alpha(t/\theta_\nu)\,\lambda^{-1}_{2,\nu}, \quad \text{with} \tag{3.4}$$

$$\theta_\nu = \lambda^{-1}_{3,\nu}, \text{ and } e^{\lambda_{3,\nu}} \leq \lambda_{2,\nu}, \quad (\lambda_{3,\nu}, \lambda_{2,\nu} \text{ are positive constants}), \tag{3.5}$$

$$(3.6)\qquad \begin{array}{l}\beta\in C^{\infty}(\mathbb{R}),\ \text{supp}\ \beta\subset(-\infty,0\,],\beta>0 \text{ on } (-\infty,0),\\ \alpha\in C^{\infty}(\mathbb{R}),\text{supp}\ \alpha\subset[\,1,+\infty),\ \alpha>0 \text{ on } (1,+\infty),\ \alpha\equiv 1 \text{ on } (2,+\infty),\\ \alpha \text{ and } \beta \text{ bounded as well as all their derivatives.}\end{array}$$

Moreover, we set-up

$$(3.7)\qquad q_v(t,x_1,\xi_1) = H(-t)\xi_1 + H(t)\,[\,\xi_1 + \tfrac{1}{2}\,\lambda_{1,v}W(\lambda_{1,v}x_1)\,]\ ,\qquad \text{where}$$

$$(3.7)'\qquad 2^{\lambda_{1,v}} \le \lambda_{0,v} = 2^{v}\qquad \text{and}$$

$$(3.8)\qquad W\in C_0^{\infty}(\mathbb{R}),\ W\ge 0,\qquad \int W(x)\,dx = 1\ .$$

Lemma 3.1

The function Q defined by (3.1) $\in S^{1}_{1,0}$ and satisfies (1.1).

We'll begin proving Q is a smooth function. Let's remark that the open rings

$$(3.9)\qquad \Delta_v = \{\ (\xi_1,\xi_2)\in\mathbb{R}^2\ ,\ \ 2^{-1} < 2^{-2v}(\,\xi_1^2+\xi_2^2\,) < 2\ \}$$

are disjoint when v runs through the integers, and that

$$(3.10)\qquad 2^{v-1} < |\,\xi_2\,| < 2^{v+1/2}\qquad \text{on } \Delta_v\cap \text{supp}\ Q\ .$$

It is thus enough to check

$\alpha_v(t)\, q_v(t,x_1,\xi_1) = \alpha_v(t)\left\{H(-t)\xi_1 + H(t)\,[\,\xi_1 + \tfrac{1}{2}\lambda_{1,v}W(\lambda_{1,v}x_1)\,]\right\}$

which is a smooth function since α_v is C^{∞} and zero on $[\,0,\theta_v\,]$. To get (1.2), we must verify

$$(3.11)\qquad |(D_t^k D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_{\xi_1}^{\beta_1} D_{\xi_2}^{\beta_2} Q)(t,x_1,x_2,\xi_1,\xi_2)\,| \le 2^{v(1-|\beta_1|-|\beta_2|)}\ C_{k\alpha_1\alpha_2\beta_1\beta_2}\ ,$$

for $(\xi_1,\xi_2)\in\Delta_v$, where the constants C do not depend on v. We may assume $\beta_1\in\{\,0,1\}$ and $\beta_2 = 0 = \alpha_2$. Since $D_t^kD_{x_1}^{\alpha_1}Q = D_t^kD_{x_1}^{\alpha_1}\left\{\beta(t)\,\xi_1 + \alpha(t/\theta_v)\,\lambda_{2,v}^{-1}\,[\,\xi_1 + \tfrac{1}{2}\,\lambda_{1,v}\,W(\,\lambda_{1,v}x_1)\,]\,\right\}$, we get for $\beta_1 = 0$,

$$(3.12)\quad |D_t^kD_{x_1}^{\alpha_1}Q| \le 2^{v+1/2}\,\|\beta^{(k)}\|_{L^\infty} + \|\alpha^{(k)}\|_{L^\infty}\,\lambda_{3,v}^{k}\,\lambda_{2,v}^{-1}\,[\,2^{v+1/2} + \tfrac{1}{2}\,\lambda_{1,v}^{1+\alpha_1}\,\|\,W^{(\alpha_1)}\|_{L^\infty}\,]\ .$$

Moreover, we have for $\beta_1 = 1$,

$$(3.13)\quad |D_t^k D_{x_1}^{\alpha_1} D_{\xi_1} Q| \leq \|\beta^{(k)}\|_{L^\infty} + \|\alpha^{(k)}\|_{L^\infty}\ \lambda_{3,\nu}^{k}\ \lambda_{2,\nu}^{-1}\ ,$$

so that (3.5), (3.7)' , (3.12) and (3.13) give (3.11). We need to prove(1.1) : assume $Q(t,x,\xi) > 0$ and $s > t$. Since ξ belongs at most to one Δ_ν , we know $Q(t,x,\xi)$(resp. $Q(s,x,\xi)$) must be the product of a positive quantity by $\alpha_\nu(t)\ q_\nu(t,x_1,\xi_1)$ (resp.$\alpha_\nu(s)\ q_\nu(s,x_1,\xi_1)$). In fact, the function α given by (3.4) is non-negative, and $t \to q_\nu(t, x_1,\xi_1)$ is non-decreasing(from $W \geq 0$ in (3.4)). This concludes the proof of lemma 3.1 .

4. Drift of operators

Following the heuristic discussion in section 1 about the drift ,we consider first the Hardy operator .

Lemma 4.1

If $\mathcal{H}$ is the Hilbert transform, i.e. the convolution with $pv(1/i\pi\xi)$, E_+ (resp E_-) the projector defined by $(E_+\kappa)(\xi) = H(\xi)\kappa(\xi)$ (resp$(E_-\kappa)(\xi) = H(-\xi)\kappa(\xi)$), where H is the characteristic function of $\mathbb{R}^+$, the Hardy operator Ω is $i\,E_+\mathcal{H}\,E_-\,C$, where $(C\kappa)(\xi) = \kappa(-\xi)$.The $\mathcal{L}(L^2)$norm of Ω is 1 and its kernel is $H(\xi)H(\eta)/\pi(\xi+\eta)$. *Set-up, for $\varepsilon > 0$,*

$$(4.1)\quad \kappa_\varepsilon(\xi) = \Gamma(\varepsilon)^{-1/2} e^{-\xi/2} \xi^{(-1+\varepsilon)/2}\ H(\xi)$$ *, where Γ stands for the gamma function. We have*

$$(4.2)\quad 1 > (\Omega\kappa_\varepsilon, \kappa_\varepsilon)_{L^2} > 1 - \varepsilon\ .$$

Proof. It is pure routine to check that the kernel of $\Omega = i\,E_+\mathcal{H}\,E_-C$ is $H(\xi)H(\eta)/\pi(\xi+\eta)$ and this factorisation implies readily that the $\mathcal{L}(L^2)$norm of Ω is smaller than 1.It is thus enough to prove (4.2): from the change of variables $t = (\xi+\eta)/2$, $t \sin\theta = (\xi-\eta)/2$, we get

$$(4.3)\quad (\Omega\kappa_\varepsilon, \kappa_\varepsilon)_{L^2} = \frac{2}{\pi}\int_0^{\pi/2} \cos^\varepsilon\theta\ d\theta$$ which satisfies (4.2) for $\varepsilon > 0$.

We consider now

$$(4.4)\quad W \in C_0^\infty(-1/2, 1/2)\quad ,\quad W \geq 0\ ,\ \int W(x)dx = 1\ , \text{and}\quad \phi(x) = \frac{1}{2}\int_{-\infty}^{x} W(t)\,dt\ .$$

We set, with κ_ε given by (4.1) , $J = [\,\delta, 1]$, $\mathbb{1}_J$ the characteristic function of J ,

$$(4.5)\quad \Omega_W(\varepsilon,\delta) = \iiint e^{2i\pi x(\xi+\eta)}\, e^{2i\pi\phi(x)}\, \kappa_\varepsilon(\xi)\, 1_J(\xi) i\, \kappa_\varepsilon(\eta)\, 1_J(\eta) dx\, d\eta\, d\xi \ .$$

Lemma 4.2

There exists a constant C_o, such that for all positive numbers ε, δ satisfying $0 < \varepsilon \le 1/2$, $0 < \delta \le \varepsilon^{2\varepsilon^{-1}}$ and any function W as in (4.4), $\mathit{Re}\ \Omega_W(\varepsilon,\delta) \ge (1 - C_o\varepsilon)$.

Proof. Noting first that $e^{2i\pi\phi} = -\,\text{sign} + (\,\text{sign} + e^{2i\pi\phi}\,)\, 1_{(-1/2,1/2)}$, we get, using (1.12) and lemma 4.1,

$$(4.6)\quad \Omega_W(\varepsilon,\delta) = (\,\Omega\, 1_J\, \kappa_\varepsilon\, ,\, 1_J\, \kappa_\varepsilon\,)_{L^2} + \iint \kappa_\varepsilon(\xi)\, 1_J(\xi)\ i\, \kappa_\varepsilon(\eta)\, 1_J(\eta) \int_{-1/2}^{1/2} (\,\text{sign}\, x + e^{2i\pi\phi(x)}\,)\, e^{2i\pi x(\xi+\eta)} dx\ \ d\xi d\eta\ .$$

To evaluate the first term in the right-hand side of (4.6), we remark

$$(4.7)\quad (\,\Omega\, 1_J\, \kappa_\varepsilon\, ,\, 1_J\, \kappa_\varepsilon\,)_{L^2} = (\,\Omega\kappa_\varepsilon\, ,\, \kappa_\varepsilon\,)_{L^2} + R(\varepsilon,\delta) \qquad \text{with}$$

$$(4.8)\quad R(\varepsilon,\delta) = \frac{-1}{\pi\Gamma(\varepsilon)} \iint_A (\xi+\eta)^{-1}\, e^{-(\xi+\eta)/2}\, \xi^{(\varepsilon-1)/2}\, \eta^{(\varepsilon-1)/2}\, d\xi d\eta \qquad \text{, where}$$

$$(4.9)\quad A = A_1 \cup A_2 \cup A_3 \cup A_4 \qquad \text{with}$$

$$(4.10)\quad \begin{array}{ll} A_1 = \{\ 0 < \xi < \delta\ ,\ 0 < \eta\ \} & A_2 = \{\ 1 < \xi\ ,\ 0 < \eta\ \} \\ A_3 = \{\ \delta < \xi < 1,\ \eta < \delta\ \} & A_4 = \{\delta < \xi < 1\ ,\ 1 < \eta\ \} \end{array} .$$

We estimate

$$(4.11)\quad R_1(\varepsilon,\delta) = \frac{1}{\pi\Gamma(\varepsilon)} \iint_{A_1} (\xi+\eta)^{-1}\, e^{-(\xi+\eta)/2}\, \xi^{(\varepsilon-1)/2}\, \eta^{(\varepsilon-1)/2}\, d\xi d\eta\ .$$

We set $R_1(\varepsilon,\delta) = R_{11} + R_{12}$ with $(\,x = \xi+\eta\ ,\ \ y = (\xi-\eta)/2\,)$

$$(4.12)\qquad R_{11} = \frac{1}{\pi\Gamma(\varepsilon)} \int_0^{\delta} x^{-1} e^{-x/2}\, 2\int_0^{x/2} \left[\frac{x^2}{4} - y^2\right]^{(\varepsilon-1)/2} dy\, dx \quad \leq \quad \delta^{\varepsilon} / \Gamma(1+\varepsilon)\, 2^{\varepsilon},$$

$$(4.13)\qquad R_{12} = \frac{1}{\pi\Gamma(\varepsilon)} \int_{\delta}^{+\infty} x^{-1} e^{-x/2} \int_{-x/2}^{\delta - x/2} \left[\frac{x^2}{4} - y^2\right]^{(\varepsilon-1)/2} dy\, dx\ .$$

The inequality (4.12) is obtained as (4.3) setting $y = (x/2)\sin\theta$. We have moreover

$$(4.14)\qquad R_{12} \leq \frac{1}{\pi\Gamma(\varepsilon)} \int_{\delta}^{+\infty} t^{-1} e^{-t}\ \left[\mathrm{Arcsin}(y/t)\right]_{y=t-\delta}^{y=t} \ \left(t^2 - (t-\delta)^2\right)^{\varepsilon/2}\ dt$$

$$+ \frac{1}{\pi\Gamma(\varepsilon)} \int_{\delta/2}^{\delta} t^{\varepsilon - 1}\ \left[\mathrm{Arcsin}(y/t)\right]_{y=t-\delta}^{y=t}\ dt\ .$$

To prove (4.14), we first change the variables $x = 2t$, $y = -y'$ in (4.13), drop the ' later on, and estimate from above $(t^2 - y^2)^{\varepsilon/2}$: this quantity can be estimated by $(t^2 - (t-\delta)^2)^{\varepsilon/2}$ whenever $y \geq t - \delta \geq 0$ and by t^{ε} if $y \in (t-\delta, t)$ and $t \in (\delta/2, \delta)$. Eventually, one gets from (4.14)

$$(4.15)\qquad R_{12} \leq \frac{1}{\pi\Gamma(\varepsilon)} \left\{\delta^{\varepsilon/2}\ \Gamma(\varepsilon/2)\ 2^{\varepsilon/2} + \varepsilon^{-1}\delta^{\varepsilon}\right\} \frac{\pi}{2} \ \leq\ C_1 \delta^{\varepsilon/2}\ ,$$

where C_1 is an absolute constant . Consequently, we have from (4.15), (4.12)

$$(4.16)\qquad R_1(\varepsilon, \delta) \leq C_2\ \delta^{\varepsilon/2}$$, where C_2 is an absolute constant.

We set , with A_2 defined in (4.10),

$$(4.17)\qquad R_2(\varepsilon, \delta) = \frac{1}{\pi\Gamma(\varepsilon)} \iint_{A_2} (\xi+\eta)^{-1}\ e^{-(\xi+\eta)/2}\ \xi^{(\varepsilon-1)/2}\ \eta^{(\varepsilon-1)/2}\ d\xi d\eta\ .$$

We have on A_2 , $(\xi+\eta)^{-1} \leq \xi^{-1}$, $e^{-(\xi+\eta)/2} \leq e^{-\eta/2}$, so we get

$$(4.18)\qquad R_2(\varepsilon, \delta) \ \leq\ C_3\ \frac{\varepsilon}{1-\varepsilon}$$, where C_3 is an absolute constant.

Consequently, the inequalities (4.16), (4.18) and their analogues for the integrals in (4.8) over A_3 (smaller than over A_1) and A_4 (smaller than over A_2) give from (4.8) the existence of an absolute constant C_4, such that, for any $\varepsilon \in (0, 1/2]$, $\delta \in (0, \varepsilon^{2/\varepsilon}]$,

(4.19) $\qquad | R(\varepsilon, \delta) | \leq C_4 \varepsilon \quad .$

We need now to check the second term in the right - hand side of (4.6), namely

(4.20) $$S(\varepsilon, \delta) = \int\int \kappa_\varepsilon(\xi)\, 1_J(\xi) i\, \kappa_\varepsilon(\eta)\, 1_J(\eta) \int_{-1/2}^{1/2} (\text{ sign } x + e^{2i\pi\phi(x)})\, e^{2i\pi x(\xi+\eta)} dx d\xi d\eta.$$

We obtain, from (4.1),

(4.21) $| S(\varepsilon, \delta) | \leq \Gamma(\varepsilon)^{-1}\, \Gamma((\varepsilon + 1)/2)^2\, 2^{\varepsilon} \leq C_5 \varepsilon$, where C_5 is an absolute constant.

Finally, we obtain the result of the lemma 4.2, collecting the inequalities (4.21), (4.19),(4.2), and the equalities (4.6),(4.7),(4.20).

We consider now , with the notations of lemmas 4.1 and 4.2, for positive μ and λ ,

(4.22) $$\omega_1 (x_1) = \int e^{2i\pi x_1 \xi}\, \kappa_\varepsilon(\xi/\mu)\, 1_J(\xi/\mu) \mu^{-1/2} d\xi \quad ,$$

(4.23) $$\omega_2 (x_1) = -i\, e^{-2i\pi\phi(\lambda x_1)} \int e^{2i\pi x_1 \eta}\, \kappa_\varepsilon(-\eta/\mu)\, 1_J(-\eta/\mu) \mu^{-1/2} d\eta \quad , \text{ so that}$$

(4.24) $$(\omega_1,\omega_2)_{L^2} =$$

$$\int\int e^{2i\pi x_1 \xi} \kappa_\varepsilon(\xi/\mu)\, 1_J(\xi/\mu)\mu^{-1/2}\, i\, e^{2i\pi\phi(\lambda x_1)} e^{2i\pi x_1 \eta}\, \kappa_\varepsilon(\eta/\mu)\, 1_J(\eta/\mu)\mu^{-1/2} d\xi d\eta\ dx_1$$

$$= \int\int e^{2i\pi x_1(\xi+\eta)}\, e^{2i\pi\phi(\lambda\mu x_1)} \kappa_\varepsilon(\xi)\, 1_J(\xi) \qquad i\kappa_\varepsilon(\eta)\, 1_J(\eta)\, d\xi d\eta\ dx_1$$

$= \Omega_{W_{\lambda\mu}}(\varepsilon,\delta)$, as given by (4.5), with $W_{\lambda\mu}(x) = \lambda\mu\, W(\lambda\mu x)$ and W given by (4.4).

We obtain the following

Lemma 4.3

There exists a constant C_o, *such that for all positive numbers* ε, δ *satisfying* $0 < \varepsilon \leq 1/2$, $0 < \delta \leq \varepsilon^{2\varepsilon^{-1}}$, *all functions* W *and* ϕ *as in (4.4), all positive numbers* λ, μ *so that* $\lambda\mu \geq 1$, *all functions* ω_1 *and* ω_2 *given by (4.22) and (4.23),*

$$(4.26) \qquad (\omega_{10}, \omega_{20})_{L^2} \geq 1 - C_o\varepsilon \quad ,$$

where $\omega_{j0} = \omega_j / \|\omega_j\|_{L^2}$, $j = 1, 2$.

Proof. The equality (4.24) , $\lambda\mu \geq 1$, lemma 4.2 and $\|\omega_j\|_{L^2} \leq 1$ give the result.

Let

$$(4.27) \qquad \begin{aligned} &\lambda_{o,\nu} = 2^{\nu} \ , \ \lambda_{1,\nu} = \nu \ , \ \lambda_{2,\nu} = \mathrm{Log}\nu \ , \\ &\lambda_{3,\nu} = (\mathrm{Log}(\mathrm{Log}\nu))^{1/8} , \varepsilon_{\nu} = (\mathrm{Log}(\mathrm{Log}\nu))^{-1/2} , \delta = (\mathrm{Log}\nu)^{-1} \end{aligned}$$

be a choice of parameters satisfying (3.5), (3.7)', and the conditions in lemmas 4.2 and 4.3 .We consider the operator

$$(4.28) \qquad \mathbb{Q}_{\nu}(t) = \beta(t)\, H(-t) D_1 + \alpha(t/\theta_{\nu})\, (\mathrm{Log}\nu)^{-1}\, H(t - \theta_{\nu})[\, D_1 + \tfrac{1}{2}\nu\, W(\nu\, x_1)]$$

where α and β satisfy (3.6), $D_1 = \frac{1}{2i\pi}\frac{\partial}{\partial x_1}$, $\theta_{\nu} = (\mathrm{Log}(\mathrm{Log}\nu))^{-1/8}$, the function W satisfies (4.4). We have

$$(4.29) \qquad \mathbb{Q}_{\nu}(t) = \alpha_{\nu}(t)\, (\, H(-t)\, Q_1 + H(t - \theta_{\nu})\, Q_2^{(\nu)}\,) \quad , \qquad \text{with}$$

$$(4.30) \qquad \alpha_{\nu}(t) = \beta(t)\, H(-t) + \alpha(t/\theta_{\nu})\, (\mathrm{Log}\nu)^{-1}\, H(t - \theta_{\nu}) \ ,$$

$$(4.31) \qquad Q_1 = D_1 \qquad , \qquad Q_2^{(\nu)} = D_1 + \tfrac{1}{2}\nu\, W(\nu\, x_1) \ .$$

We define

$$(4.32) \qquad \omega_1^{(\nu)}(x_1) = \int e^{2i\pi x_1\xi}\, \kappa_{\varepsilon}(\xi/\mu)\, 1_J(\xi/\mu)\mu^{-1/2} d\xi \quad \|\, \kappa_{\varepsilon}\, 1_J \|_{L^2}^{-1} \quad , \text{with}$$

$$(4.33) \qquad \mu = \nu\, (\mathrm{Log}\nu)^{1/2} \ , \ \varepsilon = (\mathrm{Log}(\mathrm{Log}\nu))^{-1/2} \ , \quad J = [\delta\, , 1] \ , \ \delta = (\mathrm{Log}\nu)^{-1} \ ,$$

$$(4.34)\qquad \omega_2^{(\nu)}(x_1) = -i\, e^{-2i\pi\phi(\nu x_1)} \int e^{2i\pi x_1\eta}\, \kappa_\varepsilon(-\eta/\mu)\, \mathbb{1}_J(-\eta/\mu)\mu^{-1/2}d\eta\ \| \kappa_\varepsilon \mathbb{1}_J \|_{L^2}^{-1} \quad ,$$

with ϕ defined in (4.4).We can state now

Lemma 4.4

There exist a constant C_o and an integer ν_o such that, if ν is larger than ν_o, and μ, ε, J, δ, $\omega_1^{(\nu)}$, $\omega_2^{(\nu)}$ are given by (4.32-34), we have

$$(4.35)\qquad (\omega_1^{(\nu)}, \omega_2^{(\nu)})_{L^2} \geq 1 - \varepsilon_\nu C_o \quad \text{, and } \omega_1^{(\nu)}, \omega_2^{(\nu)} \text{ are unit vectors in } L^2 .$$

Moreover, these functions have the following spectral properties :

$$(4.36)\qquad \begin{array}{l} \text{spec } \omega_1^{(\nu)} \subset [\, \nu\,(\text{Log}\nu)^{-1/2},\ \nu\,(\text{Log}\nu)^{1/2}\,], \text{with respect to } Q_1 \\ \text{spec } \omega_2^{(\nu)} \subset [\, -\nu\,(\text{Log}\nu)^{1/2},\ -\nu\,(\text{Log}\nu)^{-1/2}\,], \text{with respect to } Q_2^{(\nu)} \end{array} .$$

Finally,

$$(4.37)\qquad \varepsilon_\nu \theta_\nu^{-2} = \varepsilon_\nu^{1/2} \quad .$$

Proof. The inequality (4.35) is a reformulation of lemma 4.3, the spectral properties are obvious on formulas (4.32) and (4.34) and (4.37)follows from (4.33).

We are going to follow the lines of the construction (1.8 – 11).We define

$$(4.38)\qquad \Omega_1^{(\nu)}(x_1, x_2) = \omega_1^{(\nu)}(x_1)\, \rho_\nu(x_2) \quad , \quad \Omega_2^{(\nu)}(x_1, x_2) = \omega_2^{(\nu)}(x_1)\rho_\nu(x_2) \quad ,$$

where ρ_ν is a function with norm 1 in $L^2(\mathbb{R})$ such that

$$(4.39)\qquad \text{support } \hat{\rho}_\nu \subset [\, 2^{\nu-1/4},\ 2^{\nu+1/4}\,] \quad .$$

We set-up , with α_ν, Q_1, $Q_2^{(\nu)}$, $\Omega_1^{(\nu)}$, $\Omega_2^{(\nu)}$, θ_ν as above $\chi_\nu = \chi(2^{-\nu+\frac{1}{2}} D_1)$, χ given in (3.3)

$$(4.40)\qquad u_\nu(t) = \begin{cases} \exp\left[\int_0^t \alpha_\nu(s)ds\, Q_1 \right] \Omega_1^{(\nu)} & \text{, on } \ t < 0 , \\ \Omega_1^{(\nu)} + \frac{t}{\theta_\nu}(\chi_\nu \Omega_2^{(\nu)} - \Omega_1^{(\nu)}) & \text{, on } \ 0 < t < \theta_\nu , \\ \chi_\nu \exp\left[\int_{\theta_\nu}^t \alpha_\nu(s)ds\, Q_2^{(\nu)} \right] \Omega_2^{(\nu)} & \text{, on } \ \theta_\nu < t . \end{cases}$$

Lemma 4.5

The function $u_\nu(t)$ defined in (4.40) belongs to $L^2(\mathbb{R}^2_{x_1,x_2})$ and

(4.41) $\quad spectrum(u_\nu(t)) \subset \Delta_\nu$,

where Δ_ν is defined in (3.9) and the spectrum is the support of the Fourier transform in $\mathbb{R}^2_{\xi_1,\xi_2}$. Moreover, with $|\ |$ standing for the $L^2(\mathbb{R}^2_{x_1,x_2})$ norm, we have, with α, β given in (3.6),

$$(4.42) \qquad |u_\nu(t)|^2 \leq \exp\{-\int_t^0 2\beta(s)ds\ \nu\,(Log\nu)^{-1/2}\}\ ,\quad if\ \ t<0\ ,$$

$$(4.43) \qquad 1-(\varepsilon_\nu C_o)^{1/2} \leq |u_\nu(t)|^2 \leq 1 \qquad ,\qquad if \quad 0<t<\theta_\nu\ ,$$

$$(4.44)\ \left| \exp\left[\int_{\theta_\nu}^{t} \alpha_\nu(s)ds\ Q_2^{(\nu)}\right] \Omega_2^{(\nu)} \right|^2 \leq \exp\{-\int_1^{t/\theta_\nu} 2\alpha(s)ds\ \theta_\nu\ \nu\,(Log\nu)^{-3/2}\},\quad if\ t>\theta_\nu .$$

The commutator $[\chi_\nu, Q_2^{(\nu)}]$ is L^2 bounded and, for $\nu>\nu_0$,

$$(4.45)\ \|[\chi_\nu, Q_2^{(\nu)}]\|_{\mathscr{L}(L^2(\mathbb{R}))} \leq 2^{-\nu/2} \quad .$$

Proof. From (4.38), (4.39) and (4.36) we get that the Fourier transform of $\Omega_1^{(\nu)}$ is supported in the rectangle , $\nu\,(Log\nu)^{-1/2} \leq \xi_1 \leq \nu\,(Log\nu)^{1/2}$, $2^{\nu-1/4} \leq \xi_2 \leq 2^{\nu+1/4}$. Consequently, on $t<0$, from (4.31), (4.30), we obtain that the square of the modulus of the Fourier transform of $u_\nu(t)$ is smaller than $\exp\{-\int_t^0 2\beta(s)ds\ \nu\,(Log\nu)^{-1/2}\}\ |\mathscr{F}(\Omega_1^{(\nu)})(\xi_1,\xi_2)|^2$, where $\mathscr{F}$ stands for the Fourier transform.On the support of the unit vector $\mathscr{F}(\Omega_1^{(\nu)})$, for $\nu>\nu_o$, we have $2^{2\nu-1} \leq 2^{2\nu-1/2} \leq \xi_1^2+\xi_2^2 \leq \nu^2\,(Log\nu) + 2^{2\nu+1/2} \leq 2^{2\nu+1}$. This proves (4.42) and (4.41) for $t<0$ .The inequality (4.44) is a consequence of the spectral location of $\omega_2^{(\nu)}$ with respect to $Q_2^{(\nu)}$ in (4.36) and of (4.30) . Moreover, on $\theta_\nu<t$, the Fourier transform of $u_\nu(t)$ is supported in the rectangle , $|\xi_1| \leq 2^{\nu-1/2}$, $2^{\nu-1/4} \leq \xi_2 \leq 2^{\nu+1/4}$ which is included in $2^{2\nu-1} \leq 2^{2\nu-1/2} \leq \xi_1^2+\xi_2^2 \leq 2^{2\nu-1} + 2^{2\nu+1/2} \leq 2^{2\nu+1}$, proving (4.41) on $\theta_\nu<t$. Similarly , collecting the information on the support of the Fourier transform of $\Omega_1^{(\nu)}$ and $\chi_\nu\,\Omega_2^{(\nu)}$,we get (4.41) on $0<t<\theta_\nu$. We remark now that the commutator $[\chi_\nu, Q_2^{(\nu)}] = [\chi(2^{-\nu+\frac{1}{2}} D_1), \frac{1}{2}\nu\,W(\nu x_1)]$; since the symbols $\chi(2^{-\nu+\frac{1}{2}}\xi_1)$ and $\frac{1}{2}\nu\,W(\nu x_1)$ both belong to $S(\nu, \frac{|dx_1|^2}{\nu^{-2}} + \frac{|d\xi_1|^2}{2^{2\nu}})$ with semi-norms independent of ν (here we use Hörmander's notation (18.4.6) in [5]), we get , using theorem 18.5.4 in [5], that the commutator$[\chi_\nu, Q_2^{(\nu)}]$ has a symbol in

$$S\left(\nu^2 2^{-\nu} \nu \,,\ \frac{|\, dx_1 \,|^2}{\nu^{-2}} + \frac{|\, d\xi_1 \,|^2}{2^{2\nu}} \right) ,$$

which gives (4.45). The proof of the lemma is complete.

We shall use a modification of the operator $Q(t, x, D_x)$ defined in (3.1) : we set

$$\mathbb{Q}(t) = \sum_{\nu > \nu_0} \Phi_\nu \ \mathbb{Q}_\nu(t)\, \Phi_\nu \quad , \qquad \text{with } \mathbb{Q}_\nu(t) \text{ given in (4.28), } \Phi_\nu \text{ by}$$

$$\Phi_\nu = Op(\psi(2^{-2\nu}(\xi_1^2 + \xi_2^2))\ \chi(\xi_1/\xi_2)) ,$$

where ψ and χ are given by (3.2), (3.3), and Op(a) stands for the operator with symbol a. The operator $\mathbb{Q}(t)$ has a symbol in the $S^1_{1,0}$ class since , if Q is given by (3.1), $\mathbb{Q}(t)$ - Op($Q(t,x,\xi)$) has a symbol in the $S^0_{1,0}$ class. We set

(4.46) $\qquad L = \frac{d}{dt} - \mathbb{Q}(t) \qquad ,$

and we calculate Lu_ν , with u_ν given by (4.40) : we get easily

(4.47) $\qquad \| Lu_\nu \|^2 \le 2\, \varepsilon_\nu \theta_\nu^{-1} \quad , \qquad \| u_\nu \|^2 \ge \frac{1}{4}\, \theta_\nu \quad ,$ $\qquad$ so that

(4.48) $\qquad \| Lu_\nu \|^2 \ \| u_\nu \|^{-2} \le 8\, \varepsilon_\nu \theta_\nu^{-2} = 8\, \varepsilon_\nu^{1/2} .$

Next we consider $v_\nu(t) = \chi\,(t / A_\nu \theta_\nu) u_\nu(t)$, with

(4.49) $\qquad A_\nu = (\mathrm{LogLog}\nu)^\sigma \quad$, with $0 < \sigma < 1/8$.

We estimate $\| Lv_\nu \|^2 \ \| v_\nu \|^{-2}$ and get easily the same estimate as in (4.48), up to an absolute constant. Note that the commutator estimate (4.45) is useful to handle the fact that the Fourier transform of $\Omega_2^{(\nu)}$ is not supported in Δ_ν : we thus get to deal with

$$\chi_\nu\, \alpha_\nu(t)\, Q_2^{(\nu)} \exp\left[\int_{\theta_\nu}^{t} \alpha_\nu(s) ds\, Q_2^{(\nu)} \right] \Omega_2^{(\nu)} - \alpha_\nu(t)\, \Phi_\nu\, Q_2^{(\nu)}\ \Phi_\nu\, \chi_\nu \exp\left[\int_{\theta_\nu}^{t} \alpha_\nu(s) ds\, Q_2^{(\nu)} \right] \Omega_2^{(\nu)}$$

which is handled using (4.45). Lemma 4.5 and the above discussion yield a proof of theorem 2.1. The main point here is that, first the spectral information (4.41) shows that the operator L given in (4.46) "sees" only v_ν through its term with index ν, and second, that the spectral information (4.36) yielding (4.42) and (4.44) allows a cut-off function to enter the game. It should be noted here that the drift concerns the large values of the spectrum, which is important to violate local solvability ; the drift for the kernels would only allow to violate global solvability.

5. Remarks on the homogeneous case

We shall consider the following family of homogeneous pseudo-differential operators, depending on the parameter ν, using the notations introduced in the previous sections :

$$(5.1)\quad \tilde{q}_\nu = \left\{ \beta(t)\,\xi_1 + \frac{1}{\mathrm{Log}\nu}\,\alpha(t/\theta_\nu)\,[\,\xi_1 + \xi_2\, 2^{-\nu}\,\frac{1}{2}\,\nu W(\nu x_1)\,]\right\}\gamma_0(\xi_1,\xi_2)\,\omega(\xi_1,\xi_2)$$

where γ_0 is homogeneous with degree 0, non-negative, supported in the cone $\xi_2 > |\xi_1|$, smooth on $\mathbb{R}^2\setminus 0$, ω a smooth non-negative function in $\mathbb{R}^2$, vanishing in a neighborhood of zero, identically equal to one for $|\xi| \geq 1$. The symbols $\tilde{q}_\nu$ are homogeneous (for $|\xi| \geq 1$) , satisfy condition (1.1), and their semi-norms are bounded independently of ν. It is possible to choose ρ_ν in (4.38) such that support $\hat{\rho}_\nu \subset [\,(1-e_\nu)2^\nu,\ 2^\nu(1+e_\nu)]$, with e_ν going to zero ; the uncertainty on ξ_2 will be then $\Delta\xi_2 \sim 2^\nu e_\nu$ and we can choose $e_\nu = 2^{-\nu/2}$ and take a $\Delta x_2 \sim 2^{-\nu/2}$. Eventually, we see that no L^2 estimate can be proved for the family of operators with symbols $(\tau + i\,\tilde{q}_\nu)\,\gamma_0(\tau, \xi_2)$, in spite of the fact they uniformly satisfy condition (1.1). This suggests that the construction of the previous sections can be lifted to the homogeneous case without loss on the dimension.

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