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EQUATIONS AUX DERIVEES PARTIELLES

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Abstract

A system of three one-dimensional particles with one of pair potentials $V^\alpha(x^\alpha)$ decaying at infinity as $|x^\alpha|^{-\rho}$, $0 < \rho < 1/2$, is considered. It is shown that such a system can possess channels of scattering not included in the usual list of channels called the asymptotic completeness.

1. INTRODUCTION

An aim of the scattering theory is to find the asymptotics as $t \to \infty$ of the solution $u(t) = \exp(-iHt)f$ of the time-dependent Schrödinger equation with a Hamiltonian $H = -2^{-1}\Delta + V(x)$ in the space $\mathcal{H} = L_2(\mathbb{R}^d)$. If f is an eigenvector of H, i.e. $Hf = \lambda f$, then obviously $u(t) = \exp(-i\lambda t)f$. Suppose now that f is orthogonal to the subspace $\mathcal{H}^{(p)}$ spanned by all eigenvectors. In the two-body short-range case when $V(x) = O(|x|^{-\rho})$, $\rho > 1$, the asymptotics of $u(t)$ is the same as that for the free system, that is

$$\exp(-iHt)f \sim \exp(-iH_0t)f_0 \tag{1.1}$$

for some $f_0 \in \mathcal{H}$ and $H_0 = -2^{-1}\Delta$ ("$\sim$" means that the difference of left- and right-hand sides tends to zero in the space $\mathcal{H}$). One can rewrite (1.1) in an equivalent way as

$$(\exp(-iHt)f)(x) \sim \exp(i\Phi_0(x,t))t^{-d/2}g(x/t), \tag{1.2}$$

where $\Phi_0(x,t) = x^2(2t)^{-1}$, $g = \exp(i\pi d/4)\hat{f}_0$ and $\hat{f}_0$ is the Fourier transform of f_0.

The relation (1.2) holds true (see e.g.[1]) also for long-range potentials satisfying the condition

$$|D^\kappa V(x)| \le C(1+|x|)^{-\rho-|\kappa|}, \quad \rho > 0, \quad |\kappa| = 0,1,2. \tag{1.3}$$

In this case the phase function $\Phi_0(x,t)$ depends on a potential $V(x)$. The asymptotics (1.2) shows that, if f belongs to the absolutely continuous subspace $\mathcal{H}^{(ac)} = \mathcal{H} \ominus \mathcal{H}^{(p)}$ of the operator H, then the solution $(\exp(-iHt)f)(x)$ "lives" in the region where $|x| \sim t$.

The situation is more complicated in the many-body case when

$$H = -2^{-1}\Delta + \sum_\alpha V^\alpha(x^\alpha), \quad 1 \le \alpha \le \alpha_0 < \infty,$$

and x^α are orthogonal projections of $x \in \mathbb{R}^d$ on some given subspaces $X^\alpha \subset \mathbb{R}^d$. Set $X_\alpha = \mathbb{R}^d \ominus X^\alpha$. The three-body case is distinguished by the assumption $X_\alpha \cap X_\beta = \{0\}$ for $\alpha \neq \beta$. Suppose first that all pair potentials V^α are short-range, i.e. that they satisfy the condition $V^\alpha(x^\alpha) = O(|x^\alpha|^{-\rho})$ for some $\rho > 1$. The relation (1.1) (or (1.2) with $\Phi_0(x,t) = x^2(2t)^{-1}$) is fulfilled for f from some subspace $\mathcal{H}_0 \subset \mathcal{H}^{(ac)}$. In general, $\mathcal{H}_0 \neq \mathcal{H}^{(ac)}$ and the orthogonal complement $\mathcal{H}^{(ac)} \ominus \mathcal{H}_0$ is determined by the point spectra of pair Hamiltonians $H^\alpha = -2^{-1}\Delta_{x^\alpha} + V^\alpha$ acting in the spaces $\mathcal{H}^\alpha = L_2(X^\alpha)$. Let us introduce their eigenvalues $\lambda^{\alpha,k}$ and eigenvectors $\psi^{\alpha,k}$. For every couple $\{\alpha, k\}$ there exists a subspace $\mathcal{H}_{\alpha,k} \subset \mathcal{H}^{(ac)}$ such that for any $f \in \mathcal{H}_{\alpha,k}$

$$(\exp(-iHt)f)(x) \sim \psi^{\alpha,k}(x^\alpha)\exp(i\Phi_{\alpha,k}(x_\alpha,t))t^{-d_\alpha/2}g(x_\alpha/t), \quad d_\alpha = \dim X_\alpha, \quad (1.4)$$

for the function $\Phi_{\alpha,k}(x_\alpha,t) = x_\alpha^2(2t)^{-1} - \lambda^{\alpha,k}t$ and some $g \in L_2(X_\alpha)$. Note that the subspaces $\mathcal{H}_0$ and $\mathcal{H}_{\alpha,k}$ are constructed as images of the corresponding wave operators $\mathcal{W}_0 : \mathcal{H} \to \mathcal{H}$ and $\mathcal{W}_{\alpha,k} : L_2(X_\alpha) \to \mathcal{H}$. The basic result of the scattering theory, called the asymptotic completeness, is that the sum of all subspaces $\mathcal{H}_0$ and $\mathcal{H}_{\alpha,k}$ exhausts the absolutely continuous subspace $\mathcal{H}^{(ac)}$ of the operator H. Actually, $\mathcal{H}^{(ac)}$ can be decomposed into the orthogonal sum

$$\mathcal{H}^{(ac)} = \mathcal{H}_0 \oplus (\bigoplus_{\alpha,k} \mathcal{H}_{\alpha,k}). \quad (1.5)$$

This result was first obtained by L. D. Faddeev [2] under some additional assumptions. The optimal formulation is due to V. Enss [3].

For three-body systems with long-range pair potentials V^α the answer is almost the same if $V^\alpha(x^\alpha)$ satisfy the condition (1.3) with some $\rho > \sqrt{3} - 1$ as functions of x^α. In this case again the asymptotic completeness (1.5) holds. For $f \in \mathcal{H}_0$ the asymptotics (1.2) and for $f \in \mathcal{H}_{\alpha,k}$ the asymptotics (1.4) with suitable functions $\Phi_0(x,t)$ and $\Phi_{\alpha,k}(x_\alpha,t)$ are fulfilled. This result was obtained by V. Enss [4, 5] and carried over by J. Dereziński [6] (by a method different from [4, 5]) to an arbitrary number of particles. Note also that in the case $\rho > 1/2$ the asymptotic completeness (1.5) was established in the papers [7, 8] under some additional assumptions. We emphasize that for $f \in \mathcal{H}_{\alpha,k}$ the solution $\exp(-iHt)f$ is localized in the variable x^α and, on the contrary, $|x_\alpha| \sim t$. Thus such solutions play an intermediary role between those for $f \in \mathcal{H}^{(p)}$ and $f \in \mathcal{H}_0$.

Our goal is to show that for some three-body systems with sufficiently slowly decreasing pair potentials there exist channels of scattering different from (1.2) and (1.4). Actually, we consider one-dimensional particles with one of pair potentials $V^\alpha(x^\alpha) = -v_\alpha|x^\alpha|^{-\rho}$, where $v_\alpha > 0$, $0 < \rho < 1/2$, for large $x^\alpha > 0$. Other pair potentials can be short-range. We construct a subspace $\mathfrak{H} \subset \mathcal{H}^{(ac)}$ of initial data f such that for $f \in \mathfrak{H}$ the solution $u(t) = \exp(-iHt)f$ of the Schrödinger equation "lives" in the region where $|x_\alpha| \sim t$ and $|x^\alpha| \sim t^\sigma$ for $\sigma = (\rho+1)/3 < 1/2$. Such solutions describe a physical process where particles of the pair α are relatively close to one another and the third particle is far away. The pair α is bound by a potential depending on the position of the third particle and this bound state is evanescent as $t \to \infty$. Thus solutions $u(t)$ for $f \in \mathfrak{H}$ are intermediary between those for $f \in \mathcal{H}_{\alpha,k}$

and $f \in \mathcal{H}_0$. The subspace $\mathfrak{H}$ is orthogonal to $\mathcal{H}_0$ and $\mathcal{H}_{\alpha,k}$ and the restriction of the operator H on $\mathfrak{H}$ has the absolutely continuous spectrum which coincides with $\mathbb{R}_+$. This contradicts, of course, the asymptotic completeness (1.5).

Note that existence of new channels for three-particle systems automatically implies the same phenomena for systems of more than three particles. It suffices to take a system where all particles but three are free and the system of these three distinguished particles possesses a described channel.

It turns out that the asymptotic completeness is violated also for two-body systems with long-range potentials if one tries to relax the condition (1.3) on derivatives of $V(x)$.

Our concrete examples of existence of new channels of scattering rely on the following general construction which is similar to that of [9]. We suppose that

$$\mathbb{R}^d = X_1 \oplus X^1, \quad \dim X_1 = d_1,\ \dim X^1 = d^1,\ d_1 + d^1 = d, \tag{1.6}$$

but we do not make any special assumptions about a potential $V(x) = V(x_1, x^1)$. Let us introduce an operator

$$H^1(x_1) = -2^{-1}\Delta_{x^1} + V(x_1, x^1) \tag{1.7}$$

acting in the space $L_2(X^1)$. Suppose that the operator $H^1(x_1)$ has an eigenvalue $\lambda(x_1)$ and denote by $\psi(x_1)$ the corresponding normalized eigenfunction. In the particular case when $V(x)$ does not depend on x_1 the operator (1.7) describes a three-body system with only one non-trivial pair interaction. In this case both channels (1.2) and (1.4) (where $\alpha = 1$) exist. We are looking for a generalization of (1.4) in the case when $\lambda(x_1) \to 0$ as $|x_1| \to \infty$. In interesting situations the function $\lambda(x_1)$ decreases slower than $|x_1|^{-1}$. Let us consider it as an "effective" potential energy and associate to the long-range potential $\lambda(x_1)$ the phase function $\Phi(x_1, t)$. We will prove under some assumptions that for every $g \in L_2(X_1)$ there exists an element $f \in \mathcal{H}^{(ac)}$ such that

$$(\exp(-iHt)f)(x) \sim \psi(x_1, x^1)\exp(i\Phi(x_1, t))t^{-d_1/2}g(x_1/t), \tag{1.8}$$

A set of these elements f is a subspace $\mathfrak{H} \subset \mathcal{H}^{(ac)}$. It is constructed as the range of the corresponding wave operator $W : L_2(X_1) \to \mathcal{H}$. The subspace $\mathfrak{H}$ is orthogonal to $\mathcal{H}_0$ if the wave operator for the pair H_0, H exists.

Existence of solutions of the time-dependent Schrödinger equation with the asymptotics (1.8) requires rather special assumptions which are naturally formulated in terms of eigenfunctions $\psi(x_1, x^1)$ of the operator $H^1(x_1)$. It turns out that typically the asymptotics of $\psi(x_1, x^1)$ as $\lambda(x_1) \to 0$ has a certain self-similarity:

$$\psi(x_1, x^1) \sim |x_1|^{-\sigma d^1/2}\Psi(|x_1|^{-\sigma}x^1) \tag{1.9}$$

for some $\Psi \in L_2(X^1)$ and $\sigma > 0$. We prove the asymptotics (1.8) if (1.9) is fulfilled for $\sigma < 1/2$. On the other hand, simple examples show that (1.9) for $\sigma \geq 1/2$ does not ensure existence of solutions with the asymptotics (1.8). It is important that

$\psi(x_1, x^1)$ can be chosen as an *approximate* solution of the equation $H^1(x_1)\psi(x_1) - \lambda(x_1)\psi(x_1) = 0$.

A construction of channels (1.8) is discussed in Section 2. Concrete examples of two- and three-body systems for which solutions of the type (1.8) exist are given in Sections 3 and 4, respectively.

2. A GENERAL CONSTRUCTION

Let us, first, recall some simple results of the two-particle long-range scattering theory. We discuss here only existence of wave operators and, following [10], define them in the coordinate representation. Let us introduce a unitary operator $U_0(t)$,

$$(U_0(t)f)(x) = \exp(i\Phi_0(x,t))t^{-d/2}f(x/t),$$

in the space $\mathcal{H} = L_2(\mathbb{R}^d)$ corresponding to the right-hand side of (1.2). Set $H_0 = -2^{-1}\Delta$, $H = -2^{-1}\Delta + V$. Let $\mathbf{X}$ be the operator of multiplication by $x^2/2$ in the space $L_2(\mathbb{R}^d)$. The following assertion can be checked by an explicit calculation.

Proposition 2.1 *Let the condition (1.3) be satisfied. Then there exists a function $\Phi_0(x,t)$ such that for any $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$*

$$\|(i\partial/\partial t + 2^{-1}\Delta - V)U_0(t)f\| = O(t^{-1-\varepsilon}), \quad t \to \infty.$$

Corollary 2.2 *The wave operator*

$$\mathcal{W}_0 = s - \lim_{t\to\infty} \exp(iHt)U_0(t) \tag{2.1}$$

exists, it is isometric and the intertwining property $H\mathcal{W}_0 = \mathcal{W}_0\mathbf{X}$ holds.

Note that $\Phi_0(x,t) = x^2(2t)^{-1} + \Omega(x,t)$, where $\Omega(x,t)$ is an approximate solution of the eikonal equation

$$\partial\Omega/\partial t + t^{-1}\langle x, \nabla\Omega\rangle + 2^{-1}|\nabla\Omega|^2 + V = 0.$$

Clearly, a function Φ_0 depends on a long-range potential V. Denote by Γ the corresponding mapping $\Gamma : V \mapsto \Phi_0$.

We emphasize that $\mathcal{W}_0$ equals the product of the usual (perhaps, modified) wave operator relating H_0 and H and of the Fourier transform.

Our aim now is to establish the asymptotics (1.8). Let us formulate precise assumptions on a function $\psi(x_1, x^1)$. We suppose that $\psi(x_1, \cdot)$ is an *approximate* eigenfunction of the operator (1.7), i.e.

$$-2^{-1}\Delta_{x^1}\psi(x_1,x^1) + V(x_1,x^1)\psi(x_1,x^1) = \lambda(x_1)\psi(x_1,x^1) + Y(x_1,x^1), \tag{2.2}$$

where

$$\|\psi(x_1,\cdot)\|_{L_2(X^1)} = 1, \quad \|Y(x_1,\cdot)\|_{L_2(X^1)} = O(|x_1|^{-1-\varepsilon}), \quad |x_1| \to \infty, \tag{2.3}$$

for some $\varepsilon > 0$. Approximate "eigenvalues" $\lambda(x_1)$ should satisfy a usual condition of long-range scattering:

$$|D^\kappa \lambda(x_1)| \leq C(1+|x_1|)^{-\rho-|\kappa|}, \quad \rho > 0, \quad |\kappa| = 0,1,2. \tag{2.4}$$

With respect to $\psi(x_1, x^1)$ itself we suppose that

$$\|\nabla_{x_1}\psi(x_1,\cdot)\|_{L_2(X^1)} = O(|x_1|^{-1}), \quad \|\Delta_{x_1}\psi(x_1,\cdot)\|_{L_2(X^1)} = O(|x_1|^{-1-\varepsilon}) \tag{2.5}$$

as $|x_1| \to \infty$. Furthermore, we require that auxiliary functions

$$\tilde{\psi}(x_1, x^1) = (x^1)^2 \psi(x_1, x^1), \tag{2.6}$$

$$\xi_\sigma(x_1, x^1) = \sigma\langle \nabla_{x^1}\psi, x^1\rangle + 2^{-1}\sigma d^1 \psi + \langle \nabla_{x_1}\psi, x_1\rangle \tag{2.7}$$

satisfy the bounds

$$\|\tilde{\psi}(x_1,\cdot)\|_{L_2(X^1)} = O(|x_1|^{1-\varepsilon}), \quad \|\xi_\sigma(x_1,\cdot)\|_{L_2(X^1)} = O(|x_1|^{-\varepsilon}). \tag{2.8}$$

We emphasize that all assumptions on functions $\psi(x_1, x^1)$, $Y(x_1, x^1)$ and $\lambda(x_1)$ are used for sufficiently large $|x_1|$ only. These conditions are well adopted to treat functions $\psi(x_1, x^1)$ with the asymptotics (1.9). Actually, suppose for a moment that for some $\sigma \in \mathbb{R}$ there is the precise equality in (1.9) and that the function $(x^1)^2\Psi(x^1)$ belongs to $L_2(X^1)$. Then $\xi_\sigma = 0$, both conditions (2.5) are fulfilled and $\tilde{\psi}$ satisfies (2.8) with $1 - \varepsilon = 2\sigma$. Thus the conditions $\varepsilon > 0$ and $\sigma < 1/2$ are equivalent.

Let us reformulate the asymptotics (1.8) in terms of the corresponding wave operator. Define the function $\Phi(x_1, t)$ by the construction of Proposition 2.1 which we apply to a function $\lambda(x_1)$ (in place of $V(x)$) in the variable x_1 (in place of x), i.e. we set $\Phi = \Gamma\lambda$. Let $U_1(t)$ be the operator of a modified free evolution in the variable x_1,

$$(U_1(t)f)(x_1) = \exp(i\Phi(x_1,t))t^{-d_1/2} f(x_1/t). \tag{2.9}$$

According to Proposition 2.1 for any $f \in C_0^\infty(\mathbb{R}^{d_1} \setminus \{0\})$ there is a bound

$$\|(i\partial/\partial t + 2^{-1}\Delta_{x_1} - \lambda)U_1(t)f\|_{L_2(X_1)} = O(t^{-1-\varepsilon}), \quad t \to \infty.$$

Let an isometric operator $J : L_2(X_1) \to \mathcal{H}$ be defined by the equality

$$(Jf)(x_1, x^1) = \psi(x_1, x^1)f(x_1), \quad J = J(\psi). \tag{2.10}$$

We prove existence of the wave operator

$$W = s - \lim_{t\to\infty} \exp(iHt)JU_1(t). \tag{2.11}$$

To that end it would suffice to verify that a function $u_1(t) = JU_1(t)f$ is a "good" approximation to a solution of the time-dependent Schrödinger equation, i.e.

$$\|\frac{d}{dt}(\exp(iHt)u_1(t))\| = \|i\partial u_1/\partial t - Hu_1\| = O(t^{-1-\varepsilon}), \quad \varepsilon > 0, \quad t \to \infty. \tag{2.12}$$

Initial data f can be chosen from some set dense in the space $L_2(X_1)$. Unfortunately, for the function $\psi(x_1, x^1)$ with the asymptotics (1.9) the estimate (2.12) can be satisfied for $\varepsilon = 0$ only.

It turns out that a better approximation to a solution of the Schrödinger equation is given by the formula

$$u(x,t) = \exp(i\sigma\gamma(x^1,t))u_1(x,t), \quad \gamma(x^1,t) = (x^1)^2(2t)^{-1}, \quad u_1(t) = JU_1(t)f. \tag{2.13}$$

The following assertion can be verified by an explicit calculation.

Proposition 2.3 *Let a function $u(t)$ be defined by the equalities (2.13) where $f \in C_0^\infty(\mathbf{R}^{d_1} \setminus \{0\})$. Suppose that a function $\psi(x_1, x^1)$ satisfies for some $\sigma \in \mathbf{R}$ the conditions (2.5) - (2.8). Let functions $Y(x_1, x^1)$ and $\lambda(x_1)$, defined by (2.2), obey (2.3), (2.4). Then*

$$\|i\partial u/\partial t - Hu\| = O(t^{-1-\varepsilon}), \quad \varepsilon > 0, \quad t \to \infty.$$

This assertion implies the main result of this section. Below $\mathbf{X}_1$ is multiplication by x_1^2 in the space $L_2(X_1)$.

Theorem 2.4 *Let functions $\psi(x_1, x^1)$, $\lambda(x_1)$ and $Y(x_1, x^1)$ satisfy the assumptions of Proposition 2.3. Define the operator $U_1(t)$ by the equality (2.9), where $\Phi = \Gamma\lambda$. Then the limit (2.11) exists and the wave operator $W : L_2(X_1) \to \mathcal{H}$ is isometric. The intertwining property $HW = W\mathbf{X}_1$ holds. In particular, the restriction of H on the range $R(W)$ of W has the absolutely continuous spectrum which coincides with $\mathbb{R}_+$.*

Solutions with the asymptotics (1.1) and (1.8) "live" in different regions of $\mathbb{R}^d$. Let us give a precise formulation of this statement.

Theorem 2.5 *Suppose that both wave operators (2.11) and (2.1) exist, for some functions Φ and Φ_0, respectively. Assume that the function (2.6) satisfies the bound*

$$\|\tilde{\psi}(x_1, \cdot)\|_{L_2(X^1)} = O(|x_1|^{2-\varepsilon}), \quad \varepsilon > 0.$$

Then the ranges of the operators W and $\mathcal{W}_0$ are orthogonal.

3. TWO-BODY POTENTIALS

We construct here a concrete class of potentials $V(x)$ for which all the assumptions of the previous section are satisfied. These potentials decay at infinity and thus correspond to the two-body case.

Let (1.6) be some decomposition of $\mathbf{R}^d$ and let

$$V(x_1, x^1) = -v(< x_1 >^q + < x^1 >^q)^{-\rho/q}, \quad \rho \in (0,1), \quad q \in (0,2), \quad v > 0, \tag{3.1}$$

where we use the notation $< y > = (1 + |y|^2)^{1/2}$. The function (3.1) is infinitely differentiable and $V(x) = O(|x|^{-\rho})$ as $|x| \to \infty$. Outside of any conical neighbourhood of the planes X_1 and X^1 the bound (1.3) is fulfilled for arbitrary κ. This suffices for existence of the wave operator (2.1). If $q = 2$, then $V(x_1, x^1)$ is a radial function so that this wave operator is, of course, complete. If $1 \leq q < 2$, then the bound (1.3) is satisfied (uniformly in directions of x) for $|\kappa| = 1$ but is violated for $|\kappa| = 2$. If $0 < q < 1$, then (1.3) is violated already for $|\kappa| = 1$.

We shall prove that under the assumption

$$1 - \rho < q < 2(1 - \rho) \tag{3.2}$$

the wave operator (2.11) exists. This ensures that $\mathcal{W}_0$ fails to be complete.

Let us construct an approximate eigenfunction $\psi(x_1, x^1)$ of the operator (1.7). We consider it on spherically symmetric for $d^1 > 1$ or odd for $d^1 = 1$ functions so that $H(x_1)$ reduces to the operator

$$h(a) = -d^2/dr^2 + \delta r^{-2} - v(a^q + < r >^q)^{-\rho/q}, \quad r = |x^1|, \quad a =< x_1 >, \tag{3.3}$$

$\delta = 4^{-1}(d^1 - 1)(d^1 - 3)$, in the space $L_2(\mathbb{R}_+)$ with the boundary condition $\psi(0) = 0$. We construct $\psi(x_1, x^1)$ as an exact eigenfunction of a Schrödinger operator $h_0(a)$ with a simpler potential. To that end we replace in (3.3) $< r >$ by r and the potential $(a^q + r^q)^{-\rho/q}$ by the first two terms of its Taylor expansion at the point $r = 0$. This gives us the operator

$$h_0(a) = -d^2/dr^2 + \delta r^{-2} - va^{-\rho} + v_0 a^{-\rho-q} r^q, \quad \psi(0) = 0, \quad v_0 = v\rho q^{-1},$$

with the discrete spectrum.

Let us choose a normalized eigenfunction $\Psi(R) = \Psi_n(R)$ corresponding to one of eigenvalues $\Lambda = \Lambda_n$ of the equation

$$-\Psi'' + \delta R^{-2}\Psi + v_0 R^q \Psi = \Lambda\Psi, \quad \Lambda = \mu a^{2\sigma}, \quad \Psi(0) = 0. \tag{3.4}$$

The function $\Psi(R)$ decays exponentially as $R \to \infty$ and equals the Airy function if $q = 1$, $\delta = 0$. Then

$$\psi(a, r) = a^{-\sigma/2}\Psi(a^{-\sigma} r), \quad \sigma = (\rho + q)(2 + q)^{-1}, \tag{3.5}$$

is a normalized eigenfunction of the operator $h_0(a)$ corresponding to an eigenvalue

$$\lambda(a) = -va^{-\rho} + \Lambda a^{-2\sigma} \quad 2\sigma > \rho. \tag{3.6}$$

The "potential" (3.6) satisfies (as a function of x_1) the assumptions (2.4) and "eigenfunctions" (3.5) satisfy the conditions (2.5) - (2.8) if $\sigma < 1/2$. The inequality $\sigma < 1/2$ is equivalent to $q < 2(1 - \rho)$.

The function $Y(a, r)$ (see (2.2)) corresponding to (3.5), (3.6) equals

$$Y(a, r) = -vy(a, r)a^{-\sigma/2}\Psi(a^{-\sigma} r),$$

where

$$y(a, r) = (a^q + < r >^q)^{-\rho/q} - a^{-\rho} + \rho q^{-1} a^{-\rho-q} r^q.$$

It follows that under the left assumption (3.2) the condition (2.3) is fulfilled, i.e. $\|Y(a, \cdot)\| = O(a^{-1-\varepsilon})$.

Applying now Theorems 2.4 and 2.5 we arrive at

Theorem 3.1 *Let a potential V be defined by (3.1) where ρ and q satisfy (3.2). Let Λ be any eigenvalue and Ψ a corresponding eigenfunction of the equation (3.4). Define the function $\psi(x_1, x^1)$ and the "potential" $\lambda(x_1)$ by the equalities (3.5), (3.6), where $a =< x_1 >$, $r = |x^1|$, and set $\Phi = \Gamma\lambda$. Then the wave operator (2.11) exists. It is isometric and $HW = W\mathbf{X}_1$. The range of W is orthogonal to that of the wave operator (2.1).*

Remark that the above construction does not work in the case $q = 2$. Actually, everything goes through but for the corresponding approximate eigenfunction $\psi(a, r)$ we obtain the equality (3.5) with $\sigma = (2 + \rho)/4 > 1/2$. Thus the first condition (2.8) is not fulfilled. Of course, non-existence of the limit (2.11) should have been expected, since the wave operator (2.1) is complete now.

We emphasize that for potentials (3.1) there exists a countable set of wave operators W_n corresponding to each eigenvalue Λ_n of the equation (3.4). The ranges of these operators are, obviously, orthogonal to each other. Furthermore, one can interchange the roles of variables x_1 and x^1. This gives us a new set of wave operators $\tilde{W}_n$ whose ranges are orthogonal to those of W_n (and, of course, to that of (2.1)).

Note finally that the first assumption (3.2), i.e. $1 - \rho < q$, is of technical nature and can probably be omitted.

4. A THREE-BODY CASE

Let us now consider the three-particle Hamiltonian $H = 2^{-1}\Delta + V(x)$ where $V(x) = V^1(x^1) + V^2(x^2)$ and x^α are orthogonal projections of $x \in \mathbb{R}^d$ on given subspaces X^α, $\alpha = 1, 2$, $X^1 \neq X^2$. We suppose that $d = 2$, $\dim X^\alpha = 1$ and that X^2 is not orthogonal to X^1. Then

$$x^2 = l(x^1 + mx_1), \quad l \neq 0, \quad m \neq 0. \tag{4.1}$$

The Hamiltonian H describes three one-dimensional particles of finite masses; one of three pair interactions is zero. With respect to other pair potentials we make

Assumption 4.1 *A bounded function $V^1 \geq 0$ and $V^1(x^1) = 0$ for $x^1 \geq 0$. There exists $p_1 < 0$ such that $V^1(x^1)$ is twicely differentiable for $x^1 \leq p_1 < 0$ and*

$$V^1(x^1) \geq v_1|x^1|^{-r},\ v_1 > 0,\ |D^\kappa V^1(x^1)| \leq C|x^1|^{-r-\kappa},\ \kappa = 0, 1, 2,\ 0 < r < 2.$$

Assumption 4.2 *A bounded function V^2 equals $V^2(x^2) = -v_2|x^2|^{-\rho}$, $\rho \in (0, 1/2)$, $v_2 > 0$, for sufficiently large $x^2, x^2 \geq p_2$.*

Let us show that under these assumptions all conditions of Section 2 are satisfied. Suppose for definiteness $l = 1$, $m > 0$. For large positive values of the parameter x_1, the potential

$$V(x_1, x^1) = V^1(x^1) + V^2(x^1 + mx_1), \quad x_1 > 0,$$

contains, as a function of x^1, two wells separated from one another by a positive barrier V^1. The first of them concentrated on the negative half-axis around a point

$-mx_1$ gives rise to channels of scattering on bound states of the Hamiltonian $H^2 = -\Delta_{x^2} + V^2(x^2)$. The second well $-v_2(x^1+a)^{-\rho}$, $a := mx_1 > p_2$, is located at the positive half-axis and vanishes as $x_1 \to \infty$. Due to a barrier it is possible to construct approximate eigenfunctions $\psi(x_1, x^1)$ of the operator $H(x_1)$ concentrated on the half-axis $x^1 \geq 0$. This construction is basically similar to considerations of the previous section for $q = 1$. An additional difficulty is to show that an interaction between wells can be neglected in the limit $x_1 \to \infty$.

Let us consider the equation (2.2). We need to find a comparatively simple potential such that, first, the corresponding Schrödinger equation may be resolved, more or less, explicitly and, second, the arising error $Y(x_1, x^1)$ satisfies the estimate (2.3). We replace $V(x_1, x^1)$ for $x^1 \geq 0$ by the function $-v_2 a^{-\rho} + v_2 \rho a^{-\rho-1} x^1$ and set

$$\psi_+(x^1, \Lambda, a) = \mathrm{Ai}(va^{-\sigma}x^1 - \Lambda), \quad x^1 \geq 0, \tag{4.2}$$

where $\mathrm{Ai}(y)$ is the (exponentially decreasing as $y \to \infty$) Airy function and parameters λ and Λ are related by the equality

$$\lambda = -v_2 a^{-\rho} + v^2 \Lambda a^{-2\sigma}, \quad v = (v_2\rho)^{1/3}, \quad \sigma = (\rho+1)/3. \tag{4.3}$$

For $x^1 \leq 0$ we keep track of a barrier due to V^1 and dispense with a well at $x^1 \sim -a$. More precisely, we replace $V(x_1, x^1) - \lambda$ by

$$Q(x^1, \Lambda, a) = V^1(x^1) - v^2 \Lambda a^{-2\sigma} \zeta(x^1/A),$$

where $\zeta \in C^\infty$, $\zeta(x^1) = 1$ for $x^1 \in (-1, 0)$, $\zeta(x^1) = 0$ for $x^1 \leq -2$ and $A = a^\delta$ with sufficiently small δ. Clearly, $Q(x^1, \Lambda, a) \to V^1(x^1)$ as $a \to \infty$. We fix any $p \leq p_1 < 0$ and distinguish a solution of the equation

$$-\psi_-'' + Q(x^1, \Lambda, a)\psi_- = 0, \quad x^1 \leq 0, \tag{4.4}$$

by its asymptotics as $x^1 \to -\infty$:

$$\psi_-(x^1, \Lambda, a) \sim Q^{-1/4}(x^1, \Lambda, a) \exp\Big(-\int_{x^1}^{p} Q^{1/2}(y, \Lambda, a)dy\Big). \tag{4.5}$$

A construction of $\psi_-(x^1, \Lambda, a)$ reduces to a consideration of a Volterra integral equation (cf. [11]). Note that

$$\psi_-(x^1, \Lambda, a) \to \psi_1(x^1), \quad \psi_-'(x^1, \Lambda, a) \to \psi_1'(x^1) \tag{4.6}$$

as $a \to \infty$ for fixed x^1 (in particular, for $x^1 = 0$) uniformly for bounded Λ. Here ψ_1 is a solution of the equation $-\psi_1'' + V^1(x^1)\psi_1 = 0$ with the asymptotics (4.5) where $Q = V^1$.

We consider (cf. [12]) the matching condition for ψ_+ and ψ_- at $x^1 = 0$ as an equation for $\Lambda = \Lambda(a)$. By virtue of (4.2) it has the form

$$va^{-\sigma}\psi_-(0, \Lambda(a), a)\psi_-'(0, \Lambda(a), a)^{-1} = \mathrm{Ai}(-\Lambda(a))\mathrm{Ai}'(-\Lambda(a))^{-1}. \tag{4.7}$$

Choose any zero $-\Lambda_n$ of the Airy function: $\mathrm{Ai}(-\Lambda_n) = 0$. According to (4.6) the implicit function theorem shows that the equation (4.7) has a solution such that $\Lambda(a) \to \Lambda_n$ as $a \to \infty$. Furthemore, $\Lambda(a) = \Lambda_n + O(a^{-\sigma})$. Now we define "eigenfunctions" $\psi(x_1, x^1)$ by the equalities

$$\psi(x_1, x^1) = c_+(a)\mathrm{Ai}(va^{-\sigma}x^1 - \Lambda(a)), \quad x^1 \geq 0,$$
$$\psi(x_1, x^1) = c_-(a)\psi_-(x^1, \Lambda(a), a), \quad x^1 \leq 0,$$

where $x_1 = m^{-1}a$. The constants $c_+(a)$ and $c_-(a)$ are determined by the continuity of the function $\psi(x_1, x^1)$ at $x^1 = 0$ and by the normalization $\|\psi(x_1, \cdot)\| = 1$. The first of them yields that $c_-(a) \sim a^{-\sigma}c_+(a)$ and the second one shows that $c_+(a) \sim a^{-\sigma/2}$.

The potential $\lambda(x_1) = \lambda_n(x_1)$ defined by (4.3) where $\Lambda = \Lambda_n(a)$ obeys, of course, the estimate (2.4). A verification of conditions (2.5) - (2.8) for $\psi(x_1, x^1)$ and of the condition (2.3) for $Y(x^1, x_1)$ is different for $x^1 \geq 0$ and $x^1 \leq 0$. On the half-axis $x^1 \geq 0$ it suffices to use, as in the two-body case, the self-similarity of the function (4.2). The condition $\sigma < 1/2$ is equivalent to $\rho < 1/2$. The contribution of the half-axis $x^1 \leq 0$ is negligible because $c_-(a) \sim a^{-3\sigma/2}$ and the function $\psi_-(x^1, \Lambda(a), a), \quad x^1 \leq 0$, decays exponentially as $x^1 \to -\infty$ uniformly in a and Λ.

Let $\Phi(x_1, t) = \Phi_n(x_1, t)$ be a phase function constructed for $x_1 \geq 0$ with respect to the long-range potential (4.3) and let an identification $J_n : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}^2)$ be defined by the equality (2.10) where $\psi = \psi_n$. According to Theorem 2.4 the wave operator (2.11) exists. Actually, one can replace J_n by a simpler identification $J_n^{(0)}$ which is equivalent to J_n. Set $\psi_n^{(0)}(x^1, x_1) = c_n x_1^{-\sigma/2}\mathrm{Ai}(vm^{-\sigma}x_1^{-\sigma}x^1 - \Lambda_n)$ for $x^1 \geq 0$ and $\psi_n^{(0)}(x^1, x_1) = 0$ for $x^1 \leq 0$; the constant c_n is chosen in such a way that $\|\psi_n^{(0)}(x_1, \cdot)\| = 1$. We stress that $\psi_n^{(0)}(0, x_1) = 0$ but its derivative is not continuous at $x^1 = 0$. Let $J_n^{(0)} = J(\psi_n^{(0)})$ be defined by the equality (2.10). Since $(J_n - J_n^{(0)})U_0(t)f \to 0$ as $t \to \infty$, the wave operator W_n defined by the equality (2.11) with respect to $J_n^{(0)}$ exists and equals that for J_n.

Assumptions 4.1 and 4.2 almost guarantee the existence of the wave operators $\mathcal{W}_0$ and $\mathcal{W}_{2,k}$ corresponding (see (1.4)) to scattering on bound states $\psi^{2,k}(x^2)$ of the Hamiltonian H^2. According to Theorem 2.5 the image $R(W_n)$ of the wave operator W_n is orthogonal to that of $\mathcal{W}_0$ whenever $\mathcal{W}_0$ exists. The scalar product of right-hand sides in (1.4) for $\alpha = 2$ and in (1.8) tends to zero as $t \to \infty$. Therefore $R(W_n)$ and $R(\mathcal{W}_{2,k})$ are also orthogonal. Now we are able to formulate our main result.

Theorem 4.3 *Let Assumptions 4.1 and 4.2 hold. Define a long-range potential* $\lambda_n(x_1)$, $x_1 > 0$, *by the equality (4.3), where* $\Lambda = \Lambda_n(a)$, $a = mx_1$, *and set* $\Phi_n = \Gamma\lambda_n$. *Then the limit* W_n *defined by the equality (2.11) where* $J = J(\psi_n^{(0)})$ *exists. The wave operator* $W_n : L_2(\mathbb{R}_+) :\to L_2(\mathbb{R}^2)$ *is isometric. The ranges of operators* W_n *are orthogonal (for different n) to each other and to the subspaces* $R(\mathcal{W}_0)$ *and* $R(\mathcal{W}_{2,k})$.

Let us finally mention that if the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal then due to the separation of variables the asymptotic completeness holds. In this case our construction does not work because $x^2 = x_1$ so that the equality (4.1) fails.

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