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EQUATIONS AUX DERIVEES PARTIELLES

HYPERBOLIC OPERATORS WITH NON LIPSCHITZ COEFFICIENTS

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1. Introduction

We are concerned with the well–posedness of the Cauchy problem for second–order strictly hyperbolic operators whose coefficients are not Lipschitz continuous but only "Log–Lipschitz": for a function a to be Log–Lipschitz (LL for short) means

$$|a(x) - a(y)| \leq C|x-y| \, |\log|x-y|| \, , \tag{1.1}$$

whenever $|x-y|$ is small (say for $|x-y| \leq 1/2$). We consider wave operators with LL coefficients, and we prove two different type of results. First,we obtain a well–posedness result when the coefficients are LL, second we deal with low regularity only in the time variable. We thus go beyond the classical well–posedness result for hyperbolic operators with Lipschitz continuous coefficients. To justify the choice of this LL regularity, we show by the construction of a counterexample (modifying slightly theorem 10 in [4]) that LL comes up as the natural threshold beyond which no well–posedness could be expected : the right–hand side of (1.1) cannot be replaced by

$$|x-y| \, |\log|x-y|| \, \varphi(|x-y|)$$

with $\varphi(r) \xrightarrow[r \to 0+]{} +\infty$ without ruining the existence of a distribution solution. Let's describe now the first kind of results. We are concerned with wave equations in divergence form,

$$P \equiv \frac{\partial^2}{\partial t^2} - \sum_{1 \leq i,j \leq n} \frac{\partial}{\partial x_i} a_{ij}(t,x) \frac{\partial}{\partial x_j} \, , \tag{1.2}$$

with a_{ij} real-valued such that

$$\begin{gathered} a_{ij} = a_{ji} \text{ and there exists } \delta > 0 \text{ such that for any } \xi \in \mathbb{R}^n \\ \sum_{1 \leq i,j \leq n} a_{ij} \xi_i \xi_j \geq \delta |\xi|^2 \end{gathered} \tag{1.3}$$

and $a_{ij} \in LL$ (isotropically) i.e. ($y_i \in \mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x$)

$$\begin{cases} a_{ij} \in L^\infty, \\ |a_{ij}(y_1) - a_{ij}(y_2)| \leq C|y_1 - y_2| \, |\log|y_1 - y_2|| \\ \text{whenever } |y_1 - y_2| \leq 1/2. \end{cases} \tag{1.4}$$

We shall prove that there exists a time $T^* > 0$ such that the following energy estimates holds whenever $0 \leq t < T^*$:

$$\begin{gathered} ||u(t,\cdot)||_{H^{1-\theta-\alpha t}(\mathbb{R}^n_x)} + ||\frac{\partial u}{\partial t}(t,\cdot)||_{H^{-\theta-\alpha t}(\mathbb{R}^n_x)} \leq \\ C_0 \left\{ ||u(0,\cdot)||_{H^{1-\theta}(\mathbb{R}^n_x)} + ||\frac{\partial u}{\partial t}(0,\cdot)||_{H^{-\theta}(\mathbb{R}^n_x)} + + \int_0^t ||Pu(s)||_{H^{-\theta-\alpha s}(\mathbb{R}^n_x)} ds \right\} , \end{gathered} \tag{1.5}$$

where $\theta > 0$ is given and the constant $\alpha > 0$ will depend on θ, and on the LL norm of the coefficients of P. The energy estimate (1.5) will allow us to prove well-posedness results for the Cauchy problem for P.

We should note here, as it appears in the inequality (1.5) that the well-posedness result we get is obtained with a loss of derivatives, in contrast with the Lipschitz case. In the latter situation, when the initial data u_0, u_1 are respectively in the Sobolev spaces H^s and H^{s-1}, the solution of the standard initial value problem is such that

$$u(t,\cdot) \in H^s \quad \text{and} \quad \frac{\partial u}{\partial t}(t,\cdot) \in H^{s-1}. \tag{1.6}$$

In our case (the coefficients a_{ij} are LL), we obtain essentially

$$u(t,\cdot) \in H^{s-\alpha t} \quad \text{and} \quad \frac{\partial u}{\partial t}(t,\cdot) \in H^{s-1-\alpha t}, \qquad \alpha > 0. \tag{1.7}$$

This result can be compared to Bahouri and Chemin's result of [1] in which they conduct an investigation of vector fields with LL coefficients in connection with problems in fluid mechanics (see also [3]). The second author of the present paper wishes to thank J.–Y. Chemin for useful discussions on these topics.

The second part of our work is concerned with well-posedness in the C^∞ class for a wave operator in $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$,

$$L \equiv \frac{\partial^2}{\partial t^2} - \sum_{1\le i,j\le n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}\ , \tag{1.8}$$

where L is strictly hyperbolic i.e. (a_{ij}) satisfy (1.3). The main point here is the weak regularity assumption on the coefficients $a_{ij}(t,x)$: we assume that a_{ij} are LL in the time variable t, smooth in the space variables x. So $a_{ij}(t,\cdot)$ are smooth (C^∞) functions such that

$$\sup_x |a_{ij}(t,x) - a_{ij}(s,x)| \le C|t-s|\ |\log|t-s||, \tag{1.9}$$

when $|t-s| \le 1/2$. We consider for instance the initial value problem

$$\begin{cases} Lu = 0 \\ u(0,x) = u_0(x) \\ \frac{\partial u}{\partial t}(0,x) = u_1(x) \end{cases} \tag{1.10}$$

with smooth u_0, u_1. We find a unique solution $u(t,x)$ depending continuously on the data u_0, u_1 in such a way that

$$\partial_t^2 u(t,x) \in LL(\mathbb{R}_t, C^\infty(\mathbb{R}_x^n)). \tag{1.11}$$

More precise and general statements will be given in section 2. It should be pointed out that, whenever the coefficients depend only on the time variable, Colombini, De Giorgi and Spagnolo in [4] already proved such a result. Their paper was our starting point, and we somehow microlocalized their energy estimates, using a Littlewood–Paley decomposition. On the other hand, in [4], the authors obtained well-posedness in the Gevrey class for Hölder continuous coefficients (still depending only on the time variable). Nishitani [8] and Jannelli [7] extended these results to operators whose coefficients are Hölder continuous in time, Gevrey in the space variables. Moreover, as mentioned above, in [4] a counterexample is given showing that one-dimensional wave equations with Hölder-continuous coefficients are not well-posed: there exists $a(t) \geq 1$, $a \in \bigcap_{s<1} C^s$, (C^s is the Hölder class of index s) such that the initial value problem

$$\begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = 0 \quad t \in \mathbb{R},\ x \in \mathbb{R} \\ u(0,x) = u_0(x)\ ,\ \partial_t u(0,x) = u_1(x) \end{cases} \tag{1.12}$$

has no distribution solution for some choice of smooth u_0, u_1.

We improve the result of [4] on this matter and we show that the function a of (1.12) can be chosen satisfying

$$|a(t) - a(s)| \leq C|t-s|\,|\log|t-s||\ \ \varphi(t-s), \tag{1.13}$$

with $\varphi(r) \xrightarrow[r \to 0+]{} +\infty$.

For instance if we denote by Λ the space of functions satisfying (1.13) with $\varphi(r) = \log|\log r|$ we have,

$$\{\text{Lipschitz functions}\} \subset LL \subset \Lambda \subset C^{1-0} = \bigcap_{\epsilon>0} C^{1-\epsilon}, \tag{1.14}$$

and we see that the class Λ is too large to expect existence of a solution.

Moreover in [6] an example of a non–solvable strictly hyperbolic equation with C^{1-0} coefficients is given. Some more counterexamples are given in [5] about non uniqueness for the Cauchy problems for strictly hyperbolic equations with C^{1-0} coefficients.

2. Statement of the results

We need first to introduce a

Definition 2.1 *Let a be a function in $L^\infty(\mathbb{R}^d, \mathbb{R})$. We set*

$$
\|a\|_{LL} = \sup_{x\in\mathbb{R}^d} |a(x)| + \sup_{\substack{0<|x_1-x_2|\leq 1/2 \\ x_j\in\mathbb{R}^d}} \frac{|a(x_1)-a(x_2)|}{|x_1-x_2|\,|\log|x_1-x_2||}. \tag{2.1}
$$

We define the set of Log–Lipschitz (LL) functions as the space of functions a such that $\|a\|_{LL} < +\infty$.

Let's note that there is no difficulty to extend this definition to the case where the source and the target of a are metric spaces. We note also that the inclusions of (1.14) are strict inclusions. The LL space is a Banach space with the $\|\cdot\|_{LL}$ norm.

To state our first energy estimate we need to deal with an operator on divergence–form: Let's consider in $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$

$$
P = \frac{\partial^2}{\partial t^2} - \sum_{1\leq i,j\leq n} \frac{\partial}{\partial x_i} a_{ij}(t,x) \frac{\partial}{\partial x_j} + M(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x}), \tag{2.2}
$$

where the matrix (a_{ij}) is real, symmetric and satisfies

$$
\delta_1|\xi|^2 \geq \sum_{1\leq i,j\leq n} a_{ij}(t,x)\xi_i\xi_j \geq \delta_0|\xi|^2 \ , \quad 1\geq \delta_0 > 0 \tag{2.3}
$$

for any $\xi\in\mathbb{R}^n$, $(t,x)\in\mathbb{R}^{1+n}$. The coefficients (a_{ij}) are assumed to be $LL(\mathbb{R}^{n+1})$ (def. 2.1). The operator M is a first order operator :

$$
M = b_0(t,x)\frac{\partial}{\partial t} + b(t,x)\cdot\frac{\partial}{\partial x} + c(t,x), \tag{2.4}
$$

with

$$
b_0, b \in C^\omega(\mathbb{R}^{n+1}) \ , \ c\in C^\kappa(\mathbb{R}^{n+1}), \tag{2.5}
$$

for some positive numbers ω,κ.

Theorem 2.1 *Let $0<\theta\leq 1/4$ be given. Let P be given by (2.2–5). There exists $\beta>0$, $T^*>0$, $C>0$ such that for $0\leq t\leq T^*$, $u\in C^\infty(\mathbb{R}^{1+n})$*

$$
\begin{aligned}
&\int_0^t \|Pu(s)\|_{H^{-\theta-\beta s}(\mathbb{R}^n)}ds + \|\dot u(0)\|_{H^{-\theta}(\mathbb{R}^n)} + \|u(0)\|_{H^{1-\theta}(\mathbb{R}^n)} \geq \\
&\geq C^{-1}\left\{\sup_{0\leq s\leq t}\|\dot u(s)\|_{H^{-\theta-\beta s}(\mathbb{R}^n)} + \sup_{0\leq s\leq t}\|u(s)\|_{H^{1-\theta-\beta s}(\mathbb{R}^n)}\right\}.
\end{aligned} \tag{2.6}
$$

Here $\beta = \frac{1}{\delta_0}\alpha(P)$, where $\alpha(P)$ is a positive constant depending only on the LL norm of the (a_{ij}), the C^ω and C^κ norm of the coefficients of M (cf.(2.4), (2.5)), δ_0 given in (2.3), $T^* = \frac{1}{\beta}$.

It would possible to state theorems of well posedness with finite speed of propagation for the support for global problems. On the other hand it is also easy to derive local existence results for the Cauchy problem from the previous energy estimates. However, to get a local well–posedness result would require a local uniqueness theorem which is not a straightforward consequence the previous energy estimates.

We consider now a strictly hyperbolic operator in $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$

$$L = \frac{\partial^2}{\partial t^2} - \sum_{1\le i,j\le n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + M(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x}) \tag{2.7}$$

where, in addition to the requirements (2.2-5), the coefficients a_{ij} satisfy:

$$x \mapsto a_{ij}(t,x) \ \text{ is smooth } (C^\infty) \text{ for each fixed } t, \tag{2.8}$$

$$a_{ij}(t,x) \in L^\infty(\mathbb{R}^{1+n}) \ \text{ and } \ \frac{\partial a_{ij}}{\partial x_k}, \ \frac{\partial^2 a_{ij}}{\partial x_h \partial x_\ell} \in L^\infty(\mathbb{R}^{1+n}) \tag{2.9}$$

$$\sup_{\substack{0<|t-s|\le 1/2 \\ x\in\mathbb{R}^n}} \frac{|a_{ij}(t,x) - a_{ij}(s,x)|}{|t-s|\,|\log|t-s||} < +\infty. \tag{2.10}$$

The operator M is a first order operator as in (2.4) such that

$$x \mapsto b_0(t,x) \ , \ b(t,x), \ c(t,x) \tag{2.11}$$

are smooth (C^∞) for each fixed t,

$$b_0, b, c \in L^\infty(\mathbb{R}^{1+n}) \ \text{ and } \ \frac{\partial b_0}{\partial x_k}, \ \frac{\partial b}{\partial x_k}, \ \frac{\partial c}{\partial x_k} \in L^\infty(\mathbb{R}^{1+n}). \tag{2.12}$$

We state now our second energy estimate result.

Theorem 2.2 *Let L be given by (2.7-12). There exists $\beta > 0$, and $T^* > 0$, $C > 0$ such that for $0 \le t \le T^*$, $u \in C^\infty(\mathbb{R}^{1+n})$,*

$$\begin{aligned} &\int_0^t \|Lu(s)\|_{H^{-\beta s}(\mathbb{R}^n)} ds + \|\dot u(0)\|_{H^0(\mathbb{R}^n)} + \|u(0)\|_{H^1(\mathbb{R}^n)} \ge \\ &\ge C^{-1}\left\{ \sup_{0\le s\le t} \|\dot u(s)\|_{H^{-\beta s}(\mathbb{R}^n)} + \sup_{0\le s\le t} \|u(s)\|_{H^{1-\beta s}(\mathbb{R}^n)} \right\}. \end{aligned} \tag{2.13}$$

Here $\beta = \frac{1}{\delta_0}\alpha(L)$, *$\alpha(L)$ is a positive constant depending only on the norms of the functions in (2.9), (2.10), (2.12), δ_0 is given in (2.3)* , $T^* = \frac{1}{\beta}$.

We now state a theorem analogous to theorem 2.2 for the derivatives of u. Although its proof is rather standard, we should pay attention to the phenomenon of loss of derivatives ($\beta > 0$ in (2.13)).

We define for L given by (2.7–12)

$$\alpha(L) = \left(\|A\|_{LL} + \|\nabla_x A\|_{Lip(x)} + \|M\|_{L^\infty(\mathbf{R}, C^1(\mathbf{R}^n))}\right) \tag{2.14}$$

where $A = (a_{ij})_{1\leq i,j\leq n}$, $\|A\|_{LL}$ its LL norm ((2.1)),

$$\|\nabla_x A\|_{Lip(x)} = \|\nabla_x A\|_{L^\infty(\mathbf{R}^{n+1})} + \sup_{\substack{t\in\mathbf{R}\\ x,y\in\mathbf{R}^n}} \frac{|(\nabla_x A)(t,x) - (\nabla_x A)(t,y)|}{|x-y|}, \tag{2.15}$$

$$\|M\|_{L^\infty(\mathbf{R}^n, C^1(\mathbf{R}^n))} = \|\, |b_0| + |\nabla_x b_0| + |b| + |\nabla_x b| + |c| + |\nabla_x c|\, \|_{L^\infty(\mathbf{R}^{n+1})}. \tag{2.16}$$

Theorem 2.3 *There exists $C(n)$ depending only on the dimension such that if L is given by (2.7–12), if*

$$\beta = \frac{1}{\delta_0} C(n)\alpha(L), \tag{2.17}$$

$$T^* = \frac{1}{\beta}, \tag{2.18}$$

$$C_0 = C(n)(1+\delta_1) \quad (\delta_1 \quad \textit{given in (2.3)}), \tag{2.19}$$

we get that for any $m \geq 0$, there exists C_m such that for $u \in C^\infty(\mathbb{R}^{n+1})$,

$$\begin{aligned} &\sup_{0\leq t\leq T^*} \|u(t)\|_{H^{m+1-\beta t}(\mathbf{R}^n)} + \sup_{0\leq t\leq T^*} \|\dot u(t)\|_{H^{m-\beta t}(\mathbf{R}^n)} \\ &\leq C_0\left(1 + e^{C_m T^*}\right)\left\{\|u(0)\|_{H^{m+1}(\mathbf{R}^n)} + \|\dot u(0)\|_{H^m(\mathbf{R}^n)} + + \int_0^{T^*} \|Lu(s)\|_{H^{m-\beta s}(\mathbf{R}^n)} ds\right\} \end{aligned} \tag{2.20}$$

The important fact here is that T^* although finite is independent of m.

The next theorem will give a strong argument in favour of the class LL since we provide a counterexample for a one–dimensional wave equation whose speed has a modulus of continuity as close as we wish of the LL modulus $|t|\,|\log|t||$.

Theorem 2.4 *Let ψ be a positive function defined on a neighborhood of $+\infty$ in $\mathbb{R}$, such that ψ is increasing, concave, $\psi(+\infty) = +\infty$.*

Then, there exists a function $a(t)$, defined on $t \geq 0$ valued in $[1/2, 3/2]$ such that, for $|t-s|$ small enough,

$$|a(t) - a(s)| \leq C|t-s| \ \log|t-s|^{-1}\psi(\log|t-s|^{-1}), \tag{2.21}$$

and smooth functions $u_0, u_1 \in C^\infty(\mathbb{R})$ such that the initial value problem

$$\begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = 0 \\ u_{|_{t=0}} = u_0, \ \partial_t u_{|_{t=0}} = u_1 \end{cases} \tag{2.22}$$

has no solution in $C^0([0,1], \mathcal{D}'(\mathbb{R}))$.

3. Properties of LL functions

We give here a list of properties of LL functions without providing the proofs which will be given in a forthcoming paper. We start with a

Definition 3.1 *Let ω be a positive function defined on a neighborhood of $+\infty$ in $\mathbb{R}$. We'll say ω is a weight if*

$$\omega \text{ is monotone increasing, and } \omega(+\infty) = +\infty \tag{3.1}$$

$$\text{there exists } N_0 \text{ such that } \omega(t)t^{-N_0} \text{ is bounded,} \tag{3.2}$$

$$\text{for any positive number } \lambda \ , \frac{\omega(\lambda t)}{\omega(t)} \text{ is bounded.} \tag{3.3}$$

For the construction of our counterexample, we will use the following generalization of the LL class (see def. 2.1).

Definition 3.2 *Let ω be a weight (i.e. satisfying def. 3.1). Let u be a function in $L^\infty(\mathbb{R}^d, \mathbb{R})$. The function u will be said $L^\omega L$ if, for some $\delta > 0$, such that* $\mathrm{def}(\omega) \supset [\log\frac{1}{\delta}, +\infty[$,

$$\Omega_\delta(u,\omega) = \sup_{\substack{0<|x_1-x_2|\leq\delta \\ x_j\in\mathbb{R}^d}} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|\omega(\log(|x_1 - x_2|^{-1}))} < +\infty. \tag{3.4}$$

We set, for $u \in L^\omega L$,

$$||u||_{L^\omega L} = ||u||_{L^\infty(\mathbb{R}^d)} + \Omega_\delta(u,\omega). \tag{3.5}$$

Let's note that $L^{id}L = LL$ (see def. 2.1). We have the following characterization for the space $L^{\omega}L$. We use below the standard notations for the Littlewood-Paley decomposition

For ν integer ≥ 1 we set $\varphi_\nu(\xi) = \varphi(\frac{\xi}{2^\nu})$,with φ smooth, supported in a ring

φ_o smooth, supported in a ball, 1 near the origin,

$$S_\nu = \sum_{0 \leq \mu \leq \nu} \varphi_\mu .$$

Proposition 3.3 *Let ω be a weight (def.3.1). The following statements are equivalent:*
(i) $u \in L^\omega L$, (def. 3.2).
(ii) $u \in L^\infty$ and $\overline{\lim_{\nu \to +\infty}} \|\nabla S_\nu(D_x)u\|_{L^\infty}\omega(\nu)^{-1} < +\infty$.

Proposition 3.4 *Let ω be a weight (def. 3.1). There exists $N_0 \geq 1$ and C_0 such that the $L^\omega L$ norm (3.5) is equivalent to*

$$\|\varphi_0\left(\frac{D_x}{2^{N_0}}\right)u\|_{L^\infty} + \sup_{v > N_0} \|\nabla S_\nu(D_x)u\|_{L^\infty}\omega(\nu)^{-1}. \tag{3.6}$$

Moreover, if $u \in L^\omega L$ (φ_ν defined in (3.3)), $\nu > N_0$,

$$\|\varphi_\nu(D_x)u\|_{L^\infty} \leq C_0\|u\|_{L^\omega L}2^{-\nu}\omega(\nu). \tag{3.7}$$

Proposition 3.5 *Let ω be a weight and let $a \in L^\omega L(\mathbb{R}^n)$ and s real $|s| < 1$. Then the multiplication operator $u \mapsto au$ is continuous from $H^s \to H^s$:*

$$\|au\|_{H^s} \leq C(s,n)\|a\|_{L^\omega L}\|u\|_{H^s},$$

where $C(s,n)$ is a constant depending only on s and on the dimension.

Proposition 3.6 *Let ω be a weight (def. 3.1), $a \in L^\omega L$ (def. 3.2), then there exist N_0 and C_0 such that the following estimate holds for $\mu \geq N_0$*

$$\|[\varphi_\mu(D_x)\ ,\ a(x)]\|_{\mathcal{L}(L^2)} \leq C_0\|a\|_{L^\omega L}2^{-\mu}\omega(\mu), \tag{3.8}$$

so that, for $a \in LL$,

$$\|[\varphi_\mu(D_x)\ ,\ a(x)]\|_{\mathcal{L}(L^2)} \leq C_0\|a\|_{LL}2^{-\mu}\mu. \tag{3.9}$$

4. Sketch of Proof for the energy estimate

Let's study only a model case for our energy estimate. Let $u(t,x)$ $(t \in \mathbb{R}, x \in \mathbb{R})$ a solution to

$$\left\{ \begin{array}{l} \partial_t^2 u - \partial_x(a(t,x)\partial_x u) = f \\ u(0,x) = 0, \ \partial_t u\ (0,x) = 0, \end{array} \right. \tag{4.1}$$

with $a \geq 1$. We consider

$$\varphi \in C_0^\infty(\frac{1}{2} < |\xi| < 2) \tag{4.2}$$

and we set

$$u_\nu(t,x) = \left(\varphi\left(\frac{Dx}{2^\nu}\right)u\right)(t,x) = \int e^{2i\pi x\xi}\varphi(\frac{\xi}{2^\nu})\hat{u}(t,\xi)d\xi, \tag{4.3}$$

where $\hat{u}$ stands for the Fourier transform in the x variable,

$$\varphi_\nu = \varphi\left(\frac{Dx}{2^\nu}\right). \tag{4.4}$$

Denoting $\dot{v} = \dfrac{\partial v}{\partial t}$, we get from (1.17)

$$\left\{ \begin{array}{l} \ddot{u}_\nu - \partial_x\varphi_\nu a\partial_x u = f_\nu = \varphi_\nu f \\ u_\nu(0,x) = 0 \ \ , \ \ \dot{u}_\nu(0,x) = 0. \end{array} \right. \tag{4.5}$$

We get

$$\left\{ \begin{array}{l} \ddot{u}_\nu - \partial_x a\partial_x u_\nu = f_\nu + \partial_x[\varphi_\nu, a]\partial_x u = g_\nu \\ u_\nu(0,x) = \dot{u}_\nu(0,x) = 0. \end{array} \right. \tag{4.6}$$

For the simplicity of our exposition, let's assume $a(t,x)$ is LL in the t variable, C^∞ in the x variable. In this case, the commutator $[\varphi_\nu, a]$ doesn't give rise to any difficulty since φ_ν acts only in the x variables. The reader must be warned that the handling of this commutator is non trivial in the first part of our work when a is isotropically LL. Let's compute, with $D_x = \frac{1}{i}\frac{\partial}{\partial x}$

$$\Omega_\nu = 2\,\mathrm{Re}\int_0^T (\ddot{u}_\nu + D_x a D_x u_\nu, \dot{u}_\nu)_{L^2(\mathbb{R}_x)} e^{-\lambda_\nu t} dt, \tag{4.7}$$

where $\lambda_\nu > 0$ is to be chosen later. We have to deal with the low regularity of a in the t-variable: this leads us to introduce a mollified version for a. We set

$$a_\epsilon(t,x) = \int a(s,x)\rho(\frac{t-s}{\epsilon})\frac{ds}{\epsilon} = a \underset{t}{\ast} \rho_\epsilon \tag{4.8}$$

with $\epsilon > 0$, $\rho \in C_0^\infty(\mathbb{R})$, $\int \rho = 1$, $\rho \geq 0$, so that

$$a_\epsilon \geq 1, \tag{4.9}$$

since $a \geq 1$. It is a matter of routine to prove

$$\begin{cases} |a(t,x) - a_\epsilon(t,x)| \leq C_0 \epsilon |\log_2 \epsilon| \\ \text{and} \\ |\dot{a}_\epsilon(t,x)| \leq C_0 |\log_2 \epsilon|. \end{cases} \tag{4.10}$$

We get from (1.21)

$$\begin{aligned} \Omega_\nu = & \int_0^T e^{-\lambda_\nu t} \frac{d}{dt} \left\{ |\dot{u}_\nu(t)|^2_{L^2(\mathbb{R}_x)} + (a_\epsilon D_x u_\nu, D_x u_\nu) \right\} dt \\ & - \int_0^T e^{-\lambda_\nu t} (\dot{a}_\epsilon D_x u_\nu, D_x u_\nu) dt \\ & + 2 \operatorname{Re} \int_0^T e^{-\lambda_\nu t} ((a - a_\epsilon) D_x u_\nu, D_x \dot{u}_\nu) dt. \end{aligned} \tag{4.11}$$

Using the initial conditions, we get, integrating by parts, using $a_\epsilon \geq 1$,

$$\begin{aligned} \Omega_\nu \geq & \, e^{-\lambda_\nu T} \left\{ |\dot{u}_\nu(T)|^2 + |D_x u_\nu(T)|^2 \right\} \\ & + \int_0^T e^{-\lambda_\nu t} \lambda_\nu \left\{ |\dot{u}_\nu(t)|^2 + |D_x u_\nu(t)|^2 \right\} dt \\ & - C_0 \int_0^T e^{-\lambda_\nu t} \left\{ |\log_2 \epsilon| \, |D_x u_\nu(t)|^2 + \epsilon |\log_2 \epsilon| \, |D_x u_\nu(t)| \, |D_x \dot{u}_\nu(t)| \right\} dt. \end{aligned} \tag{4.12}$$

We choose now

$$\epsilon = 2^{-\nu} \quad , \quad \lambda_\nu = \beta \nu \tag{4.13}$$

with $\beta > 0$ to be chosen later. We get

$$\begin{aligned} \Omega_\nu \geq & \, e^{-\lambda_\nu T} \left\{ |\dot{u}_\nu(T)|^2 + |D_x u_\nu(T)|^2 \right\} \\ & + \int_0^T e^{-\lambda_\nu t} \beta \left\{ |\dot{u}_\nu(t)|^2 \nu + |D_x u_\nu(t)|^2 \nu \right\} dt \\ & - C_0 \int_0^T e^{-\lambda_\nu t} \left\{ \nu \, |D_x u_\nu(t)|^2 + 2^{-\nu} \nu \, |D_x u_\nu(t)| \, |D_x \dot{u}_\nu(t)| \right\} dt. \end{aligned} \tag{4.14}$$

Using now the spectral localization of u_ν we obtain:

$$\begin{aligned} \Omega_\nu \geq & \, e^{-\lambda_\nu T} \left\{ |\dot{u}_\nu(T)|^2 + |D_x u_\nu(T)|^2 \right\} \\ & + \int_0^T e^{-\lambda_\nu t} \beta \left\{ |\dot{u}_\nu(t)|^2 \nu + 2^{2\nu - 2} \nu |u_\nu(t)|^2 \right\} dt \\ & - C_0 \int_0^T e^{-\lambda_\nu t} \left\{ \nu 2^{2\nu+2} |u_\nu(t)|^2 + 2^{2\nu+2} 2^{-\nu} \nu |u_\nu(t)| \, |\dot{u}_\nu(t)| \right\} dt. \end{aligned} \tag{4.15}$$

Now, if β is chosen so that

$$\beta \geq 8C_0 \tag{4.16}$$

we obtain

$$\begin{aligned} &2\,\mathrm{Re}\int_0^T (\ddot{u}_\nu + D_x a D_x u_\nu, \dot{u}_\nu) e^{-\lambda_\nu t} dt \geq \\ &e^{-\lambda_\nu T}\{|\dot{u}_\nu(T)|^2 + |D_x u_\nu(T)|^2\} \\ &+\frac{\beta}{2}\int_0^T e^{-\lambda_\nu t}\{\nu|\dot{u}_\nu(t)|^2 + 2^{2\nu-2}\nu|u_\nu(t)|^2\}dt. \end{aligned} \tag{4.17}$$

This energy estimate gives a control of $e^{-\lambda_\nu t}|\dot{u}_\nu(t)|^2 + e^{-\lambda_\nu t}2^{2\nu}|u_\nu(t)|^2$ and, since $\lambda_\nu = \beta\nu$, this amounts to control

$$\||D_x|^{-\alpha t}\dot{u}(t)\|_{L^2(\mathbf{R}_x)} + \||D_x|^{1-\alpha t}u(t)\|_{L^2(\mathbf{R}_x)}, \tag{4.18}$$

which explains the loss of derivatives we referred to earlier.

For commutation purposes, we have to consider the following kernel on $\ell^2(\mathbb{N})$

$$k^{(\theta)}_{\nu\mu} = \|[\varphi_\nu, a]\varphi_\mu\|_{\mathcal{L}(L^2)} 2^{\nu+3}(\nu+1)^{-1/2}(\mu+1)^{-1/2} e^{(\lambda_\mu - \lambda_\nu)t/2} 2^{(\mu-\nu)\theta} \tag{4.19}$$

and we state the following

Lemma 4.1 *The operator $K(\theta)$ from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ with kernel $k^{(\theta)}_{\nu\mu}$ given by (4.13) is bounded with a norm independent of t, provided that $0 \leq t \leq T < 1/8\beta$*

$$\|K(\theta)\|_{\ell^2(\mathbf{N})\to\ell^2(\mathbf{N})} \leq C(\theta)\|A\|_{LL(x)}.$$

The norm $\|A\|_{LL(x)}$ involves only the x-regularity, that is

$$\|A\|_{LL(x)} = \|A\|_{L^\infty(\mathbf{R}^{n+1})} + \sup_{\substack{t \\ |x-y|\leq 1/2}} \frac{|A(t,x) - A(t,y)|}{|x-y|\,|\log|x-y||}. \tag{4.20}$$

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