Séminaire Équations aux dérivées partielles - École Polytechnique

V. Buslaev
A. Fedotov

The monodromization and Harper equation

Séminaire Équations aux dérivées partielles (Polytechnique) (1993-1994), exp. nº 21, p. 1-21
<http://www.numdam.org/item?id=SEDP_1993-1994____A22_0>

CENTRE
DE
MATHEMATIQUES

Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE

F-91128 PALAISEAU Cedex (FRANCE)
Tél. (1) 69 33 40 91
Fax (1) 69 33 30 19 ; Télex 601.596 F

Séminaire 1993-1994

EQUATIONS AUX DERIVEES PARTIELLES

THE MONODROMIZATION AND HARPER EQUATION

V. BUSLAEV and A. FEDOTOV

Exposé n° XXI

22 mars 1994

THE MONODROMIZATION AND HARPER EQUATION

Vladimir Buslaev and Alexander Fedotov

Department of Mathematical Physics,
St.Petersburg University
1, Ulianovskaja st., St. Petersburg,
198904, Russia
buslaev@imi.imi.spb.su and
fedotov@onti.phys.lgu.spb.su

Contents

1 Introduction

This paper is devoted to the investigation of the spectrum of Harper equation,

$$\frac{\psi(x+h)+\psi(x-h)}{2}+\cos x\ \psi(x)=E\psi(x), \tag{1.1}$$

on $L_2(\mathbb{R})$. In (1.1) h is a fixed positive number, E is the spectral parameter. Harper equation appeared as a model for the investigation of Bloch electron in a crystal placed in the weak constant magnetic field [Ho]. It was discovered that for a wide class of numbers h the spectrum can be a Cantor set and (1.1) became one of the most popular models of the spectral theory. There are many papers devoted to this equation. We give just few references allowing to "reconstruct" the history and the up-to-date state of the problem: [C-F-K-S, H-K-S, S, L].

Our paper consists of two parts. In the first part we shortly describe a general tool, the monodromization operation which we hope is very natural for the spectral analysis of difference equations with periodic coefficients. In the second part of the paper we apply this tool to investigate Harper equation.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

The monodromization. To investigate the spectrum of an ordinary differential equation with periodic coefficients one constructs its Bloch solutions. For that one introduces the monodormy matrix M corresponding to a basis in the space of the solutions. The spectral analysis comes to the analysis of the matrix equation

$$\Psi(x+s) = M \cdot \Psi(x), \tag{1.2}$$

where s is constant. It is almost trivial since M does not depend on x.

Consider a difference equation with periodic coefficients. In this case one can introduce the Bloch solution and the monodromy matrix notions as well. But now the monodromy matrices appear to be periodic in x and constructing the Bloch solutions one comes to a new difference equation of the form (1.2) with a periodic matrix M. We say that the original difference equation and this new difference equation are connected by the monodromization.

It appears that there are many relations between important properties of the difference equations connected by the monodromization. But if the analysis of the new difference equation does not yet give all the answers, one has to apply the monodromization once more and so on.

In result, instead of only one auxiliary simplest difference equation one has to investigate a sequence of difference equations with periodic coefficients.

The case of Harper equation. To investigate Harper equation we rewrite it in the form

$$\psi(x+h) = \begin{pmatrix} 2(E-\cos x) & -1 \\ 1 & 0 \end{pmatrix} \cdot \psi(x). \tag{1.3}$$

We show that this equation can be considered as a member of a family of equations

$$\psi(x+\alpha) = \mathcal{M}(x,w)\psi(x),$$

$$\mathcal{M}(x,w) \in SL(2,\mathbb{C}),$$

$$w = (s,t), \quad s \in \mathbb{C}, \quad |s| = 1, \quad it \in \mathbb{R}.$$

This family appears to be invariant with respect to the monodromization (Theorem 3.1). It allows to investigate Bloch solutions of Harper equation and to get a geometrical description of its spectrum in terms of an dynamical system,

$$F: \begin{matrix} w_l \\ \alpha_l \end{matrix} \longmapsto \begin{matrix} w_{l+1} \\ \alpha_{l+1} \end{matrix}. \tag{1.4}$$

The spectrum is contained in a set $\mathcal{S}_h$ consisting of the all points $E \in [-2,2]$ such that the corresponding trajectories do not leave a certain domain of the phase space, see Theorem 3.6. It implies that $\mathcal{S}_h$ has the structure similar to one of the classical Cantor set: it can be described by successive removing of the subintervals from the interval $[-2,2]$.

If $h/2\pi$ is rational the above process consists only of a finite number of steps. In this case we obtain complete description of the spectrum.

The transformation of α in (1.4) can be interpreted in terms of the continuous fraction

$$\frac{h}{2\pi} = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \cfrac{1}{p_3 + \dots}}}. \tag{1.5}$$

It reflects the intimate connection between the monodromization and continuous fractions in general case.

Semi-classical asymptotics. In section 4 we begin asymptotical analysis of the spectrum, i.e. we begin to investigate the case when all the denominators p_l in (1.5) are bounded from below by a big constant P. In this case the spectrum appeared to be a Cantor set having zero Lebeague mesure. But the volume of this paper does not allow to discuss these questions in details.

The papers of other authors. Our work has close connections with ones of Wilkinson and B.Hellfer, J.Sjöstrand, see, for example, [W, H-S]. These papers are devoted to the semi-classical analysis of the problem. To check that the spectrum is a Cantor set, having zero Lebeague mesure, these authors suggested a semi-classical renormalization procedure. It appeared that Harper equation almost reproduces itself in course of this procedure.

The papers of Wilkinson give an heuristic treatment of the problem. The papers of B.Hellfer and J.Shöstrand contain the rigorous mathematical analysis.

The central point of our work is the exact reproduction in a wider class of difference equations.

The methods and the restrictions. Our technique is based on the ideas of the complex WKB method. It is a known tool in the theory of ordinary differential equations. In [B-F$_1$ – B-F$_4$] we developed a version of this method for difference equations. Our method gives the proofs only in the case when all the denominators p_l in (1.5) are bounded from below by some constant

$$0 < P < p_l. \tag{1.6}$$

We hope that the results are independent of this condition.

As we already mentioned, our analysis is connected with constructing of Bloch solutions of Harper equation. Now we can construct them only when the number $h/2\pi$ is rational or Diophantine. Consequently, only for these two cases we obtain the geometrical description of the spectrum.

Note that (1.6) and the assumption for $h/2\pi$ to be rational or Diophantine are not contradictory. This number is rational if and only if the continuous fraction (1.5) is finite, [K], and it is Diophantine if and only if for all l one has $p_l \leq C^{\beta^l}$ with some positive constants C and β, see [Y].

Acknowledgements. Authors are very grateful to A.Grigis, B.Helffer, R.Seiler and J.Sjöstrand for many important remarks and stimulating discussions. This work supported by Russian Foundation of Fundamental Research. The part of the work was done when the first author was in Technische Universität, Berlin. The work was finished during our stay at University Paris 13 and Ecole Normale Superieur, Paris. We are very grateful for their support and hospitality.

2 The monodromization.

In this section we consider the matrix difference equation

$$\Psi(x+h_1) = M_0(x)\cdot\Psi(x), \tag{2.1}$$

where $x\in\mathbb{R}$ is a real variable, $M_0(x)$ is a given matrix function,

$$M_0(x)\in SL(2,\mathbb{C}),\quad M_0(x+h_0)=M_0(x). \tag{2.2}$$

The constants h_0 and h_1 satisfy the inequality

$$0<h_1<h_0. \tag{2.3}$$

2.1 Set of the solutions.

Let $\Psi_0(x)$ be a matrix solution of equation (2.1) and let

$$\Psi_0(x)\in SL(2,\mathbb{C}).$$

This solution is called *fundamental.*

A matrix function $\Psi(x)$ is a solution of (2.1) if and only if it can be represented in the form [B-F$_2$]:

$$\Psi(x)=\Psi_0(x)\cdot C(x),$$

where $C(x)$ is an h_1-periodic matrix,

$$C(x)\in Mat(2,\mathbb{C}),\quad C(x+h_1)=C(x).$$

2.2 The monodromy matrix.

In view of (2.2) $\Psi_0(x+h_0)$ is again a solution of (2.1) and one can write:

$$\Psi_0(x+h_0)=\Psi_0(x)\cdot M_1^t(x),$$

where ^{t} is just a symbol of the transposition. The matrix $M_1(x)$ is called the *monodromy* matrix corresponding to the fundamental solution Ψ_0.

Obviously,

$$M_1(x)\in SL(2,\mathbb{C}),\quad M_1(x+h_1)=M_1(x). \tag{2.4}$$

2.3 Bloch solutions.

We call a fundamental solution $\chi_0(x)$ of equation (2.1) a *Bloch solution*, if the corresponding monodromy matrix $\Gamma_1(x)$ is diagonal:

$$\chi_0(x+h_0)=\chi_0(x)\cdot\Gamma_1(x),\quad \Gamma_1(x)=\begin{pmatrix} e^{ik(x)} & 0\\ 0 & e^{-ik(x)}\end{pmatrix}.$$

The function $k(x)$ in the last formula is called the *Bloch quasi-momentum.*

We call a Bloch solution *monotonous* if $|Imk(x)|\geq c>0$ for some positive constant c.

2.4 The monodromization.

Represent the number h_0 in the form

$$h_0 = p_1\, h_1 + h_2, \quad p_1 \in \mathbb{N}, \tag{2.5}$$

$$0 \le h_2 < h_1. \tag{2.6}$$

This representation is unique. Let $\Psi_0(x)$ be a fundamental solution of (2.1) and $M_1(x)$ be the corresponding monodromy matrix. If

$$h_2 > 0$$

one can consider the equation

$$\Psi_1(x + h_2) = M_1(x) \cdot \Psi_1(x). \tag{2.7}$$

In view of (2.4) this equation is analogous to equation (2.1).

We say that equation (2.7) appears from equation (2.1) as a result of *the monodromization.* We emphasize that the matrix $M_1(x)$ in (2.7) depends on the choice of the fundamental solution $\Psi_0(x)$.

Let us investigate the properties of equations connected by the monodromization.

2.5 The chain identity.

Let h_0/h_1 be rational,

$$\frac{h_0}{h_1} = \frac{n_0}{n_1}, \tag{2.8}$$

where n_0 and n_1 are mutually prime natural numbers. The theorem 2.2 from [B-F$_3$] can be formulated as follows.

Theorem 2.1 Let Ψ_0 be a fundamental solution of (2.1) and $M_1(x)$ be the corresponding monodromy matrix. Then

$$\begin{aligned} &tr\ (M_0(x + (n_0 - 1)h_1)\, M_0(x + (n_0 - 2)h_1)\ \ldots\ M_0(x)) = \\ &= tr\ (M_1(x + (n_1 - 1)h_2)\, M_1(x + (n_1 - 2)h_2)\ \ldots\ M_1(x)) . \end{aligned} \tag{2.9}$$

We call formula (2.9) *the chain identity.*

2.6 Lyapunov exponents.

In this subsection we assume that the ratio h_1 / h_0 is irrational. Under very general assumptions about the matrix $M_0(x)$ for a.e. x there exist the limits

$$\lim_{n\to+\infty} \frac{\ln ||M_0(x + nh_1) \cdot M_0(x + (n-1)h_1)\ \ldots\ M_0(x)||}{n}$$

and

$$\lim_{n\to+\infty} \frac{\ln ||M_0^{-1}(x - nh_1) \cdot M_0^{-1}(x - (n-1)h_1)\ \ldots\ M_0^{-1}(x)||}{n}$$

independent of x, see [O]. These limits are called *the Lyapunov exponents.* We denote them by $\lambda_\pm(M_0, h_1)$.

Note that the ratio h_1 / h_2 is irrational as well. One can try to calculate the Lyapunov exponents $\lambda_\pm(M_1, h_2)$. It leads to the following statement.

Theorem 2.2 Let the fundamental solution $\Psi_0(x)$ of equation (2.1) be locally bounded. The Lyapunov exponents $\lambda_\pm(M_1, h_2)$ exist if and only if the Lyapunov exponents $\lambda_\pm(M_0, h_1)$ exist and

$$\lambda_\pm(M_0, h_1) = \frac{h_1}{h_0}\, \lambda_\mp(M_1, h_2). \tag{2.10}$$

We do not prove this theorem here and remark only that it can be considered as a generalization of Theorem 2.1 for the irrational case.

2.7 The monodromization and the Bloch solutions.

For h_1 / h_0 being irrational the existence of a Bloch solution of equation (2.1) from a "good" functional space is usually not obvious. The monodromization can help to solve the problem.

The monodromization of equation (2.1) leads to equation (2.7). Assume that it has a Bloch solution χ_1. By definition,

$$\chi_1(x + h_2) = M_1(x) \cdot \chi_1(x), \quad \chi_1(x + h_1) = \chi_1(x) \cdot \Gamma_2(x),$$

Γ_2 is a diagonal matrix. Let $\varepsilon_1(x) \in SL(2, \mathbb{C})$ be a diagonal matrix such that

$$\varepsilon_1(x + h_1) = \varepsilon_1(x) \cdot \Gamma_2(x). \tag{2.11}$$

One can write

$$\chi_1(x) = \Phi_1(x) \cdot \varepsilon_1(x), \tag{2.12}$$

where $\Phi_1(x) \in SL(2, \mathbb{C})$ is an h_1-periodic matrix:

$$\Phi_1(x + h_1) = \Phi_1(x).$$

Of course, the matrices Φ_1 and ε_1 in (2.12) are not uniquely defined.

One can easily check the theorem:

Theorem 2.3. Let there exist a fundamental solution Ψ_0 of equation (2.1), a Bloch solution χ_1 of equation (2.7) arising after the monodromization of (2.1) and let χ_1 can be represented in the form (2.12). The matrix

$$\chi_0(x) = \Psi_0 \cdot \sigma_2 \cdot \Phi_1(x), \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{2.13}$$

is a Bloch solution of (2.1) and the corresponding monodromy matrix is given by the formula

$$\Gamma_1 = \varepsilon_1(x) \cdot \varepsilon_1^{-1}(x + h_2), \tag{2.14}$$

Normal Bloch solutions. Monotonous Bloch solutions are important for spectral applications. Are there monotonous Bloch solutions of equation (2.7) such that the corresponding Bloch solutions of equation (2.1) are also monotonous? There is one case when one can guarantee it.

Consider equation (2.1). We call its monotonous Bloch solution $\chi_0(x)$ *normal* if the corresponding quasi-momentum is h_0-periodic,

$$k_1(x + h_0) = k_1(x). \tag{2.15}$$

Note the definition of the quasi-momentum and formula (2.15) imply that the quasi-momentum of a normal Bloch solution is both h_0-periodic and h_1-periodic in x. Thus, in the case when h_1/h_0 is irrational the quasi-momentum is independent of x.

Lemma 2.4 Let equations (2.1) and (2.7) are connected by the monodromization and let there exist a normal Bloch solution of (2.7). Then there exists a normal Bloch solution χ_0 of equation (2.1).

Proof. Let χ_1 be a normal Bloch solution of (2.7) and let $k_2(x)$ be the corresponding Bloch quasi-momentum. One can easily represent $\chi_1(x)$ in the form (2.12) just by choosing

$$\varepsilon_1 = \begin{pmatrix} e^{ik_2(x)\frac{x}{h_1}} & 0 \\ 0 & e^{-ik_2(x)\frac{x}{h_1}} \end{pmatrix}.$$

Therefore, by Theorem 2.3, one can construct a Bloch solution χ_0 of equation (2.1). Formula (2.14) implies that

$$k_1(x) = -\frac{h_1}{h_0}\, k_2(x). \tag{2.16}$$

Therefore $\chi_0(x)$ is monotonous. Since $h_0 = p_1 h_1 + h_2$ and $k_2(x)$ is both h_1- and h_2-periodic, formula (2.16) implies that $k_1(x + h_0) = k_1(x)$. It means that χ_0 is a normal Bloch solution.

□

2.7 The monodromization procedure.

As we have already seen, the investigation of different objects related to equation (2.1) comes to the analysis of the analogous objects for equation (2.7). If, nevertheless, the analysis of equation (2.7) does not answer the questions concerning equation (2.1), it is natural to apply the monodromization to equation (2.7). These arguments lead to the idea of *the monodromization procedure.*

Define the sequence $h_2, h_3, \ldots$, by the formulae

$$h_{l-1} = p_l\, h_l + h_{l+1}, \quad 0 \le h_{l+1} < h_l, \quad p_l \in \mathbb{N}. \tag{2.17}$$

If $h_1 \,/\, h_0 \in \mathbb{Q}$ then at some step of this calculation we get $h_{L+1} = 0$, and thus, the sequence h_l appears to be finite.

Note that the ratio $h_1 \,/\, h_0$ can be uniquely represented as a continuous fraction with the elements from $\mathbb{N}$. The numbers $p_1, p_2, p_3, \ldots$ coincide with the denominators of this continuous fraction:

$$\frac{h_1}{h_0} = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \cfrac{1}{p_3 + \ldots}}}. \tag{2.18}$$

If $h_1 \,/\, h_0 \in \mathbb{Q}$ then the continuous fraction (2.7) is finite.

Consider the sequence of the equations

$$\Psi_l(x + h_{l+1}) = M_l(x) \cdot \Psi_l(x), \tag{2.19}$$

where M_l is a monodromy matrix corresponding to the fundamental solution Ψ_{l-1}. By construction,

$$M_l(x) \in SL(2, \mathbb{C}), \quad M_l(x + h_l) = M_l(x) \quad \forall l \in \mathbb{N}. \tag{2.20}$$

One can say that all of these equations are generated by the equation (2.1). We call this process the monodromization procedure. If $h_1 \,/\, h_0 \in \mathbb{Q}$ it consists of a finite number of steps.

The general properties of the monodromization procedure imply from the properties of its elementary step, i.e. the monodromization. We discuss specially only the case $h_1 \,/\, h_0 \in \mathbb{Q}$.

2.8 The monodromization procedure in the rational case.

Denote the number of the last non-zero h_l by L. Suppose that we can construct fundamental solutions for all the L equations (2.19).

The chain identity implies that

$$tr\ (M_0(x + (n_0 - 1)h_1)\, M_0(x + (n_0 - 2)h_1) \ldots M_0(x)) = tr\ (M_L(x)), \tag{2.20}$$

where $M_L(x)$ is the monodromy matrix constructed at the last step of the monodromization procedure.

One can not construct a *Bloch solution* of the equation

$$\psi(x + h_L) = M_{L-1}(x) \cdot \psi(x), \tag{2.21}$$

as in the subsection 2.6: since $h_{L+1} = 0$ one can not apply the monodromization to (2.21). Consider a fundamental solution of (2.19), $\Psi_{L-1}(x)$, the corresponding monodromy matrix $M_L(x)$ and the equation for eigen-vectors and eigen-values of $M_L(x)$:

$$\Phi_L(x) \cdot \Gamma_{L+1}(x) = M_L(x) \cdot \Phi_L(x). \tag{2.22}$$

The matrix $\Gamma_{L+1}(x)$ is diagonal, $\det \Phi_L = 1$, both the matrices are h_L-periodic,

$$\Gamma_{L+1}(x + h_L) = \Gamma_{L+1}(x), \quad \Phi_L(x + h_L) = \Phi_L(x).$$

In terms of matrix $\Phi_L(x)$ one can easily construct a Bloch solution of (2.21):

$$\chi_{L-1}(x) = \Psi_{L-1}(x) \cdot \sigma_2 \cdot \Phi_L(x). \tag{2.23}$$

The corresponding monodromy matrix is given by the formula:

$$\Gamma_L(x) = \Gamma_{L+1}^{-1}(x). \tag{2.24}$$

The above Bloch solution exists if $\operatorname{tr} M_L(x) \neq \pm 2$.

3 Monodromization procedure for the class $\mathbb{M}$

3.1 Class $\mathbb{M}$.

Consider the set of the unimodular matrices

$$\mathcal{M} = \begin{pmatrix} a - 2\cos x & s + t\,e^{-ix} \\ -s - t\,e^{ix} & d \end{pmatrix}. \tag{3.1}$$

Here we describe explicitly the dependence $\mathcal{M}$ on x : the parameters a, d, s and t are constant. Since $\det \mathcal{M} = 1$ one has

$$a = \frac{1}{st} - \frac{s}{t} - \frac{t}{s}, \quad d = st.$$

In the above set we separate the manifold $\mathbb{M}$

$$|s| = 1, \quad t \in i\,\mathbb{R}. \tag{3.2}$$

We denote the pair (s,t) by w and the corresponding matrices by $\mathcal{M}(x,w)$.

In the sequel we call the value

$$\mathcal{E}\,(w) = \frac{a+d}{2} = \frac{st + \dfrac{1}{st} - \dfrac{s}{t} - \dfrac{t}{s}}{2}$$

the spectral parameter. Note that for $w \in \mathbb{M}$

$$\mathcal{E}(w) \in \mathbb{R}.$$

At the "curve" $t = 0$ formula (3.2) does not make sense. Fix the value of the spectral parameter $\mathcal{E}\,(w)$ and direct t to 0. It makes the s to tend to ∓ 1. When

$$\mathcal{E}\,(w) = E, \quad s \to \mp 1 \quad \text{and} \quad t \to 0 \tag{3.3}$$

the matrix $\mathcal{M}(x,w)$ turns into the matrix

$$\mathcal{M}_{\pm}(x,E) = \begin{pmatrix} 2E - 2\cos x & \mp 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Note that any of the equations

$$\Psi(x+h) = \mathcal{M}_{\pm}(x,E)\cdot\Psi(x) \tag{3.4}$$

is equivalent to Harper equation (1.1). Thus, one can say that there are two *singular points* in $\mathbb{M}$ corresponding to Harper equation.

3.2 The monodromization on $\mathbb{M}$.

Consider the equation

$$\Psi(x+h) = \mathcal{M}(x,w)\Psi(x), \tag{3.5}$$

Here h and w are two fixed parameters,

$$0 < h < 2\pi, \quad w \in \mathbb{M}.$$

One can prove the following statement.

Theorem 3.1. Let c_1, c_2 and c_3 be fixed constants, $-2 < c_2 < c_3$, $c_1 > 0$. There exists an h^* such that if

$$0 < h \leq h^* \tag{3.6}$$

and

$$|t| \leq c_1, \tag{3.7}$$

$$-2 < c_2 \leq \mathcal{E}(w) \leq c_3 \tag{3.8}$$

one can construct a fundamental solution $\Psi(x,w,h)$ of equation (3.5) so that the corresponding monodromy matrix $M_1(x)$ has the form

$$M_1(x) = \mathcal{M}\left(\frac{2\pi}{h_1}x, w_1\right), \tag{3.9}$$

where $w_1 \in \mathbb{M}$ is a coefficient independent of x,

$$w_1 = f(w,h),$$

The fundamental solution $\Psi(x,w,h)$ is entire in x, both $\Psi(x,w,h)$ and $f(w,h)$ smoothly depend on w and h.

Remark. Note that all the coefficients of the matrix $M_1(x)$ smoothly depend on w. Therefore, even if $t_1(w) = 0$ for some w, the "new spectral parameter"

$$\mathcal{E}(w_1) = \frac{1}{2}\left(\frac{1}{s_1 t_1} + s_1 t_1 - \frac{s_1}{t_1} - \frac{t_1}{s_1}\right),$$

remains finite.

Theorem 3.1 was proved in [B-F$_3$] for the case when instead of (3.8) one has $0 < c_2 \leq \mathcal{E}(w) \leq c_3$. Extending the proof to the case $c_2 < 0$ we found out that $\Psi(x,w,h)$ can have singularities when the value $\mathcal{E}(w)$ is close to -2.

We call the solution $\Psi(x,w,h)$ mentioned in this theorem *standard*.

Theorem 3.1 remains true for equations (3.4). But in this case, of course, condition (3.7) disappears and condition (3.8) has to be substituted by

$$-2 < c_2 \leq E \leq c_3. \tag{3.10}$$

Now the standard solution and the coefficient smoothly depend on E,

$$w_1 = f_0(E,h), \quad \Psi = \Psi(x,E,w). \tag{3.11}$$

The proof for the case $0 < c_2 \leq E \leq c_3$ can be found in [B-F$_2$].

To use Theorem 3.1 for the spectral analysis of Harper equation we have to take care of the cases when conditions (3.7) – (3.8) are violated.

3.3 Symmetries.

Let us define the operations:

$$j_h\, w \equiv j_h\,(s,t) = \left(-\frac{e^{\frac{ih}{2}}}{s},\, -\frac{1}{t}\right),$$

and

$$\tau\, w \equiv \tau\,(s,t) = (s,\, -t)\,.$$

Obviously, these operations preserve the properties (3.2).

Consider equation (3.5). One can easily prove the following statement.

Lemma 3.2 Let $\psi(x,w)$ be a matrix solution of equation (3.5) then

$$\sigma_3 \cdot \psi(x-\pi, w)\cdot Q(x),$$

where

$$\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q(x) = \begin{pmatrix} e^{\frac{i\pi x}{h}} & 0 \\ 0 & e^{-\frac{i\pi x}{h}} \end{pmatrix}, \tag{3.12}$$

is a solution of the equation with the matrix $\mathcal{M}(x,\, \tau\, w)$.

Moreover the matrix

$$\mu(x-\frac{h}{2}, w)\cdot \psi(x-\frac{h}{2}, w),$$

where

$$\mu(x,w) = \begin{pmatrix} -se^{-\frac{ix}{2}} - te^{\frac{ix}{2}} & ste^{-\frac{ix}{2}} \\ e^{\frac{ix}{2}} & 0 \end{pmatrix}, \tag{3.13}$$

is a solution of the equation with the matrix $\mathcal{M}(x,\, j_h\, w)$.

There is an analog of the first part of this Lemma for Harper equation.

Lemma 3.3. Let $\psi(x,E)$ be a solution of (1.1) then

$$e^{\frac{\pi i x}{h}}\psi(x-\pi, E)$$

is a solution of equation (1.1) with E changed by $-E$.

These Lemmas lead to beautiful properties of the functions $f(w,h)$ and $f_0(E,h)$. They are also connected with internal symmetries of the spectrum of Harper equation. We are planning to discuss it in an separate paper. Here we just mention only that Lemma 3.3 implies the relation

$$f_0(-E,h) = j_{\frac{2\pi^2}{h}}\,(f_0(E,h))\,. \tag{3.14}$$

4 Bloch solutions and the spectrum of Harper equation

To get a geometrical description of the spectrum of Harper equation we investigate the existence of monotonous Bloch solutions for the matrix equation (3.5). Now we can do it only in two cases: when $\frac{h_1}{2\pi}$ is a Diophantine number and when it is a rational number.

To prove our main results we assume that the ratio $h/2\pi$ can be expanded into continuous fraction (1.5) so that all the denominators p_l are bounded from below by some positive constant P,

$$0 < P \leq p_l \quad \forall l \in \mathbb{N}. \tag{4.1}$$

4.1 Bloch solutions. Preliminary results.

Consider equation (3.5). Without using the idea of the monodromization one can prove the following theorem.

Theorem 4.1. Let $c_1 > 0$ and $c_4 > 2$ be some fixed constants. There exists an h^* such that if

$$0 < h \leq h^*, \tag{4.2}$$

$$|t| \leq c_1 \tag{4.3}$$

and

$$2 < c_4 \leq |\mathcal{E}(w)| \tag{4.4}$$

then for $h/2\pi$ being rational or Diophantine one can construct a smooth normal Bloch solution $\chi(x, w, h)$ of equation (3.5).

Does there exist any Bloch solution in the case when the number $\frac{h_1}{2\pi}$ is irrational but not Diophantine? Such numbers are extremely close to rationals [K]. We note that the Bloch quasi-momentum does depend on x in the rational case and is independent of x in the Diophantine case. Thus, the answer for the "boundary" case can be quite subtle.

4.2 The Bloch solutions and the monodromization.

In this subsection we continue the investigation of Bloch solutions. Now we use the idea of the monodromization and Theorem 3.1. The conditions (3.7)–(3.8) restrict the direct application of this theorem and we have to use an auxiliary tool: the invariance properties of equation (3.5) described by Lemma 3.2.

Put

$$|w|_h = \begin{cases} w & \text{if } |t| \leq 1, \ \mathcal{E}(w) \geq 0, \\ j_h w & \text{if } |t| > 1, \ \mathcal{E}(j_h w) \geq 0, \\ \tau w & \text{if } |t| \leq 1, \ \mathcal{E}(w) < 0, \\ \tau j_h w & \text{if } |t| > 1, \ \mathcal{E}(j_h w) < 0. \end{cases} \tag{4.5}$$

Consider equation (3.5). Let $h_0 = 2\pi$, $h_1 = h$. Construct the sequence h_0, h_1, h_2, ... as in section 2.3. Recall that this sequence is finite,

$$h_0,\ h_1,\ \dots,\ h_L,$$

when $h/2\pi \in \mathbb{Q}$.

Define, when possible, the sequence

$$w_{l+1} = \left| f\left(w_l,\ 2\pi \frac{h_l}{h_{l-1}}\right) \right|_{2\pi \frac{h_{l+1}}{h_l}}, \quad w_1 = |w|_{h_1},$$

$$l \in \mathbb{N}.$$

Theorem 4.2 Let c be a fixed positive constant. There exists $P > 0$ such that for h satisfying condition (4.1):

(1) If
$$\mathcal{E}(w_l) \le 2 + c, \quad 1 \le l \le m,$$
then there exists w_{m+1}. In the case $h/2\pi \in \mathbb{Q}$, of course, $m < L$.

(2) If
$$\mathcal{E}(w_l) \le 2 + c, \quad 1 \le l < m, \quad \text{and} \quad \mathcal{E}(w_m) > 2 + c$$
and $h/2\pi$ is Diophantine, then there exists a normal Bloch solution of equation (3.5). If $h/2\pi$ is rational then $m \le L$ but instead of the L-th condition it suffices to check that
$$\mathcal{E}(w_L) > 2.$$

Proof. We give here the proof of this theorem to illustrate how our main "tools" do work.

We shall use Theorems 3.1 and 4.1. Fix the constants in (3.7) – (3.8) and (4.3) – (4.4). Let

$$c_1 = 1, \quad c_2 = 0, \quad c_3 = c_4 = 2 + c.$$

Chose h^* so that for $h \le h^*$ and the chosen values of constants c_1, c_2, c_3 and c_4 both the theorems be valid simultaneously.

Put

$$P = \frac{2\pi}{h^*}.$$

Now (4.1) and (2.17) imply that

$$2\pi \frac{h_{l+1}}{h_l} < h^* \quad \forall l.$$

It is easily seen that if $h/2\pi$ is Diophantine then all the ratios h_{l+1}/h_l are Diophantine and if $h/2\pi$ is rational then all the ratios h_{l+1}/h_l are rational as well.

The first statement of the theorem follows from the definition of $|.|_h$ and Theorem 3.1.

To prove the second statement we use Lemma 4.3. In this lemma and in the sequel we denote by $\chi(x,w,h)$ a normal Bloch solution of equation (3.5) and by $k(x,w,h)$ the corresponding quasi-momentum.

Lemma 4.3 Let there exist a normal Bloch solution $\chi(x,|w|_h,h)$. Then there exists also a normal Bloch solution $\chi(x,w,h)$.

Proof of Lemma 4.3. Recall that $|.|_h$ is defined by (4.5). The case when $w=|w|_h$ is trivial. There are, however, three other possibilities.

The case $|w|_h = j\,w$. It happens when the second component of w satisfies the inequality $|t|>1$. Put

$$\chi(x,w,h)=\frac{i}{\sqrt{ts}}\mu(x-\frac{h}{2},jw)\cdot\chi(x-\frac{h}{2},jw,h), \tag{4.6}$$

where μ is the matrix given by (3.13). Since $j^2=1$ then by Lemma 3.2 formula (4.6) gives a solution of equation (3.5). Obviously, $\det\chi(x,w,h)=\det\chi(x,jw,h)$. Thus, the solution $\chi(x,w,h)$ is fundamental. But

$$\chi(x+2\pi,w,h)=\frac{i}{\sqrt{ts}}\mu(x+2\pi-\frac{h}{2},w)\cdot\psi(x+2\pi-\frac{h}{2},w)=$$

$$=-\frac{i}{\sqrt{ts}}\mu(x-\frac{h}{2},w)\cdot\psi(x-\frac{h}{2},w)\cdot\Gamma(x-\frac{h}{2},jw,h),$$

where $\Gamma(x,jw,h)$ is the monodromy matrix corresponding to $\chi(x,jw,h)$. Therefore $\chi(x,w,h)$ is a Bloch solution of equation (3.5) and corresponding monodromy matrix is given by the formula:

$$\Gamma(x,w,h)=-\Gamma(x-\frac{h}{2},jw,h).$$

It implies the statement of the lemma.

The case when $w_1=\tau\,w$. It happens when

$$|t|\le 1 \quad \text{but} \quad \mathcal{E}(w)<0.$$

Put

$$\chi(x,w,h)=\sigma_3\cdot\chi(x-\pi,|w|_h,h)\cdot Q(x),$$

where σ_3 and $Q(x)$ are given by (3.12). Again by means of Lemma 3.2 one can check that $\chi(x,w,h)$ is a fundamental solution of (3.5). But

$$\chi(x+2\pi,w,h)=\sigma_3\cdot\chi(x+\pi,|w|_h,h)\cdot Q(x+2\pi)=$$

$$=\sigma_3\cdot\chi(x-\pi,|w|_h,h)\cdot Q(x)\Gamma(x,w,h), \tag{4.7}$$

where

$$\Gamma(x,w,h) = Q^{-1}(x)\cdot\Gamma(x-\pi,|w|_h,h)\cdot Q(x+2\pi) \tag{4.8}$$

and $\Gamma(x,|w|_h,h)$ is the monodromy matrix corresponding to $\chi(x,|w|_h,h)$. Since $Q(x)$ is diagonal, the formulae (4.7) and (4.8) imply that $\chi(x,w,h)$ is a Bloch solution of equation (3.5). Moreover,

$$k\,(x,w,h) = \frac{2\pi^2}{h} + k\,(x,\tau w,h).$$

It means that $\chi(x,w,h)$ is normal.

The case when $w_1 = \tau\, j\, w$ is a simple combination of the two previous cases.

□

Now we can come back to the proof of the Theorem. Let for a given w one has from the beginning:

$$\mathcal{E}(w_1) > 2+c, \quad w_1 = |w|_h. \tag{4.9}$$

By Theorem 4.1 one can construct $\chi(x,w_1,h)$. But then, since $w_1 = |w|_h$, there exists $\chi(x,w,h)$. It proves the theorem in the case (4.9).

If

$$\mathcal{E}(w_1) \le 2+c$$

then by Theorem 3.1 there exists the standard solution $\Psi(x,w_1,h)$ of the equation

$$\Psi_0(x+h_0) = \mathcal{M}(x,w_1)\Psi_0(x) \tag{4.10}$$

and applying to (4.10) the monodromization one comes to the equation

$$\Psi_1(x+h_2) = \mathcal{M}\left(\frac{2\pi}{h_1}x,\, f(w_1,h_1)\right)\cdot\Psi_1(x). \tag{4.11}$$

Recall that

$$w_2 = |f(w_1,h_1)|_{2\pi\frac{h_2}{h_1}}\,.$$

Consider at first the case when

$$\mathcal{E}(w_2) > 2+c. \tag{4.12}$$

In this case by Theorem 4.1 one can construct a normal Bloch solution

$$\chi(x,\, w_2,\, 2\pi\frac{h_2}{h_1})$$

and then, by Lemma 4.3, a Bloch solution $\chi(x,\, f(w_1,h_1),\, 2\pi\frac{h_2}{h_1})$. Obviously,

$$\chi_1(x) = \chi(\frac{2\pi}{h_1}x,\, f(w_1,h_1),\, 2\pi\frac{h_2}{h_1})$$

is a normal Bloch solution of equation (4.11).

Since χ_1 is a normal Bloch solution of (4.11), then by Lemma 2.4 one can construct a normal Bloch solution $\chi(x,w_1,h_1)$. Now the proof of the theorem follows from Lemma 4.3

If instead of (4.12) one has $\mathcal{E}(w_2) \le 2 + c$, then to prove the theorem it is necessary to apply the monodromization once more, now to equation (4.11), and so on. It leads directly to the proof in the irrational case. To finish the proof in the rational case one has to construct also a normal Bloch solution of the equation

$$\chi_{L-1}(x + h_L) = \mathcal{M}\left(\frac{2\pi}{h_{L-1}} x,\, w_{L-1}\right) \cdot \chi_{L-1}(x),$$

where h_L is the last non-zero h_l. For that one has to use the results of subsection 2.8.

□

4.3 The spectrum of Harper equation.

It appears that for Harper equation as for ordinary differential Schrödinger equations one can prove that if for a given E the equation has monotonous Bloch solutions, i.e. solutions having the form

$$\psi^{\pm}(x) = e^{\pm ik \frac{x}{2\pi}}\, v(x), \tag{4.13}$$

where $v \in L_2(0, 2\pi)$ is a 2π-periodic function and $Imk \neq 0$, then Harper operator has the bounded inverse on $L_2(\mathbb{R})$. The existence of the solutions (4.13) can be investigated by means of Theorem 4.2. It leads to the following result.

Theorem 4.4 Let c be a positive constant. There exists a positive constant P such that under condition (4.1) the spectrum of Harper equation (1.1) is contained in the set $\mathcal{S}_h$ described by the conditions:

$$\begin{gathered} |E| \le 2, \\ E_1(E) \le 2 + c, \\ \dots \\ E_l(E) \le 2 + c, \\ \dots, \end{gathered} \tag{4.14}$$

where

$$E_l(E) = \mathcal{E}(w_l(E)),$$

$$w_{l+1} = |f(w_l, \alpha_l)|_{\alpha_{l+1}}, \tag{4.15}$$

$$\alpha_{l+1} = \frac{4\pi^2}{\alpha_l} \bmod 2\pi, \quad 0 \le \alpha_{l+1} < 2\pi, \tag{4.16}$$

$$l \in \mathbb{N},$$

and

$$w_1 = |f_0(|E|,\, h)|_{\alpha_1},$$

$$\alpha_0 = h.$$

Remarks

1. Note that

$$\alpha_l = 2\pi \frac{h_l}{h_{l-1}}.$$

where $\{h_l\}$ is the sequence of the "monodromization shifts" from subsection 4.2.

2. By definition all the w_l are even functions of E.

The theorem implies that the set $\mathcal{S}_h$ has the structure similar to one of the classical Cantor set: it can be described by successive removing of the subintervals from the interval $[-2, 2]$.

Formulae (4.14)-(4.15) describe a dynamical system. The set Σ_h can considered as a "generalized Julia set": it consists of the all points $E \in [-2, 2]$ such that the corresponding trajectories do not leave the "domain" $\mathcal{E}(w) \leq 2 + c$ in the "phase space" $\mathbb{M}$.

Does the set $\mathcal{S}_h$ coincide with the spectrum of Harper equation? In the rational case one can obtain the exact condition for E to be in the spectrum:

$$|tr\ (M_0(x + (n-1)h, E)\, M_0(x + (n-2)h, E)\ \ldots\ M_0(x, E))\,| \leq 2, \quad \text{for some } x, \tag{4.17}$$

where M_0 is the matrix from (1.3) and n is the denominator of the fraction $h/2\pi$. In this case the monodromization procedure consists of a finite number L of steps. By means of the chain identities and Lemma 3.2 one can prove that (4.17) equivalent to the inequality:

$$E_L(E) \equiv \mathcal{E}(w_L) \leq 2.$$

5 Semi-classical asymptotics

To describe the semi-classical structure of the spectrum, i.e. its structure when all the α_l from (4.16) are small or, equivalently, when P in (4.1) is big, it suffices to get the asymptotics of the function $f(w, h)$ as $h \to 0$. But the volume of the paper does not allow to discuss it entirely. As we already mentioned the set $\mathcal{S}_h$ can be described by successive removing of the subintervals from the interval $[-2, 2]$. Here we describe the first step of this removing. It gives correct idea about the whole process.

We begin by writing down the asymptotics as $h \to 0$ for the function $E_1(E)$ which enters into the second of the conditions (4.14). Obviously, it suffices to consider the case $E \geq 0$.

Semi-classical asymptotics of the function $E_1(E)$.

The leading terms of the asymptotics can be described in terms of the complex isoenergetic curve

$$\cos p + \cos z = E. \tag{5.1}$$

The function $p(z)$ defined by (5.1) is a many-valued function. The picture of its branching points is 2π-periodic. There are always only four branching points in the

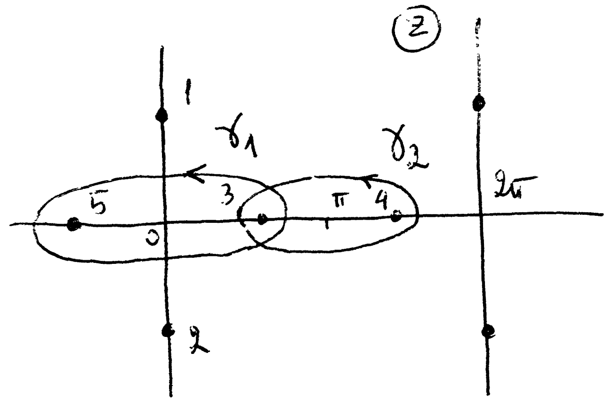

FIG.1

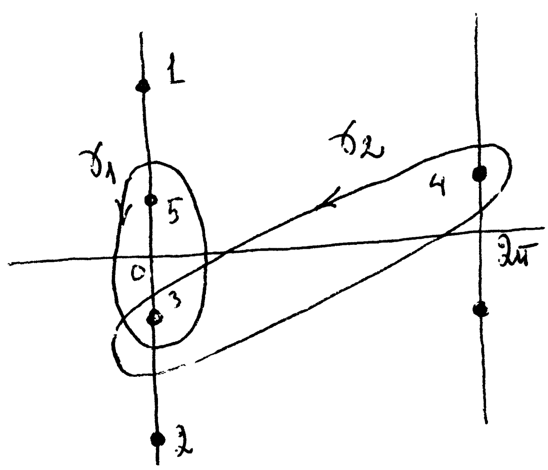

FIG.2

strip $-\pi \leq Re\, z < \pi$. In fig. 1 we show branching points for $0 \leq E \leq 2$, fig. 2 corresponds to $E \geq 2$.

Put

$$\Phi(E) = \int_{\gamma_1} p\, dz \tag{5.2}$$

and

$$S(E) = \int_{\gamma_2} p\, dz. \tag{5.3}$$

The contours γ_1 and γ_2 for different values of E are shown in fig. 1 – 2. The branches of the momentum p in (5.2) and (5.3) have to be chosen so that $S(E) > 0$ for $E > 0$ and $\Phi(E) > 0$ for $0 < E < 2$. Note that the function $S(E)$ is monotonously increasing, $S(0) = 0$, the function $\Phi(E)$ is monotonously decreasing and $\Phi(0) = 2\pi^2$, $\Phi(2) = 0$.

To write down the asymptotic formulae we have to introduce also two standard functions $U(\xi)$ and $V(\xi)$, $\xi \in \mathbb{R}$. The function U is given by the formula:

$$U(\xi) = \frac{\Gamma\left(\frac{i\xi}{\pi} + \frac{1}{2}\right) e^{-\frac{i\xi}{\pi}\left(\ln\frac{|\xi|}{\pi} - 1\right)}}{\Gamma\left(-\frac{i\xi}{\pi} + \frac{1}{2}\right) e^{\frac{i\xi}{\pi}\left(\ln\frac{|\xi|}{\pi} - 1\right)}},$$

it possesses the properties: $|U(\xi)| = 1$ and $U(\xi) \to 1$ as $\xi \to +\infty$.

The function $V(\xi)$ is given by the formula

$$V(\xi) = \sqrt{2\pi}\frac{e^{\frac{\xi}{\pi}\left(\ln\frac{|\xi|}{\pi} - 1\right)}}{\Gamma\left(\frac{\xi}{\pi} + \frac{1}{2}\right)},$$

it is real and $V(\xi) \to 1$ as $\xi \to +\infty$ and $V(\xi) \sim 2\cos\xi$ as $\xi \to -\infty$.

Now we can write down asymptotic formulae for $E_1(E)$. Let $0 < \varepsilon < 1$ be a fixed constant. For $0 \le E \le \varepsilon$:

$$E_1(E) = 2\cosh\left(\frac{S(E)}{2h}\right) \cdot \left\{\cos\left(\frac{\Phi(E)}{2h} + \arg U(\frac{S(E)}{2h})\right) + O(h\ln h)\right\}.$$

For $\varepsilon \le E \le 2 - \varepsilon$

$$E_1(E) = \exp\left(\frac{S(E)}{2h}\right) \cdot \left\{\cos\frac{\Phi(E)}{2h} + O(h)\right\}.$$

In the case $2 - \varepsilon \le E \le 2 + \varepsilon$

$$E_1(E) = \exp\left(\frac{S(E)}{2h}\right) \cdot \left\{\cos\frac{\Phi(E)}{2h}\left(V(\frac{\Phi(E)}{2h})\right)^{-1} + O(h\ln h)\right\}.$$

Finally, when $2 + \varepsilon \le E \le 2 + Const$ one has

$$E_1(E) = \frac{1}{2}\, exp\left(\frac{S(E)}{2h}\right) \cdot \{1 + O(h)\}.$$

All these formulae are uniform in E.

The asymptotic formulae allow, in particular, to draw a typical graph of the $E_1(E)$. We show it in fig. 3.

Note that due to (3.14) $E_1(0) = 2\cos(\frac{2\pi^2}{h})$ and that by the definition of E_1

$$E_1(-E) = E_1(E).$$

4.2 Asymptotic properties of the spectrum.

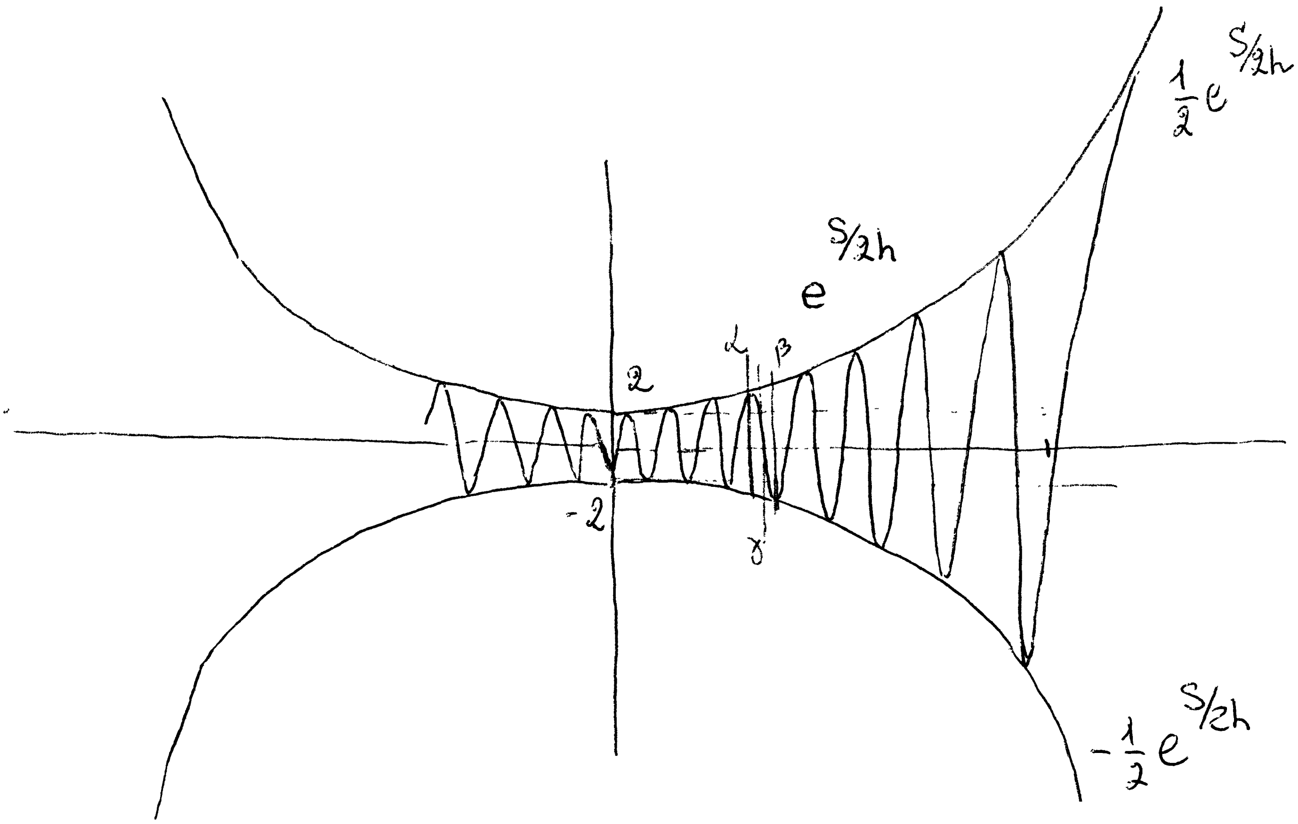

FIGURE 3.

Let us come back to the Theorem 4.3 In view of (4.14) the graph of the function $E_1(E)$ shows that as $\alpha_0 \equiv h \to 0$ outside the small vicinity of the point $E = 0$ the spectrum can be covered by a system of exponentially small intervals.

If we want to investigate the spectrum with more details, we have to take into the account the second condition from (4.14),

$$E_2(E) \leq 2 + c.$$

For that we have to separate from $[-2, 2]$ all the subintervals where the graph $E_1(E)$ is monotonous and $E_1(E) \leq 2 + c$. Denote one of them by $[a, b]$. On this interval we have "to replace" the graph of $E_1(E)$ by the graph of the function $E_2(E)$. The further asymptotic analysis shows that if α_1 is small the graph of $E_2(E)$ on the interval $[a, b]$ has, up a natural change of scale, the same character as the graph of $E_1(E)$ on the whole interval $[-2, 2]$. It means that the spectrum of Harper equation on the interval $[a, b]$ is contained in the set of subintervals which are discribed by the condition $E_2(E) \leq 2 + c$. Outside the small vicinity of some point in $[a, b]$ they are exponentially small with respect to the natural new scale.

Repeating these arguments in the case of an Diophantine $h/2\pi$, one can see that the whole spectrum of Harper equation is contained in a Cantor set having zero Lebeague mesure.

In the case $2\pi/h = q$, $q \in \mathbb{N}$ just one step of the above process leads to the complete asymptotical description of the spectrum. In the case $h/2\pi \in \mathbb{Q}$ one has to make a finite number of steps.

References

[B-F_1] V. Buslaev, A. Fedotov, *Complex WKB method for Harper equation*, Reports of Mittag-Leffler Institute **11** (1993).

[B-F_2] V. Buslaev, A. Fedotov, *The functional structure of a monodormy matrix for Harper equation*, Operator Theory: Advances and Applications (I.Gohberg, eds.), 1994.

[B-F_3] V. Buslaev, A. Fedotov, *On a class of matrices related to Harper equation*, Reports of Mittag-Leffler Institute **19** (1993).

[B-F_4] V. Buslaev, A. Fedotov, *Complex WKB method for Harper equation. I.*, Algebra and Analysis (in russian) **No 4** (1994).

[C-F-K-S] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrdinger operators*, Springer-Verlag, 1987.

[Ho] D. Hofstadter, *Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*, Phys. Rev. B **14** (1976), 2239–2249.

[H-S] B. Helffer, J. Sjöstrand, *Analyse semi-classique pour l'equation de Harper (avec application a'l etude de l'equation de Schrödinger avec champ magnétique)*, Mémoires de la SMF **34** (1988).

[H-K-S] B. Helffer, P. Kerdelhué, J. Sjöstrand, *Le papillon de Hofstadter revisité*, Mémoires de la SMF **43** (1990).

[K] A.Ya.Khinchin, *Continued fractions*, Phoenix Science Series, 1964.

[L] Y.Last Zero mesure spectrum for the almost Mathieu operator, Preprint (March 1994).

[O] V.I.Oseledets, *A multiplicative ergodic theorem*, Trans. Mosc. Math. Soc. **19** (1969), 197-231.

[Wi] M. Wilkinson, *Critical properties of electron eigenstates in incommensurate systems.*, Proc. Royal Society of London. A **391** (1984), 305–350.

[Y] J.C.Yoccoz, Ph.D. thesis.