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EQUATIONS AUX DERIVEES PARTIELLES

DIRECT IMAGES OF ELLIPTIC PAIRS AND MICROLOCALIZATION

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Direct images of elliptic pairs and microlocalization

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1 Introduction

Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. In [7], we introduced the notion of a proper f-elliptic pair $(\mathcal{M},F)$ on X, and proved that the direct image of such a pair is an object of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$ with coherent cohomology. When f is projective and $F = \mathbb{C}_X$, one recovers the classical direct image theorem of Kashiwara [3] (as well as its generalization to the non proper case of [2]). When $Y = \{\mathrm{pt}\}$, M is a compact real analytic manifold, X a complexification of M, $F = \mathbb{C}_M$ and $\mathcal{M}$ is elliptic on M in the classical sense, one recovers the classical finiteness theorem for solutions of elliptic systems.

In this paper, we shall prove that direct image commutes with microlocalization. More precisely, denote by $\mathcal{E}_X$ the sheaf of (finite order) microdifferential operators on T^*X ([5] or see [6] for a detailed exposition), and still denote by $\underline{f}_!$ the direct image for $\mathcal{E}$-modules (see below). Then, we prove that

$$\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \simeq \underline{f}_![\pi_X^{-1}(\mathcal{M} \otimes F) \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X].$$

This result was established by Kashiwara [3] when $F = \mathbb{C}_X$ and f is projective. It was also announced in a non proper case in [2].

In the last section, we show that this result has interesting applications in the study of correspondences for $\mathcal{D}$-modules, as for example, in the case of the Penrose transform considered by [1].

2 Direct image of $\mathcal{D}$ and $\mathcal{E}$ modules

Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds.

Recall that the proper direct image of a right $\mathcal{D}_X$-module $\mathcal{M}$ is defined through the formula

$$\underline{f}_!(\mathcal{M}) = Rf_!(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$$

where $\mathcal{D}_{X\to Y}$ denotes the differential transfer module associated to f.

At the microlocal level, we consider the following diagram:

$$T^*X \xleftarrow[{}^t f']{} X \times_Y T^*Y \xrightarrow[f_\pi]{} T^*Y$$

and recall that the microlocal proper direct image of a right $\mathcal{E}_X$-module $\mathcal{M}$ is defined through the formula

$$\underline{f}_!(\mathcal{M}) = Rf_{\pi!}({}^t f'^{-1}\mathcal{M} \otimes^L_{{}^t f'^{-1}\mathcal{E}_X} \mathcal{E}_{X\to Y}),$$

where $\mathcal{E}_{X\to Y}$ denotes the micro-differential transfer module associated to f.

The microlocalization of a right $\mathcal{D}_X$-module $\mathcal{M}$ is the right $\mathcal{E}_X$-module $\mathcal{M}\mathcal{E}$ defined on T^*X by setting

$$\mathcal{M}\mathcal{E} = \pi_X^{-1}\mathcal{M} \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X.$$

3 The topology of the sheaf $\mathcal{C}_{Y|X}(0)$

Proposition 3.1 *Let X be a complex analytic manifold. Assume Y is a complex submanifold of X and denote by $\mathcal{C}_{Y|X}(0)$ the sheaf of holomorphic microfunctions of order 0 on $T^*_Y X$. Then, for any compact subset $K \subset T^*_Y X$, the space*

$$\Gamma(K; \mathcal{C}_{Y|X}(0))$$

has a canonical DFN topology.

Proof: Working locally, we may use a coordinate system $(x_1, \ldots, x_d, y_1, \ldots, y_{n-d})$ where Y is defined by the equations

$$x_1 = 0; \cdots; x_d = 0.$$

Denote by $(\xi_1, \ldots, \xi_d)$ the corresponding coordinates on $T^*_Y X$. It follows from [5, Thm 1.4.5] that, for any open subset U of $T^*_Y X$, the formula

$$\int \delta(p - \langle x, \xi\rangle) u(x,y) dx = \sum_{j=-\infty}^{0} a_j(y,\xi)\delta^{(j)}(p) \tag{3.1}$$

establishes a one to one correspondence between holomorphic microfunctions

$$u(x,y) \in \Gamma(U; \mathcal{C}_{Y|X}(0))$$

and sequences of homogeneous holomorphic functions

$$a_j(x,\xi) \in \Gamma(U; \mathcal{O}_{T^*_Y X}(j)) \qquad (j \le 0)$$

such that for any compact subset $K \subset U$

$$\sum_{j=-\infty}^{0} |a_j(x,\xi)|_K \frac{\epsilon^{-j}}{(-j)!} < +\infty$$

for some $\epsilon > 0$.

Let us first construct the requested DFN topology in two special cases.

Case a. Assume K is a convex compact subset of $T^*_Y X$ on which $\xi_k \neq 0$. Denote by $p : \dot{T}^*_Y X \longrightarrow P^*_Y X$ the canonical projection. The preceding discussion shows that the map

$$\begin{array}{rcl} \Gamma(K; \mathcal{C}_{Y|X}(0)) & \longrightarrow & \Gamma(p(K) \times \{0\}; \mathcal{O}_{P^*_Y X \times \mathbb{C}}) \\ u(x,y) & \longmapsto & f_k(y,\xi,\tau) = \sum_{j=0}^{+\infty} a_{-j}(y, \xi/\xi_k) \dfrac{\tau^j}{j!} \end{array}$$

is an isomorphism. Using this isomorphism, we endow $\Gamma(K; \mathcal{C}_{Y|X}(0))$ with the usual DFN topology of $\Gamma(p(K) \times \{0\}; \mathcal{O}_{P^*_Y X \times \mathbb{C}})$. If, moreover, $\xi_\ell \neq 0$ on K, one has

$$f_k(\xi,\tau) = f_\ell(\xi, \tau\xi_k/\xi_\ell).$$

Hence, the DFN topology of $\Gamma(K; \mathcal{C}_{Y|X}(0))$ does not depend on k.

Case b. Let π denote the canonical projection of the bundle $T^*_Y X$ on its base Y identified to the zero section. Assume K is a convex compact subset of $T^*_Y X$ such that $\pi(K) \subset K$. It follows from (3.1) that

$$\begin{array}{rcl} \Gamma(K; \mathcal{C}_{Y|X}(0)) & \longrightarrow & \Gamma(\pi(K); \mathcal{O}_Y) \\ u(x,y) & \longmapsto & a_0(y,0) \end{array}$$

is an isomorphism. We use this isomorphism to transport on $\Gamma(K; \mathcal{C}_{Y|X}(0))$ the usual DFN topology of $\Gamma(\pi(K); \mathcal{O}_Y)$.

One checks easily that, if $K_1 \subset K_2$ are two compact subsets of $T^*_Y X$ of the kind treated in case (a) or (b) above, then the restriction map

$$\Gamma(K_2; \mathcal{C}_{Y|X}(0)) \longrightarrow \Gamma(K_1; \mathcal{C}_{Y|X}(0))$$

is continuous.

Let K be an arbitrary compact subset of $T^*_Y X$. The preceding discussion shows that we can find a finite covering $(K_i)_{i\in I}$ of K by compact subsets such that $\Gamma(K_i; \mathcal{C}_{Y|X}(0))$ and $\Gamma(K_i \cap K_j; \mathcal{C}_{Y|X}(0))$ are DFN spaces. Thanks to the exact sequence

$$0 \longrightarrow \Gamma(K; \mathcal{C}_{Y|X}(0)) \xrightarrow{\alpha} \prod_{i\in I} \Gamma(K_i; \mathcal{C}_{Y|X}(0)) \xrightarrow{\beta} \prod_{i,j\in I} \Gamma(K_i \cap K_j; \mathcal{C}_{Y|X}(0)),$$

we may use α to transport on $\Gamma(K;\mathcal{C}_{Y|X}(0))$ the DFN topology of $\ker\beta$. To show that this topology is independent of the chosen covering, it is sufficient to show that it is equivalent to the topology induced by a finer covering. Since such a topology is obviously weaker, the conclusion follows from the closed graph theorem.

Since a direct computation shows that the above defined topology is independent of the chosen coordinate systems, the conclusion follows easily. □

Corollary 3.2 *Let X be a complex analytic manifold. Assume K is a compact subset of T^*X. Then*

$$\Gamma(K;\mathcal{E}_X(0))$$

has a canonical DFN topology.

Proof: Apply the preceding proposition to $\mathcal{C}_{\Delta_X|X\times X}(0)$. □

Proposition 3.3 *Let X, Z be complex analytic manifolds and let Y be a complex submanifold of X. We identify $T^*_{(Z\times Y)}(Z\times X)$ and $Z\times T^*_YX$. We denote by $q: Z\times T^*_YX \longrightarrow T^*_YX$ the second projection. Then, for any Stein compact subset $K\subset Z$, one has*

$$Rq_![(\mathcal{C}_{Z\times Y|Z\times X}(0))_{K\times T^*_YX}]\simeq\Gamma(K;\mathcal{O}_Z)\hat{\otimes}\,\mathcal{C}_{Y|X}.$$

Proof: Let S be a complex manifold. Denote by $p_S: Z\times S\longrightarrow S$ the second projection. By classical results of analytic geometry, we know that

$$Rp_{S!}[(\mathcal{O}_{Z\times S})_{K\times S}]\simeq\Gamma(K;\mathcal{O}_Z)\hat{\otimes}\,\mathcal{O}_S.$$

Using the explicit isomorphisms constructed in the proof of the preceding proposition, the conclusion follows easily. □

Corollary 3.4 *Let Z, Y be complex analytic manifolds and denote by*

$$f: Z\times Y\longrightarrow Y$$

the second projection. Assume K is a Stein compact subset of Z. Then

$$Rf_{\pi!}[(\mathcal{E}_{Z\times Y\to Y}(0))_{K\times T^*Y}]\simeq\Gamma(K;\mathcal{O}_Z)\hat{\otimes}\,\mathcal{E}_Y(0).$$

Proof: Apply the preceding proposition to $\mathcal{C}_{Z\times\Delta_Y|Z\times(Y\times Y)}(0)$. □

4 Main result

Theorem 4.1 *Assume $f : X \longrightarrow Y$ is a morphism of complex analytic manifolds and $(\mathcal{M}, F)$ is an f-elliptic pair on X with f-proper support. Then the canonical map*

$$[\underline{f}_!(\mathcal{M} \otimes F)]\mathcal{E} \longrightarrow \underline{f}_!([\mathcal{M} \otimes F]\mathcal{E})$$

is an isomorphism in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_Y)$.

Proof: Recall that we have the commutative diagram

$$\begin{tikzcd} T^*X \arrow[d, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "{}^t f'"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*Y \arrow[d, "\pi_Y"] \\ X & X \arrow[l, "\sim"'] \arrow[r, "f"'] & Y \end{tikzcd}$$

Hence, we have successively

$$\begin{aligned} &\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\ &\quad = \quad Rf_{\pi !}[\pi^{-1}(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y})] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\ &\quad = \quad Rf_{\pi !}[\pi^{-1}(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y] \\ &\quad = \quad Rf_{\pi !}[\pi^{-1}(\mathcal{M} \otimes F) \otimes^L_{\pi^{-1}\mathcal{D}_X} (\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y)]. \end{aligned}$$

Note that there is a canonical map

$$\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y \longrightarrow \mathcal{E}_{X\to Y}. \tag{4.1}$$

Hence, we get a canonical morphism

$$\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \longrightarrow \underline{f}_![\pi_X^{-1}(\mathcal{M} \otimes F) \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X]. \tag{4.2}$$

When f is a closed embedding, (4.1) is an isomorphism. Hence (4.2) is an isomorphism for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and any $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$.

In the general case, consider the graph embedding

$$i : X \longrightarrow X \times Y$$

and the projection

$$p : X \times Y \longrightarrow Y.$$

Since $(\mathcal{M}, F)$ is an f-elliptic pair, the pair $(\underline{i}_!\mathcal{M}, F \boxtimes \mathbb{C}_Y)$ is p-elliptic. Since our result holds for closed embeddings and

$$\underline{i}_!\mathcal{M} \otimes (F \boxtimes \mathbb{C}_Y) \simeq \underline{i}_!(\mathcal{M} \otimes F),$$

we are reduced to prove the theorem for the pair $(\underline{i}_!\mathcal{M}, F \boxtimes \mathbb{C}_Y)$ and the map p.

We may thus assume that f is the second projection from $X = Z \times Y$ to Y and that $F = G \boxtimes \mathbb{C}_Y$ where G is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$. Moreover, working as in [7], we may also assume that $\mathcal{M} = \mathcal{N} \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_X$ where $\mathcal{N}$ is a coherent $\mathcal{D}_{X|Y}$-module. In this case,

$$\begin{aligned}
&\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\
&\qquad = \; Rf_{\pi!}[\pi^{-1}(\mathcal{M} \otimes F) \otimes^L_{\pi^{-1}\mathcal{D}_X} (\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y)] \\
&\qquad = \; Rf_{\pi!}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes^L_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y)]
\end{aligned}$$

and

$$\underline{f}_![\pi_X^{-1}(\mathcal{M} \otimes F) \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X] \;=\; Rf_{\pi!}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}].$$

Hence, we are reduced to show that the canonical arrow

$$\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0) \longrightarrow \mathcal{E}_{X\to Y}(0)$$

induces an isomorphism

$$\begin{aligned}
(4.3) \qquad & Rf_{\pi!}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes^L_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0))] \\
& \xrightarrow{\sim} \; Rf_{\pi!}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0)]
\end{aligned}$$

As a matter of fact, $\mathcal{E}_{X\to Y} \simeq \mathcal{E}_{X\to Y}(0) \otimes_{f_\pi^{-1}\mathcal{E}_Y(0)} f_\pi^{-1}\mathcal{E}_Y$ as a $(\mathcal{D}_{X|Y}, \mathcal{E}_Y)$-bimodule and a scalar extension of 4.3 gives the theorem.

Using the realification process developed in [7], we may assume from the beginning that Z is a complexification of a real analytic manifold M and that G is supported by M.

Since the result is local on T^*Y (hence on Y), we may assume also that $\mathcal{N}$ has a projective resolution $\mathcal{L}^\cdot$ by finite free $\mathcal{D}_{X|Y}$-modules (see [7, Prop. 3.1]).

As for G, we may assume it is isomorphic to a bounded complex $T^\cdot$ of the type

$$0 \longrightarrow \cdots \bigoplus_{i_a \in I_a} \mathbb{C}_{K_{a,i_a}} \longrightarrow \cdots \bigoplus_{i_k \in I_k} \mathbb{C}_{K_{k,i_k}} \longrightarrow \cdots \bigoplus_{i_b \in I_b} \mathbb{C}_{K_{b,i_b}} \longrightarrow 0$$

where the sets I_k are finite and K_{k,i_k} is a subanalytic compact subset of M (see [7, Prop. 3.10]).

Hence,

$$\mathcal{N} \otimes (F \boxtimes \mathbb{C}_Y) \simeq \mathcal{L}^\cdot \otimes (T^\cdot \boxtimes \mathbb{C}_Y)$$

and the components of this last complex are finite direct sums of sheaves of the type

$$\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}$$

where K is a subanalytic compact subset of M.

Note that

$$\begin{aligned}(4.4)\qquad & \pi^{-1}(\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}) \otimes^{L}_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0)) \\ & \quad \xrightarrow{\sim} \pi^{-1}(\mathcal{O}_X)_{K\times Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0) \\ (4.5)\qquad & \pi^{-1}(\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0) \\ & \quad \xrightarrow{\sim} (\mathcal{E}_{X\to Y}(0))_{K\times T^*Y}\end{aligned}$$

The right hand side of (4.4) is acyclic for $f_{\pi!}$ thanks to usual properties of Stein compact subsets. Moreover, Corollary 3.4 shows that the right hand side of (4.5) is also acyclic for $f_{\pi!}$. Hence, the morphism (4.3) of $\mathbf{D}^b(\mathcal{E}_Y(0))$ is represented in $C^b(\mathcal{E}_Y(0))$ by the morphism

$$\begin{aligned}(4.6)\qquad & f_{\pi!}[\pi^{-1}(\mathcal{L}^\cdot \otimes (T^\cdot \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0))] \\ & \quad \xrightarrow{\sim} f_{\pi!}[\pi^{-1}(\mathcal{L}^\cdot \otimes (T^\cdot \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0)]\end{aligned}$$

Let us denote by $R^\cdot$ the complex

$$f_![\mathcal{L}^\cdot \otimes (T^\cdot \otimes \mathbb{C}_Y) \otimes_{\mathcal{D}_{X|Y}} \mathcal{O}_X].$$

Its components are direct sums of sheaves of the type

$$f_![(\mathcal{O}_X)_{K\times Y}] \simeq \Gamma(K;\mathcal{O}_Z) \hat{\otimes} \mathcal{O}_Y$$

which are DFN-free $\mathcal{O}_Y$-modules. As in [7], it is easy to check that the $\mathcal{O}_Y$-linear differential of $R^\cdot$ is continuous with respect to the these natural topologies. Hence, we may consider $R^\cdot$ as a topological complex of DFN-free $\mathcal{O}_Y$-modules. Using Corollary 3.4, we have successively

$$\begin{aligned} Rf_{\pi!}[(\mathcal{E}_{X\to Y}(0))_{K\times T^*Y}] &\simeq \Gamma(K;\mathcal{O}_Z) \hat{\otimes} \mathcal{E}_Y(0) \\ &\simeq [\Gamma(K;\mathcal{O}_Z) \hat{\otimes} \pi_Y^{-1}\mathcal{O}_Y] \hat{\otimes}_{\pi_Y^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0) \\ &\simeq \pi_Y^{-1} f_![(\mathcal{O}_X)_{K\times Y}] \hat{\otimes}_{\pi_Y^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0)\end{aligned}$$

and (4.6) is represented as the canonical morphism

$$\pi^{-1}R^\cdot \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0) \longrightarrow \pi^{-1}R^\cdot \hat{\otimes}_{\pi^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0).$$

Since $R^\cdot$ has $\mathcal{O}_Y$-coherent cohomology, Lemma 4.2 below allows us to conclude the proof. □

The following lemma is easily deduced from the results in §1–2 of [4].

Lemma 4.2 *Let S be a complex analytic manifold. Assume $\mathcal{F}$ is a DFN $\mathcal{O}_S$-module and $\mathcal{M}^\cdot$ is a complex of DFN-free $\mathcal{O}_S$-modules. If $\mathcal{M}^\cdot$ is bounded from above and has $\mathcal{O}_S$-coherent cohomology then the natural morphism*

$$\mathcal{M}^\cdot \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{M}^\cdot \hat{\otimes}_{\mathcal{O}_S} \mathcal{F}$$

is a quasi-isomorphism in $C^-(\mathcal{O}_S)$.

5 Applications

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds.

Corollary 5.1 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module endowed with a good filtration. Assume:*

(i) f is proper on $\operatorname{supp} \mathcal{M}$,

(ii) f_π is finite on ${}^t f'^{-1}(\mathrm{char}\mathcal{M}) \cap X \times_Y \dot{T}^*Y$, where $\dot{T}^*Y = T^*Y \setminus T^*_Y Y$.

Then, for $j \neq 0$, $H^j(\underline{f}_!\mathcal{M})$ is a flat connection (i.e. its characteristic variety is contained in the zero section).

Proof: The second hypothesis implies that $\underline{f}_!(\mathcal{M}\mathcal{E})$ is concentrated in degree zero on $\dot{T}^*Y$. The first hypothesis and Theorem 4.1 imply that

$$(\underline{f}_!\mathcal{M})\mathcal{E} \simeq \underline{f}_!(\mathcal{M}\mathcal{E}).$$

Hence, for $j \neq 0$, $\operatorname{supp} H^j[(\underline{f}_!\mathcal{M})\mathcal{E}]$ is contained in the zero section. Since $\mathcal{E}$ is flat over $\pi^{-1}\mathcal{D}$, the conclusion follows easily. $\square$

As we shall see now, we may apply this last result in the study of correspondences

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ Z & & X \end{tikzcd}$$

when assuming

$$(5.1) \qquad \begin{cases} g \text{ is smooth,} \\ f \text{ is proper,} \\ (g,f) : Y \longrightarrow Z \times X \text{ is a closed embedding.} \end{cases}$$

Let

$$\Lambda = T^*_Y(Z \times X) \cap (\dot{T}^*Z \times \dot{T}^*X)$$

and denote by p_2 the projection $T^*(Z \times X) \longrightarrow T^*X$. Assume:

$$(5.2) \qquad p_2 \text{ is finite on } \Lambda.$$

Corollary 5.2 *Assume (5.1) and (5.2) and let $\mathcal{N}$ be a coherent $\mathcal{D}_Z$-module endowed with a good filtration. Then for $j \neq 0$, $H^j(\underline{f}_!\underline{g}^*\mathcal{N})$ is a flat connection.*

The proof follows immediately from Corollary 5.1, since g being smooth, $\underline{g}^*\mathcal{N}$ is concentrated in degree zero and endowed with a good filtration.

Example 5.3 (Penrose Correspondence (see [1])) Let T be a $\mathbb{C}$-vector space of dimension 4, $F(1,2)$ the flag manifold of type $(1,2)$, i.e. the set of couples of linear subspaces $l \subset p$ of T of dimension 1 and 2 respectively. Define similarly $F(1)$ (the complex projective space of dimension 3) and $F(2)$ (the Minkowski compactification of $\mathbb{C}^4$). Then, we get a correspondence:

$$\begin{tikzcd} & Y = F(1,2) \arrow[dl, "g"'] \arrow[dr, "f"] & \\ Z = F(1) & & X = F(2) \end{tikzcd}$$

and one checks easily that hypothesis (5.1) and (5.2) are satisfied. Moreover, p_2 induces an isomorphism $\Lambda \xrightarrow{\sim} V$, where V is a smooth regular involutive submanifold of $\dot{T}^*X$. Note that V is the characteristic variety of the wave equation. We refer the reader to [1] for a detailed study of the Penrose correspondence in the framework of $\mathcal{D}$-modules.

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