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G. PERELMAN

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Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE

F-91128 PALAISEAU Cedex (FRANCE)
Tél. (1) 69 33 40 91
Fax (1) 69 33 30 19 ; Télex 601.596 F

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EQUATIONS AUX DERIVEES PARTIELLES

STABILITY OF SOLITARY WAVES FOR NONLINEAR SCHRÖDINGER EQUATION

G. PERELMAN

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0. Introduction

The results of this work can be considered as a generalisation of the previous works of V. Buslaev and the author [BP1,2] that were denoted to the nonlinear scattering of the states close to a soliton.

Under some general conditions on the function F the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} + F(|\psi|^2)\psi \ , \ \psi(x,t) \in \mathbb{C} \ , x \in \mathbb{R}, t \in \mathbb{R}_+ \tag{1}$$

admits a class of bounded solutions $w(x, \sigma(t))$ which parameters $\sigma = \sigma(t) \in \mathbb{R}^4$ depend explicitly on time t. In [BP1,2] we considered the Cauchy problem for equation (1) with the initial data

$$\psi_{|t=0} = w(\cdot, \sigma_0(0)) + \chi_0 \ ,$$

χ_0 being sufficiently small in suitable sense. It was proved that if the spectrum of the linearization of equation (1) on the soliton $w(., \sigma_0(t))$ has the simplest structure in some natural sense, the asymptotic behavior of ψ as $t \to +\infty$ is given by the formula (in L_2-norme) :

$$\psi = w(\cdot, \sigma_+(t)) + \exp(-il_0 t) f_+ + o(1) \ ,$$

here $\sigma_+(0)$ is close to $\sigma_0(0), \ell_0 = -\partial_x^2, f_+ \in L_2(\mathbb{R})$ and is sufficiently small.

In this work the following situation will be considered. Assume that one has a set of solitons $w(x, \sigma_{0_j}(t)), j = 1, \cdots, N$, their initial velocities v_{0j} being ordered in the same way as the initial coordinates :

$$v_{01} < v_{02} < \cdots < v_{0N} \ , \ c_{01} < c_{02} < \cdots < c_{0N} \ .$$

Moreover, assume that the initial coordinates are sufficiently well separated. Consider the Cauchy problem for equation (1) with the initial data

$$\psi_{|t=0} = \sum_{j=1}^{N} w(\cdot, \sigma_{0_j}(0)) + \chi_0 \ ,$$

where χ_0 is sufficiently small. If all the linearizations constructed independently from the solitons $w(x, \sigma_{oj})$ satisfy the spectral conditions used in the case of one soliton, one expects that as $t \to +\infty$

$$\psi = \sum_{j=1}^{N} w(\cdot \sigma_{+j}(t)) + \exp(-il_0 t) f_+ + o(1), \tag{2}$$

$\sigma_{+j}(0)$ being close to $\sigma_{0j}(0)$ and $f_+ \in L_2(\mathbb{R})$ being small.

In this note we prove formula (2) in the case of large relative velocities $v_{0j} - v_{0j+1}$.

1. Preliminary facts and formulation of the result.

1.1. Solitons consider the nonlinear Schrödinger equation (1). Assume that

i) F is a smooth ($\in C^\infty$) real-valued function obeying to the estimate

$$F(\xi) \geq -F_1\xi^q \ , \ F_1 > 0, \xi \geq 1, q < 2 \ ;$$

ii) point $\xi = 0$ is a root of F :

$$F(\xi) = F_2\xi^p(1 + 0(\xi)), p > 0 \ .$$

The further assumptions about F will be given in terms of the function

$$\mathcal{U}(\varphi, \alpha) = -\frac{1}{8}\alpha^2\varphi^2 - \frac{1}{2}\int_0^{\varphi^2} F(\xi)d\xi \ .$$

It is assumed that

iii) for α from some interval, $\alpha \in A \subset \mathbb{R}_+$, the function $\varphi \mapsto \mathcal{U}(\varphi, \alpha)$ has a positive root and the smallest positive root $\varphi_0 = \varphi_0(\alpha)$ is simple : $\mathcal{U}_\varphi(\varphi_0, \alpha) \neq 0$.

Under assumptions ii) - iii) there exists an unique even positive solution $\varphi(y)$ of the equation

$$\varphi_{yy} = -\mathcal{U}_\varphi$$

vanishing at infinity : $\varphi(y, \alpha) \sim \varphi_0 \exp(-\frac{\alpha|y|}{2}), |y| \to \infty$.

The functions $w(x, \sigma) = \exp(-i\beta + i\frac{vx}{2})\varphi(x - \beta, \alpha)$, $\sigma = (\beta, \omega, b, v), \omega = -\frac{\alpha^2}{4} + \frac{v^2}{4}$ will be called soliton sates.

If $\sigma = \sigma(t)$ is a solution of the Hamilton system

$$\beta' = \omega \ , \ \omega' = 0 \ , b' = v \ , v' = 0 \qquad (1.1)$$

the function $w(x, \sigma(t))$ is a solution of equation (1) called solitary wave, or simpler, soliton.

2.1. Linearization on the soliton.

Consider the linearization of equation (1) on the soliton $w(x, \sigma(t))$:

$$\psi \sim w + \chi$$

$$i\chi_t = (-\partial_x^2 + F(|w|^2))\chi + F'(|w|^2)(|w|^2\chi + w^2\bar{\chi}) .$$

Instead of χ introduce the function $\vec{f}$:

$$\vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \chi(x,t) = \exp(i\phi) f(y,t) ,$$

$$\phi = -\beta(t) + \frac{vx}{2} , \quad y = x - b(t) .$$

It satisfies the equation

$$i\vec{f}_t = L(\alpha)\vec{f} , \ L(\alpha) = L_0(\alpha) + V(\alpha), L_0(\alpha) = (-\partial_y^2 + \frac{\alpha^2}{4})\sigma_3,$$

$$L(\alpha) = (F(\varphi^2) + F'(\varphi^2)\varphi^2)\sigma_3 + iF'(\varphi^2)\varphi^2\sigma_2$$

σ_2, σ_3 are the standart Pauli matrices :

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Consider $L(\alpha)$ as a linear operator in $L_2(\mathbb{R} \to \mathbb{C}^2)$ defined on the domain where $L_0(\alpha)$ is selfadjoint. The continuous spectrum of $L(\alpha)$ lies on two half-axis $(-\infty, -E_0], [E_0, \infty)$, $E_0 = \frac{\alpha^2}{4}$. $L(\alpha)$ can have also finite and finite dimensional discrete spectrum on real axis and on imaginary one. The point $E = 0$ is always a point of the discrete spectrum. One can indicate two eigenfunctions

$$\vec{\xi}_1 = -i\varphi \begin{pmatrix} 1 \\ -1 \end{pmatrix} , \vec{\xi}_3 = -\varphi_y \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

and two adjoint functions

$$\vec{\xi}_2 = -\frac{2}{\alpha}\varphi_\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \vec{\xi}_4 = \frac{i}{2} y\varphi \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

They satisfy the relations

$$L\vec{\xi_1} = L\vec{\xi_3} = 0 \ , L\vec{\xi_2} = i\vec{\xi_1} \ , L\vec{\xi_4} = i\vec{\xi_3} \ .$$

Generically the equation $L(\alpha)\psi = \pm E_0(\alpha)\psi$ do not have solutions bounded at infinity. If, nevertheless, they exist, the points $\pm E_0$ are called resonances.

1.3 Description of the problem.

To simplity the computations we will discuss only the case of two solitons. Consider the Cauchy problem for equation (1) with the initial data

$$\psi_{|t=0} = \psi_0 \ , \ \psi_0 = w(\cdot, \sigma_{01} + w(\cdot, \sigma_{02}) + \chi_0 \ , \tag{1.2}$$

$$\sigma_{0i} = (\beta_{0i}, \omega_{0i}, b_{0i}, v_{0i}), \omega_{0i} = \frac{1}{4}(v_{0i}^2 - \alpha_{0i}^2), i = 1, 2 \ .$$

Let $v_{01} > v_{02}, b_{01} > b_{02}$. Without lost of generality one can suppose that $v_{01} = -v_{02} = v_0, b_{01} = -b_{02} = b_0$, $v_0 > 0, b_0 > 0$. Our goal is to describe the asymptotic behavior of the solution ψ as $t \to +\infty$.

Assume that the following conditions hold

T_1) the norm

$$N = (1 + v_0)^\beta (1 + b_0)^\delta (\|\chi_0'\|_2 + \|\rho_0(x)\chi_0\|_2), \beta = 3/4, \delta = 3/4 \ ,$$

$$\rho_0(x) = \theta(x)(1 + (x - b_0)^2) + \theta(-x)(1 + (x + b_0)^2) \ ,$$

is sufficiently small. [1]

Here $\theta(x)$ is the standard Heviside function.

T_2) $n = v_0 \exp(-b_0^{1/2})$ is sufficiently small [1]

T_3) v_0 is sufficiently large [1]

T_4) $E = 0$ is the only point of the discrete spectrum of $L(\alpha_{0i}), i = 1, 2$, and the dimention of the corresponding root subspace is equal four.

T_5) The points $\mp E_0(\alpha_{0i})$ are not resonances.

T_6) The function F is a polynomial [2] and $p > 4$.

[1] sufficiently small (large) "assumes the constants that depend only on $\alpha_{0i}, i = 1, 2$.

[2] This assumption is not essential, and is made only to simplity the computations.

Then there exist $\sigma_{+i}, i = 1,2$ and $f_+ \in L_2 \cap L_\infty$ such that

$$\psi = w(\cdot, \sigma_{+1}(t)) + w(\cdot, \sigma_{+2}(t)) + \exp(-il_0 t) f_+ + o(1) \tag{1.3}$$

as $t \to +\infty$.

Here $\sigma_{+i}(t)$ is the trajectory of the system (1.1) with the initial data $\sigma_{+i}(0) = \sigma_{+i}, o(1)$ assumes the L_2-norm. Moreover σ_{+i} is sufficiently close to σ_{0i} and f_+ is sufficiently small.

It is worth mentioning that for α sufficiently close to α_{0i} the operator $L(\alpha)$ also satisfies the conditions $T_2), T_3)$.

1.4. The plan of the work.

The main idea will repeat the main idea of [BP 1,2], but there will appear some technique modifications. Following [BP1,2] we introduce some new coordinates for the description of the solution with initial data (1.2). The new coordinates possess the important property : for all the time t they admit only small deviations from their initial values. We consider this coordinates in Section 2. We conclude this section by a system of equation for the new coordinates. In section 4 we prove that the new coordinates indeed admit only the small deviations. For this purpose we use a method of majorants. In 5 we derive asymptotic formula (1.3). Section 3 contains a list of some auxiliary estimates which are used in 4.

The list of publications related to the problem is essentially the same as in $[BP1]$, so we again refer our reader to this work.

2. Separation of the motions

2.1. Orthogonality conditions.

Write the solution ψ of the Cauchy problem (1), (1.2) as the sum

$$\psi = \sum_{i=1,2} w(x, \sigma_i(t)) + \chi(x,t) \ , \tag{2.1}$$

$$w(x, \sigma_i(t)) = \exp(i\phi_i)\varphi(y_i, \alpha), \phi_i = -\beta_i(t) + \frac{1}{2} v_i(t) x \ ,$$

$$y_i = x - b_i(t) \ , \ \alpha_i = \alpha_i(t) \ , \ i = 1,2 \ .$$

Here $\sigma_i(t) = (\beta_i(t), w_i(t), b_i(t), v_i(t))$ is an arbitrary trajectory in the set of admissible values of the parameters, it is not a solution of (1.1) in general.

The decomposition (2.1) can be fixed by the conditions of orthogonality :

$$\langle \vec{\chi}(t), \sigma_3 \vec{w}_\sigma(\cdot, \sigma_i(t)) \rangle = 0 \ , \tag{2.2}$$

where $\vec{\chi} = \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w \\ \bar{w} \end{pmatrix}$, and $\langle \cdot, \cdot \rangle$ is the inner produnct in $L_2(\mathbb{R} \to \mathbb{C}^2)$. We remark that

$$\vec{w}_\beta = \exp(i\sigma_3\phi)\vec{\xi}_1 \ , \ \vec{w}_\omega = \exp(i\sigma_3\phi)\vec{\xi}_2$$

$$\vec{w}_b = \exp(i\sigma_3\phi)\vec{\xi}_3 \ , \ \vec{w}_v = \exp(i\sigma_3\phi)(\vec{\xi}_4 - \frac{v}{2}\vec{\xi}_2) \ .$$

Thus the conditions (2.2) can be written in the form

$$\langle \vec{f}_i(t), \sigma_3 \vec{\xi}_j(\alpha_i(t)) \rangle = 0 \ , i = 1, 2, j = 1, \cdots, 4 \ ,$$

$$\vec{\chi}(x,t) = \exp(i\phi_i\sigma_3)\vec{f}_i(y_i, t) \ , \vec{\xi}_j(\alpha_i(t)) = \vec{\xi}_j(y_i, \alpha_i(t)) \tag{2.3}$$

Geometrically these conditions mean that that for each t the function $\vec{f}_i(t)$ belongs to the subspace of the continuous spectrum of $L(\alpha_i(t))$.

2.2. Differential equations.

Write down the system for σ_i and χ in more explicit form. Let us pass from σ to a new system of parametres (γ, ω, c, v) :

$$\beta(t) = \int_0^t \omega(\tau)d\tau + \gamma(\tau) \ , \ b(t) = \int_0^t v(\tau)d\tau + c(\tau) \ .$$

In term of the new variables (1.1) takes the form

$$\gamma' = 0 \ , \ \omega' = 0 \ , c' = 0 \ , v' = 0 \ .$$

Below σ_i will also denote the vector $(\gamma_i, \omega_i, c_i, v_i)$.

We rewrite (1) as an equation for $\vec{\chi}$:

$$i\vec{\chi}_t = H(\vec{\sigma}(t))\vec{\chi} + N_1(\vec{\sigma}, \vec{\chi}), \tag{2.4}$$

where

$$\vec{\sigma}(t) = (\sigma_1(t), \sigma_2(t)) \in \mathbb{R}^8 \ ,$$

$$H(\vec{\sigma}) = -\partial_x^2\sigma_3 + \sum_{i=1,2} W(\sigma_i) \ , \tag{2.5}$$

$$W(\sigma) = (F(|w|^2) + F'(|w|^2)|w|^2)\sigma_3 +$$
$$+F'(|w|^2)\begin{pmatrix} 0 & w^2 \\ -\bar{w}^2 & 0 \end{pmatrix}, w = w(x,\sigma) \ .$$

The nonlinearity N_1 is given by the following expression

$$N_1 = N + \sum_{=1,2} e^{i\sigma_3\phi_i} i\ell(\sigma_i)\vec{\xi}_1(y_i, \alpha_i) \ , \tag{2.6}$$

$$N = F(|w_1 + w_2 + \chi|^2)\begin{pmatrix} w_1 + w_2 + \chi \\ -\bar{w}_1 - \bar{w}_2 - \bar{\chi} \end{pmatrix} -$$
$$-\sum_{i=1,2}\left(F(|w_i|^2)\begin{pmatrix} w_i \\ -\bar{w}_i \end{pmatrix} + W(\sigma_i)\vec{\chi}\right), w_i = w(x,\sigma_i) \ ,$$
$$\ell(\sigma) = (bv' - \gamma) + \frac{v'}{2}(x-b) + ic'\sigma_3\partial_x - i\alpha'\sigma_3\partial_\alpha \ .$$

Substitute the expression for $\vec{\chi}_t$ from (2.4) into the derivative of the orthogonality conditions :

$$A(\alpha i)\vec{\lambda}_i = \vec{g}_i \quad i = 1,2 \tag{2.7}$$

Here

$$A(\alpha) = -\frac{i}{2}\{\langle\vec{\xi}_i(\alpha), \sigma_3\vec{\xi}_j(\alpha)\rangle\}_{i,j=1}^4,$$
$$\lambda_i = \left(\gamma_i' - \frac{1}{2}b_i v_i' \ , \ \omega_i' - \frac{1}{2}v_i v_i' \ , \ c_i' \ , \ v_i'\right) \ ,$$
$$g_{ij} = \langle N + ie^{i\sigma_3\phi_{\hat{i}}}\ell(\sigma_{\hat{i}})\vec{\xi}_1(y_{\hat{i}}, \alpha_{\hat{i}}), \sigma_3 e^{i\sigma_3\phi_i}\vec{\xi}_j(y_i,\alpha_i)\rangle +$$
$$+\langle\sigma_3\vec{f}_i \ , \ \ell(\sigma_i)\sigma_3\vec{\xi}_j(\alpha_i)\rangle \ , j = 1,\cdots,4 \ , \hat{i} = 3 - i \ , i = 1,2$$

One can obtain the explicit expression for the matrix A :

$$A(\alpha) = \begin{pmatrix} 0 & e & 0 & 0 \\ -e & 0 & 0 & 0 \\ 0 & 0 & 0 & -n \\ 0 & 0 & n & 0 \end{pmatrix}, n = -\frac{1}{4}\|\varphi\|_2^2 \ , e = \frac{4}{\alpha}\frac{dn}{d\alpha} \ .$$

Under assumption T_4), $\det A(\alpha_{0i}) \neq 0$ (see [BP 1]).

On principe system (2.7) can be solved with respect to the derivatives and, jointly with equation (2.4), constitues the complete system for $\vec{\sigma}, \vec{\chi}$:

$$\vec{\sigma}' = G(\vec{\sigma}, \vec{\chi}) \ , \tag{2.8}$$

$$i\vec{\chi}_t = H(\vec{\sigma}) + N_1(\vec{\sigma}, \vec{\chi}) \tag{2.9}$$

Equation (2.8) is not a complete equivalent of conditions (2.2) = (2.3). To get the equivalence one has to add to (2.8) conditions (2.2) = (2.3) at the time-moment $t = 0$:

$$\langle \vec{\chi}_0 \ , \ \sigma_3 \vec{w}_\sigma(\cdot, \sigma_{0i}) \rangle = 0 \ , \ i = 1, 2. \tag{2.10}$$

Generally these conditions are not satified by the given decomposition (1.2) of the initial data ψ_0. But if χ_0 is sufficiently small and the solitons are sufficiently well separated it is possible to reconstruct decomposition (1.2) in order to satisfy (2.10) (see [BP1]). So one can assume that (1.2) obeys to (2.10).

2.3. Equations on the finite interval.

Following [BP1] we consider equations equations (2.8.9) on some finite interval $[0, t_1]$ and later investigate the limite $t_1 \to \infty$. On the interval $[0, t_1]$ one can pick out the leading term of (2.9) in the form

$$i\vec{\chi}_t = H(\vec{\sigma}_1(t))\vec{\chi} \tag{2.11}$$

where $\vec{\sigma}_1(t) = (\sigma_{11}(t), \sigma_{12}(t))$ is the solution of (1.1) with the initial data $\vec{\sigma}_1(t_1) = \vec{\sigma}(t_1)$.

Rewrite full equation (2.9) in order to get the operator $H(\vec{\sigma}_1)$ as the main term of the right side :

$$i\vec{\chi}_t = H(\vec{\sigma}_1)\vec{\chi} + D \ , \tag{2.12}$$
$$D = N + \sum_{i=1,2} W(\sigma_i) - W(\sigma_{1i}) \ .$$

3. Preliminary estimates.

We give here without proof a list of estimates for the solution of nonlinear equation (1) and of linear equation (2.11). These estimates for the nonlinear evolution can be found in [GV1,2]. The estimates of the linear evolution are enough transparent and can be proved by means of simple (but unfortunately not short) computations.

Let the initial data ψ_0, see (1.2) belongs to H^1, H^1 being the standard Sobolev-space. Then the Cauchy problem has an unique solution $\psi(\cdot,t)\in C(\mathbb{R}_+\to H^1)$ which satisfies the conservation laws

$$\int dx|\psi|^2=\text{const},\ \int dx[\frac{1}{2}|\psi_x|^2+\mathcal{U}_0(|\psi|^2)]=\text{const}, \tag{3.1}$$

$$\mathcal{U}_0(\xi)=\frac{1}{2}\int_0^\xi F(s)ds\ .$$

Moreover, if $\|(1+|x|)\psi_0\|_2<\infty$ then the solution ψ also has the finite, but growing in time, similar norm and

$$\int dx|(x+2it\partial_x)\psi|^2+4t^2\int\mathcal{U}_0(|\psi|^2)dx-4\int_0^t dss\int dxG(|\psi|^2)=$$

$$=\text{const}\ ,G(\xi)=6\mathcal{U}_0(\xi)-\xi F(\xi)\ . \tag{3.2}$$

Consider the linear equation

$$i\vec{\chi}_t=H(\vec{\sigma}(t))\vec{\chi}\ , \tag{3.3}$$

$$\vec{\sigma}=(\sigma_1,\sigma_2)\ ,$$

where $\sigma_i(t)=(\beta_i(t),\omega_i(t),b_i(t),v_i)$ satisfies the system (1.1).

If T_3) holds (with $v=\frac{v_1-v_2}{a}>0$ instead of v_0) one can construct eight solution $\vec{\rho}_{ij}(x,t), i=1,2,\ j=1,\cdots,4$, of (3.3) with the following asymptotic behavior as $t\to+\infty$:

$$\|\vec{\rho}_{ij}(x,t,\vec{\sigma})-e^{i\sigma_3\phi_i}\vec{\xi}_j(y_i,\alpha_i)\|_X=0(v^2e^{-\gamma b(t)}),j=1,3 \tag{3.4}$$

$$\|\vec{\rho}_{ij}(x,t,\vec{\sigma})-e^{i\sigma_3\phi_i}(\vec{\xi}_j(y_i,\alpha_i)+t\vec{\xi}_{j-1})\|_X=0(v^2e^{-\gamma b(t)}),j=2,4$$

$$b(t)=b_1(t)-b_2(t)\ ,\ \gamma>0\ .$$

Here $\|\ .\ \|_X$ assumes the following norm

$$\|f\|_X=v^3\|(1+|x|^3)f\|_2+v^2\|(1+x^2)f'\|_2+v\|(1+|x|)f''\|_2+\|f'''\|_2\ .$$

Let $N(t)$ be the subspace generated by $\vec{\rho}_{ij}(\cdot,t)$. Introduce the projection operators $P^d(t,\vec{\sigma}),P(t,\vec{\sigma})$:

$$\text{Ker}\ P^d(t)=L_2\ominus\sigma_3N(t)\ ,\ \text{Im}\ P^d(t)=N(t),P=I-P^d\ .$$

The existence of P is guaranteed by the assumption T_4) and estimates (3.4).

Let $\mathcal{U}(t,\tau,\sigma)$ be the evolution operator corresponding to (3.3), then the following estimates are true :

$$\|\mathcal{U}(t,\tau)P(\tau)h\|_2 \le c\|h\|_2 \ ,$$

$$\|\mathcal{U}(t,\tau)P(\tau)h\|_\infty \le c \begin{cases} \|h\|_{H^1} \\ (t-\tau)^{-1/2}(\|h\|_2 + \|h\|_{1,1}) \ , \end{cases}$$

$$\|\rho(x-b_i(t))\mathcal{U}(t,\tau)P(\tau)h\|_2 \le$$

$$\le \frac{c}{(1+t-\tau)^{1/2}} \quad \frac{b(\tau)}{b(\tau)+t-\tau}\left[(1+ve^{-\gamma b(\tau)})\|h\|_2 + \right.$$

$$\begin{cases} \|h\|_{2,2} \\ \|h\|_{2,1} + \|h\|_{1,2} \end{cases} , i=1,2, t \ge \tau \ge 0 \ .$$

Here $\rho(y) = \frac{1}{(1+|y|)^\nu}, \nu > 4.5$,

$$\|h\|_{\kappa,e} = \|\theta(x-b_c)(1+|x-b_1|^\kappa)h\|_e +$$

$$+ \|\theta(b_c - x)(1+|x-b_2|^\kappa)h\|_e \ ,$$

$$b_c(\tau) = \frac{b_1(\tau)+b_2(\tau)}{2} \ ,$$

constants c depend only on $\alpha_i, b(0), v$ and are bounded on a finite vicinity of the point $\alpha_i = \alpha_{0i}, b(0) = +\infty, v = +\infty$.

4. Estimates of the majorants.

4.1. Estimates of the solitons parameters.

Introduce a natural system of norms for the components of the solution ψ.

$$M_0(t) = \sum_{i=1,2} \|\lambda_i(t)\| \ , M_1(t) = \sum_{i=1,2} \|\rho(y_i)\chi\|_2, M_2(t) = \|\chi\|_\infty \ .$$

These norms generate the system of majorants

$$\mathbb{M}_0(t) = \sup_{\tau \le t} \varepsilon^{-1}(1+\tau)(1+\varepsilon\tau)^{2\beta} M_0(\tau) \ ,$$

$$\mathbb{M}_1(t) = \sup_{\tau \le t} (1+\tau)^{1/2}(1+\varepsilon\tau)^\beta M_1(\tau)$$

$$\mathbb{M}_2(t) = \sup_{\tau \le t} (1+\tau)^{1/2} M_2(\tau) \ , \ \varepsilon = b_0^{-1}, \beta = 3/4$$

Finally, set $\mathbb{M}_j = \mathbb{M}_j(t_1)$.

From the definition of $\mathbb{M}_0(t)$ one has

$$|\alpha_i(t) - \alpha_{0i}| \ , |v_i(t) - v_{0i}| \ , |c_i(t) - b_{0i}| \le c\mathbb{M}_0(t) \ ,$$

$$|b_i(t) - b_{1i}(t)| \le c\mathbb{M}_0(\varepsilon t + 1)^{1-2\beta}$$

$$|b_i(t) - b_{0i}(t)| \le c\mathbb{M}_0(t)(1+t), i = 1,2 \ .$$

These estimates and relation (2.7) lead more or less directly to the inequality

$$\mathbb{M}_0 \le W(\mathbb{M})(e^{-\gamma b_0} + \mathbb{M}_0\mathbb{M}_1 + \varepsilon^{-1}\mathbb{M}_1^2) \ , \gamma > 0 \ . \tag{4.1}$$

Here $W(\mathbb{M})$ is a function of $\mathbb{M}_0, \cdots, \mathbb{M}_2$, which are bounded on some finite vicinity of the point $\mathbb{M}_j = 0$ and, may be acquire infinite value $+\infty$ out some larger vicinity. It depends only on $\alpha_{0i}, i = 1,2$ and it is possible to give an explicit expression for it, but, in fact this expression is useless for our aims.

4.2. Estimates of the dispersive part of χ.

Write the solution χ of (2.12) as the sum $\vec{\chi} = \vec{h} + \vec{k}$ of the projections on the subspaces corresponding to the "continuous" and the "discrete spectra" of

$$H(\vec{\sigma}_1(t)) :$$

$$\vec{h} = P_1\vec{\chi} \ , \ \vec{k} = P_1^d\vec{\chi} \ , P_1 = P(t, \vec{\sigma}_1) \ .$$

Using (2.12) one can write down the following integral representation for $\vec{h}$:

$$\vec{h} = \mathcal{U}_1(t,0)\vec{h}_0 - i\int_0^t \mathcal{U}_1(t,\tau)P_1(\tau)Dd\tau \tag{4.2}$$

Here $\mathcal{U}_1(t,\tau)$ is the propagator corresponding to $H(\vec{\sigma}_1(t))$ and $\vec{h}_0 = P_1(0)\vec{\chi}_0$.

In order to estimate the integral term of (4.2) one represent D as the sum

$$D = D_1 + D_2 + D_3 \ ,$$

$$D_1 = \sum_{i=1,2} ie^{i\sigma_3\phi_i}\ell(\sigma_i)\vec{\xi}_1(y_i, \alpha_i) \ ,$$

$$D_3 = F(|\chi|^2)\sigma_3\vec{\chi} \ ,$$

D_2 being the remainder.

The direct computations show

$$\|D_1\|_{2,2} \le W(\mathbb{M})(1+t)(1+\varepsilon t)^{2\beta}\varepsilon \mathbb{M}_0 \ , \tag{4.3}$$

$$\|D_2\|_{2,2} \le W(\mathbb{M})[e^{-\gamma b_0(t)} + \mathbb{M}_0\mathbb{M}_1(1+t)^{1/2}(1+\varepsilon t)^{3\beta-1}+$$
$$+\mathbb{M}_1\mathbb{M}_2(1+t)(1+\varepsilon t)^{\beta}]. \tag{4.4}$$

In order to estimate D_3 one has to take into account conservation laws (3.1-2).

As result one can prove that D_3 admits the following estimates

$$\begin{array}{ll} \|D_3\|_2 & \le W(\mathbb{M})(1+N)\mathbb{M}_2^{2p}(1+t)^{-p} \ , \\ \|D_3\|_{1,1} & \le W(\mathbb{M})(1+N)^2\mathbb{M}_2^{2p-1}(v_0t+b_0)(1+t)^{1/2-p} \ , \\ \|D_3\|_{1,2} & \le W(\mathbb{M})(1+N)^2\mathbb{M}_2^{2p}(v_0t+b_0)(1+t)^{-p} \ , \\ \|D_3\|_{2,1} & \le W(\mathbb{M})(1+N)^2\mathbb{M}_2^{2p-1}(v_0t+b_0)^2(1+t)^{1/2-p} \ , \end{array} \tag{4.5}$$

where N is N-norm of χ_0.

Using equation (4.2) and combining estimates (4.3-4), (4.5) one can get finally

$$\|h\|_\infty(1+t)^{1/2} \le W(\mathbb{M})(N+1)^2[v_0^{-\beta}\varepsilon^{\delta}N + e^{-\gamma b_0} + \varepsilon^{1/2}\mathbb{M}_0$$
$$+\varepsilon^{-1/2}\mathbb{M}_0\mathbb{M}_1 + \varepsilon^{-1/2}\mathbb{M}_1\mathbb{M}_2 + \varepsilon^{-1}v_0\mathbb{M}_2^{2p-1}] \ ,$$
$$\|\rho(x-b_i(t))h\|_2(1+t)^{1/2}(1+\varepsilon t)^{\beta} \le W(\mathbb{M})(N+1)^2(1+n)\times$$
$$\times[v_0^{-\beta}\varepsilon^{\delta}N + e^{-\gamma b_0} + \varepsilon \ell n\frac{1}{3}v_0^{\beta}\mathbb{M}_0 + \varepsilon^{-1/2}v_0^{\beta}\mathbb{M}_0\mathbb{M}_1$$
$$+\ell n\frac{1}{\varepsilon}v_0^{\beta}\mathbb{M}_1\mathbb{M}_2 + v_0^{2+\beta}\varepsilon^{-2-\beta}\mathbb{M}_2^{2p-1}] \ . \tag{4.6}$$

Just here it is important to assume $p > 4$.

4.3. Closing of the estimates.

Here we will obtain the estimates for $\vec{k}$ and it will close the system of the inequalities for the majorants.

The 8-dimentional component $\vec{k}$ can be expressed in term of χ as follows

$$\langle \vec{k}, \sigma_3\rho_{ij}(\vec{\sigma}_1)\rangle = \langle \vec{\chi}, \sigma_3\rho_{ij}(\vec{\sigma}_1)\rangle, i=1,2, j=1,\cdots,4 \ .$$

Since

$$\vec{k} = \vec{k}_1 + \vec{k}_2 \ , \ \vec{k}_i = \sum_{j=1}^{4} k_{ij}\rho_{ij}(\vec{\sigma}_1)$$

and

$$\{\langle \rho_{ij}(\vec{\sigma}_1), \sigma_3 \rho_{\kappa\ell}(\vec{\sigma}_1)\rangle\}_{\substack{i,\kappa=1,2\\ j,\ell=1,\cdots,4}} = -2i \begin{pmatrix} A(\alpha_{11}) & 0 \\ 0 & A(\alpha_{12}) \end{pmatrix}$$

a system for the coefficients k_{ij} arises, having the form

$$A(\alpha_{1i})\mathbb{K}_i = \ell_i \ , \ \mathbb{K}_i = (k_{i1}, k_{i2}, k_{i3}, k_{i4}), i = 1,2$$

$$\ell_i = (\langle \vec{\chi}, \sigma_3 \rho_{ij}(\vec{\sigma}_1)\rangle)_{j=1,\cdots,4} \tag{4.7}$$

It is clear that the main term in (4.7) is given by the system

$$A(\alpha_{1i})\mathbb{K}_i^0 = \ell_i^0 \ , \ \mathbb{K}_i^0 = (k_{i1}^0, k_{i2}^0, k_{i3}^0 \ , k_{i4}^0) \ ,$$

$$\ell_i^0 = \Big(\langle \vec{\chi} \ , \ \sigma_3 e^{i\sigma_3\phi_{1i}} \vec{\xi}_j(\alpha_{1i})\rangle\Big)_{j=1,\cdots,4,}$$

$$\phi_{1i} = -\beta_{1i}(t) + \frac{v_{1i}x}{2} \ .$$

Owing to the orthogonality conditions (2.2) one can rewrite ℓ_i^0 as follows

$$\ell_i^0 = \Big(\langle \vec{\chi}, \sigma_3(e^{i\sigma_3\phi_{1i}}\vec{\xi}_j(\alpha_{1i}) - e^{i\sigma_3\phi_i}\vec{\xi}_j(\alpha_i))\rangle\Big)_{j=1,\cdots,4,}$$

which leads to the estimate

$$||\mathbb{K}_i^0|| \leq W(\mathbb{M})\mathbb{M}_0\mathbb{M}_1(1+t)^{-1/2}(1+\varepsilon t)^{-\beta} \tag{4.8}$$

It is simple to check that the difference $\vec{k}_i - \vec{k}_i^0 \ , \ \vec{k}_i^0 = \sum_{j=1}^4 k_{ij} e^{i\sigma_3\phi}\xi_j(\alpha_{1i})$ admits the following estimate

$$||\vec{k}_i - \vec{k}_i^0||_\infty, ||\vec{k}_i - \vec{k}_i^0||_2 \leq W(\mathbb{M})(1+N)v_0^{-1}e^{-\gamma b_0(t)} \tag{4.9}$$

Now we can combine estimates (4.6, 8-9) for all the components of χ :

$$\mathbb{M}_1 \leq W(\mathbb{M})(N+1)^2(n+1)\Big[\varepsilon^{3/4}v_0^{-\beta}N + e^{-\gamma b_0} + \varepsilon^{1/2}\mathbb{M}_0 +$$

$$+ v_0^\beta\varepsilon^{-1/2}\mathbb{M}_0\mathbb{M}_1 + v_0^\beta\varepsilon^{-1/2}\mathbb{M}_1\mathbb{M}_2 + v_0^{2+\beta}\varepsilon^{-2-\beta}\mathbb{M}_2^{2p-1}\Big] \ ,$$

$$\mathbb{M}_2 \leq W(\mathbb{M})(N+1)^2\Big[v_0^{-\beta}\varepsilon^{3/4}N + e^{-\gamma b_0} + \varepsilon^{1/2}\mathbb{M}_0 + \varepsilon^{-1/2}\mathbb{M}_0\mathbb{M}_1$$

$$+ \varepsilon^{-1/2}\mathbb{M}_1\mathbb{M}_2 + v_0\varepsilon^{-1}\mathbb{M}_2^{2p-1}\Big] \ . \tag{4.10}$$

Introduce new scales :

$$\mathbb{M}_0 = v_0^{-2\beta}\varepsilon^{1/2}\widehat{\mathbb{M}}_0 \ , \ \mathbb{M}_1 = v_0^{-\beta}\varepsilon^{3/4}\widehat{\mathbb{M}}_1 , \mathbb{M}_2 = v_0^{-\beta}\varepsilon^{3/4}\widehat{\mathbb{M}}_2 \ .$$

Then inequalities (4.1,10) takes the form

$$\begin{aligned} \widehat{\mathbb{M}}_0 &\le W(\mathbb{M})(n + \widehat{\mathbb{M}}_0\widehat{\mathbb{M}}_1 + \widehat{\mathbb{M}}_1^2), \\ \widehat{\mathbb{M}}_1 &\le W(\mathbb{M})(N+1)^2(n+1)(n + N + \widehat{\mathbb{M}}_0 + \widehat{\mathbb{M}}_1\widehat{\mathbb{M}}_2 + \widehat{\mathbb{M}}_2^{2p-1}) \\ \widehat{\mathbb{M}}_2 &\le W(\mathbb{M})(N+1)^2(n + N + \widehat{\mathbb{M}}_0 + \widehat{\mathbb{M}}_1\widehat{\mathbb{M}}_2 + \widehat{\mathbb{M}}_2^{2p-1}) \ . \end{aligned}$$

So one has obtained a closed set of the inequalities for the majorants and can try to solve it. It is clear that if N and n are sufficiently small, then $\widehat{\mathbb{M}}_j, j = 0, \cdots, 2$ can belongs either a small vicinity of the point $\widehat{\mathbb{M}}_j = 0$ or some domain which distance from $(0,0,0)$ is limited from below uniformly with respect to N and n. Since all the norms $\mathbb{M}_j$ are continuous in t_1 and for $t_1 = 0$ are sufficiently small, only the first possibility can be realised. It means that the functions $\mathbb{M}_0, \mathbb{M}_1, \mathbb{M}_2$ admit the following estimates (uniformly in $t_1 \in \mathbb{R}_+$) :

$$\widehat{\mathbb{M}}_0 \le \mu(n,N)(n + N^2) \tag{4.11}$$

$$\widehat{\mathbb{M}}_j \le \mu(n,N)(n+N), j = 1,2 \tag{4.12}$$

$\mu(n,N)$ being a bounded function defined for small (n,N).

5. Scattering.

Estimate (4.11) for $\mathbb{M}_0$ shows that

$$\|\lambda_i\| \le \frac{\mu(n,N)N^2}{1+t}(\frac{\varepsilon}{1+\varepsilon t})^{3/2}, t \in \mathbb{R}_+,$$

which implies that all variables $\gamma_i, \omega_i, c_i, v_i, i = 1,2$, have limits $\gamma_{\infty i}, \omega_{\infty i}, c_{\infty i}, v_{\infty i}$ as $t \to +\infty$. So one can introduce the limiting trajectories $\sigma_{+i}(t)$:

$$\beta_{+i}(t) = \omega_{+i}t + \gamma_{+i}, \omega_{+i} = \omega_{\infty i}, \gamma_{+i} = \gamma_{\infty i} + \int_0^\infty (\omega_i(\tau) - \omega_{\infty i})d\tau \ ,$$

$$b_{+i}(t) = v_{+it} + c_{+i}, v_{+i} = v_{\infty i}, c_{+i} = c_{\infty i} + \int_0^\infty (v_i(\tau) - v_{\infty i})d\tau \ .$$

It is clear that

$$w(x, \sigma_i(t)) - w(x, \sigma_+(t)) = 0(t^{-1/2}) \tag{5.1}$$

in the space $L_2 \cap L_\infty$.

In order to study the asymptotical behavior of χ one can use the following integral representation

$$\vec{\chi} = \exp(-it\sigma_3\ell_0)\vec{\chi}_0 - i\int_0^t \exp[-i(t-\tau)\sigma_3\ell_0]N_2 d\tau \ ,$$

$$N_2 = N_1 + \sum_{i=1,2} W(\sigma_i(t)) \ ,$$

here $W(\sigma)$ and N_1 are given by the formulas (2.5-6)

From (4.3-5, 4.11-12) it follows that

$$N_2 = 0(t^{-5/4}) \ ,$$

in the space $L_2 \cap L_1$, which allows us to represent χ in the following form

$$\vec{\chi}(t) = \exp(-it\sigma_3\ell_0)\vec{f}_+ + \vec{\chi}_R(t) \ , \tag{5.2}$$

$$\vec{f}_+ = \vec{\chi}_0 - i\int_0^\infty \exp(i\tau\sigma_3\ell_0)N_2 d\tau \ ,$$

$$\vec{\chi}_R(t) = i\int_t^\infty \exp -i(t-\tau)\sigma_3\ell_0]N_a d\tau \ .$$

It is clear that

$$\|\chi_R\|_a = 0(t^{-1/4}) \ , \|\chi_R\|_\infty = 0(t^{-3/4}) \tag{5.3}$$

Combining this with (5.2.1) one gets finally

$$\psi = w(.,\sigma_{+1}(t)) + w(.,\sigma_{+2}(t)) + e^{-\text{ilot}}f_+ + R \ ,$$

where R admits the estimates

$$R = 0(t^{-1/4}) \text{ in } L_a - \text{norm}, \ \ R = 0(t^{-1/2}) \text{ in } L_\infty - \text{norm.}$$

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Université de Reims
Département de Mathématiques
Chemin de la Houssinière
F - 51 REIMS