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On the local well-posedness of the KP equations

N. Tzvetkov*

1 Introduction

Our goal here is to describe some recent results on the Kadomtsev-Petviashvili (KP) equations obtained in [13], [21].

The KP equations are two dimensional generalizations of the famous Korteweg-de Vries (KdV) equation. They occur in many physical contexts as "universal" models for the propagation of weakly nonlinear dispersive long waves which are essentially one-directional, with weak transverse effects. The soliton structure of the KdV equation is not broken down by the transverse perturbation and inverse scattering transform techniques could be applied to the Cauchy problem associated to the KP equations under appropriate decay assumptions on the initial data (cf. [25]). In addition (cf. [26]) there exists an infinite number of quantities which are conserved by the KP evolution. These conservation laws may be useful to obtain global solutions of the KP equations once a low regularity local (in time) well-posedness theory of the respective Cauchy problem is established. Here we will study this issue. The Cauchy problems for the KP equations read

$$(u_t + uu_x + u_{xxx})_x \pm u_{yy} = 0, \quad u(0, x, y) = \phi(x, y), \quad (t, x, y) \in \mathbb{R}^3. \tag{1}$$

The (+) sign corresponds to the "defocusing" KP-II equation, while the (−) sign corresponds to the "focusing" KP-I equation. In the context of water waves, the KP-II equation models long gravity waves with weak surface tension effects while the KP-I equation arises for capillary gravity waves, in the presence of strong surface tension effects. We can write (1) as integral equations

$$u(t, x, y) = U^{\pm}(t)(\phi(x, y)) - \int_0^t U^{\pm}(t - t')[u(t', x, y)u_x(t', x, y)]dt', \tag{2}$$

where $U^{\pm}(t) = \exp(-t(\partial_x^3 \pm \partial_x^{-1}\partial_y^2))$ are the unitary groups defining the free KP evolutions. The operators $U^{\pm}(t)$ are actually convolution operators

$$U^{\pm}(t)(\phi(x, y)) = (E^{\pm}(t) \star \phi)(t, x, y),$$

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where the convolution is with respect to x, y and $E^{\pm}(t)$ are defined by the oscillatory integrals

$$E^{\pm}(t) = c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(ix\xi + iy\eta + it(\xi^3 \mp \frac{\eta^2}{\xi})\right)d\xi d\eta.$$

One can prove (cf. [16]) that

$$|E^{\pm}(t)|_{L^{\infty}_{xy}} \lesssim |t|^{-1}. \tag{3}$$

Using (3) one derives the following set of Strichartz inequalities for the free KP evolutions

$$\|U^{\pm}(t)(\phi(x,y))\|_{L^q_t(L^r_{xy})} \lesssim \|\phi\|_{L^2}, \quad \frac{1}{q}+\frac{1}{r}=\frac{1}{2}, \quad 2 \le r < \infty.$$

We note that the above inequalities are the same as the Strichartz estimates for the two dimensional Schrödinger equation. They play an important role in the local well-posedness analysis for the KP equations. In the periodic case similar estimates are not available. However in [4] some versions of the Strichartz inequalities localized in the frequency spaces are used in the context of the KP-II equation posed on the two dimensional torus $\mathbb{T}^2$.

In order to motivate the choice of the functional space for the initial data in (1), we discuss some scale invariance properties of the KP equations. Note that if $u(t,x,y)$ solves (1) with initial data $\phi(x,y)$ then (1) is also solved by $u_\lambda(t,x,y) := \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$ with initial data $\phi_\lambda(x,y) = \lambda^2\phi(\lambda x, \lambda^2 y)$. In addition we have that for $(s_1, s_2) \in \mathbb{R}^2$

$$\|u_\lambda(t,\cdot,\cdot)\|_{\dot{H}^{s_1,s_2}(\mathbb{R}^2)} = \lambda^{s_1+2s_2+1/2}\|u(\lambda^3 t,\cdot,\cdot)\|_{\dot{H}^{s_1,s_2}(\mathbb{R}^2)},$$

where $\dot{H}^{s_1,s_2}(\mathbb{R}^2)$ are homogeneous anisotropic Sobolev spaces equipped with the norm

$$\|\phi\|_{\dot{H}^{s_1,s_2}(\mathbb{R}^2)} = \|(-\partial_x^2)^{s_1/2}(-\partial_y^2)^{s_2/2}\phi\|_{L^2_{x,y}}.$$

Therefore a natural set for the initial data of the KP equations are the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ equipped with the norm

$$\|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)} = \|(1-\partial_x^2)^{s_1/2}(1-\partial_y^2)^{s_2/2}\phi\|_{L^2_{x,y}}.$$

This talk is devoted to the following question :

For which $(s_1, s_2) \in \mathbb{R}^2$ can one solve (2) in a suitable functional space via the contraction mapping principle for data $\phi \in H^{s_1,s_2}(\mathbb{R}^2)$?

The answer to the above question is satisfactory enough in the KP-II context, while for the KP-I equation we have only negative results. Define the bilinear forms $B^{\pm}(u,v)$ as

$$B^{\pm}(u,v) := \frac{1}{2}\int_0^t U^{\pm}(t-t')\partial_x[u(t',x,y)v(t',x,y)]dt'.$$

Boundedness of the bilinear forms in some functional framework are an essential ingredient of the proof of local well-posedness results for (1) via a Picard fixed point argument. Now we state our result concerning the KP-II equation.

Theorem 1 (The KP-II case, cf. [4, 21])
Fix $s_1 > -1/3$, $s_2 \geq 0$ and a positive real number T. Then there exists a space X_T continuously embedded in $C([-T,T], H^{s_1,s_2}(\mathbb{R}^2))$ such that

$$\|U^+(t)(\phi(x,y))\|_{X_T} \lesssim \|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)}, \quad \phi \in H^{s_1,s_2}(\mathbb{R}^2) \tag{4}$$

and

$$\|B^+(u,v)\|_{X_T} \lesssim \|u\|_{X_T}\|v\|_{X_T}, \quad u,v \in X_T. \tag{5}$$

We note that the implicit constants in (4), (5) may depend on T. In addition by $\|U^+(t)(\phi(x,y))\|_{X_T}$ we mean the X_T norm of the restriction of the free evolution $U^+(t)(\phi(x,y))$ to $[-T,T] \times \mathbb{R}^2$.

Theorem 1 was first proved by Bourgain [4] for $s_1 \geq 0$ and $s_2 \geq 0$ (in [4] the same result is established in the periodic setting). The result of [4] was further improved in [21]. The proof performed in [21] is typical for the continuous case (problem posed on $\mathbb{R}^2$). In particular it uses the Strichartz inequalities and some "simple calculus arguments" due to Kenig, Ponce, Vega, first introduced in the KdV context (cf. [10]). A consequence of Theorem 1 is the following local well-posedness result.

Theorem 2 (cf. [21]) *Let $s_1 > -1/3$ and $s_2 \geq 0$. Then for any $\phi \in H^{s_1,s_2}(\mathbb{R}^2)$, there exist a positive $T = T(\|\phi\|_{H^{s_1,s_2}})$($\lim_{\rho\to 0} T(\rho) = \infty$) and a unique solution $u(t,x,y)$ of the KP-II equation with initial data ϕ on the time interval $[-T,T]$ satisfying $u \in X_T$.*

We note that, due to the L^2 conservation law, in the case $s_1 = s_2 = 0$ Theorem 2 implies the global well-posedness of the KP-II equation for data in $L^2(\mathbb{R}^2)$. Further improvements of this result are possible. More precisely in [8], [20], [24] global well-posedness of the KP-II equation with data below $L^2(\mathbb{R}^2)$ are obtained. The method of the proof, due to Bourgain (cf. [6]), is based on a decomposition of the initial data into low and high Fourier modes and the studying the high-low frequency interactions. We will not further discuss this issue here.

We now state our result concerning the KP-I equation.

Theorem 3 (The KP-I case, cf. [13])
Fix $(s_1, s_2) \in \mathbb{R}^2$ and a positive real number T. Then there exists no space X_T continuously embedded in $C([-T,T], H^{s_1,s_2}(\mathbb{R}^2))$ such that

$$\|U^-(t)(\phi(x,y))\|_{X_T} \lesssim \|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)}, \quad \phi \in H^{s_1,s_2}(\mathbb{R}^2) \tag{6}$$

and

$$\|B^-(u,v)\|_{X_T} \lesssim \|u\|_{X_T}\|v\|_{X_T}, \quad u,v\in X_T. \tag{7}$$

It is shown in [17] that the KP-I equation is locally ill-posed for data in $H^{s,0}(\mathbb{R}^2)$, $s<-1/2$ (the scaling exponent). The proof uses the existence of solitary wave solutions

$$u_c(t,x,y)=\frac{8c(1-\frac{c}{3}(x-ct)^2+\frac{c^2}{3}y^2)}{(1+\frac{c}{3}(x-ct)^2+\frac{c^2}{3}y^2)^2}$$

of the KP-I equation. The special form of $u_c(t,x,y)$ is needed in the proof as well as the scaling properties if $u_c(t,x,y)$ and its decay at infinity. As far as we know the idea of using the solitary wave solution to prove local ill-posedness results was first employed in [1, 2]. In general it gives ill-posedness below the scaling exponent. In a recent work of Kenig, Ponce, Vega (cf. [11]) it is shown that some additional symmetry (Galilean invariance) can be used in order to obtain ill-posedness results above the scaling exponent for several dispersive models such as the cubic 1D nonlinear Scrödinger equation.

Notations. We denote by $\widehat{\cdot}$ or $\mathcal{F}$ the Fourier transform. For any positive A and B the notation $A\lesssim B$ (resp. $A\gtrsim B$) means that there exists a positive constant c such that $A\leq cB$ (resp. $A\geq cB$). The notation $A\sim B$ means that $A\lesssim B\lesssim A$. The notation $a\pm$ means $a\pm\varepsilon$ for arbitrary small $\varepsilon>0$.

2 On the proof of Theorem 1

The proof of Theorem 1 is based on the Fourier transform restriction method introduced by Bourgain in [3]. Fix $s_1>-1/3$, $s_2\geq 0$. This method has a special advantage for the quadratic nonlinearities and has been first applied to the nonlinear Schrödinger equation and to the KdV equation. An essential element of the method is the introduction of Fourier transform restriction spaces strongly related to the symbol of the respective equation. We now describe these spaces in the context of the KP-II equation. For $b\in\mathbb{R}$, we define the space $X^b(\mathbb{R}^3)$, equipped with the norm

$$\|u\|_{X^b}=\|\langle\tau-\xi^3+\xi^{-1}\eta^2\rangle^b\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}(1+\langle\xi\rangle^{-\frac{1}{3}}\langle\tau-\xi^3+\xi^{-1}\eta^2\rangle^{\frac{1}{6}+})\widehat{u}(\tau,\xi,\eta)\|_{L^2_{\tau\xi\eta}}$$

We note there is an extra factor $(1+\langle\xi\rangle^{-\frac{1}{3}}\langle\tau-\xi^3+\xi^{-1}\eta^2\rangle^{\frac{1}{6}+})$ in the definition of the space X^b comparing to the corresponding spaces in the KdV case. This factor is typical in the KP context in order to obtain the needed integrability with respect to η, the Fourier dual variable of y, in the integral representation of the crucial bilinear estimate (cf. (ii) below). It is also used in order to deal with the small frequency interactions. Further for a positive T we define the space X^b_T equipped with the norm

$$\|u\|_{X^b_T}=\inf_{w\in X^b}\left\{\|w\|_{X^b}, \quad w(t,x,y)=u(t,x,y) \text{ on } [-T,T]\times\mathbb{R}^2\right\}.$$

The space X^b_T should be understood as a restriction space.
A one dimensional Sobolev embedding yields that for $b > 1/2$ the space X^b_T is continuously embedded in $C([-T,T], H^{s_1,s_2}(\mathbb{R}^2))$. Since

$$\|u\|_{X^b} = \|\langle\tau\rangle^b\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}(1+\langle\xi\rangle^{-\frac{1}{3}}\langle\tau\rangle^{\frac{1}{6}+})\mathcal{F}(U^+(-\cdot)(u))(\tau,\xi,\eta)\|_{L^2_{\tau\xi\eta}}$$

one can easily obtain that

$$\|U^+(t)(\phi(x,y))\|_{X^b_T} \lesssim \|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)}, \quad \phi \in H^{s_1,s_2}(\mathbb{R}^2).$$

In order to prove the bound for $B^+(u,v)$, we write $B^+(u,v) = (B_1 o B_2)(u,v)$, where $B_2(u,v) = \partial_x(uv)$ and $B_1(F) = \frac{1}{2}\int_0^t U^+(t-t')F(t')dt'$. The proof of Theorem 1 results from the next two statements:

(i) For $b = \frac{1}{2}+$ the linear operator B_1 is continuous from X^{b-1}_T to X^b_T.

(ii) For $b = \frac{1}{2}+$ the bilinear operator B_2 is continuous from $X^b_T \times X^b_T$ to X^{b-1}_T.

We note that (i) describes quite a general property which does not depend on the particular structure of $U^+(t)$. The analysis is one dimensional. If one replace $U^+(t)$ by the identity operator (no dispersion) then (i) becomes a trivial statement : an integration gains a derivative. On the other hand to prove (ii) one needs to use heavily the special dispersive relation and nonlinearity of the KP-II equation. An important role is played by a smoothing relation for the symbol of the linearized KP-II operator. This relation is needed to compensate the loss of a derivative in the nonlinear term (cf. [4]). The corresponding arithmetic relation in the KP-I context is an essential ingredient in the constructions involved in the proof of Theorem 3 (cf. the next section). After duality and polarization arguments (ii) can be seen as a weighted convolution multilinear estimate[1]. One uses the asymmetric structure of this estimate, arguments due to Kenig, Ponce, Vega and the Strichartz estimates injected into the framework of the Fourier transform restriction spaces X^b. An interpolation argument is needed in order to deal with the small frequencies interactions. We refer to [21] for the details of the proof. In [21] we also discuss the sharpness of the couple $(s_1,s_2) = (-1/3,0)$ with respect to the boundedness of the bilinear form B_2.

In [9], Kenig, Ponce, Vega have developed a method for studying the local well-posedness of the generalized KdV equation based on some estimates on the unitary group describing the free KdV evolution. This method differs from the Fourier transform restriction method of Bourgain in the choice of the functional space where one applies a fixed point argument (the space X_T). The choice of the spaces in [9] is made

[1]We refer to the recent paper [22] for a systematic study of such estimates related to the KdV, Schrödinger and wave equations.

according to the following three type of estimates : energy estimates, sharp version of Kato smoothing effect, maximal function inequalities. The method of [9] does not make a special advantage of the quadratic nonlinearities. Since the KP equations are generalizations of the KdV equation one may try to adapt some of the arguments of [9] in a KP context.

3 On the proof of Theorem 3

Suppose that a space X_T such that (6) and (7) hold exists. Take $u = v = U^-(\cdot)\phi$ in (7). Then using (6) one obtains

$$\|B^-(U^-(\cdot)\phi, U^-(\cdot)\phi)\|_{X_T} \lesssim \|\phi\|^2_{H^{s_1,s_2}(\mathbb{R}^2)}, \quad \phi \in H^{s_1,s_2}(\mathbb{R}^2).$$

Since X_T is supposed continuously embedded in $C([-T,T], H^{s_1,s_2}(\mathbb{R}^2))$ one has for a fixed $t > 0$

$$\left\| \int_0^t U^-(t-t') \left[(U^-(t')\phi(x,y))(U^-(t')\phi(x,y))_x \right] dt' \right\|_{H^{s_1,s_2}(\mathbb{R}^2)} \lesssim \|\phi\|^2_{H^{s_1,s_2}(\mathbb{R}^2)}. \qquad (8)$$

The goal is to show that (8) fails. Define $\phi_{\alpha,N}$ by its Fourier transform as

$$\widehat{\phi_{\alpha,N}}(\xi,\eta) = \alpha^{-\frac{3}{2}} \mathbb{1}_{D_1}(\xi,\eta) + \alpha^{-\frac{3}{2}} N^{-s_1-2s_2} \mathbb{1}_{D_2}(\xi,\eta).$$

Here the positive parameters N and α are such that $N \gg 1$, $\alpha \ll 1$ and D_1, D_2 are the rectangles in $\mathbb{R}^2_{\xi,\eta}$:

$$D_1 = [\alpha/2, \alpha] \times [-6\alpha^2, 6\alpha^2], \quad D_2 = [N, N+\alpha] \times [\sqrt{3}N^2, \sqrt{3}N^2 + \alpha^2].$$

We note that $\|\phi_{\alpha,N}\|_{H^{s_1,s_2}} \lesssim 1$. In order to compute the left hand-side of (8) we need the following lemma.

Lemma 1 *(cf. [13]) The following identity holds*

$$\int_0^t U^-(t-t')F(t',x,y)dt' = c \int_{\mathbb{R}^3} e^{ix\xi + iy\eta + it(\xi^3 + \frac{\eta^2}{\xi})} \frac{e^{it(\tau - \xi^3 - \frac{\eta^2}{\xi})} - 1}{\tau - \xi^3 - \frac{\eta^2}{\xi}} \widehat{F}(\tau,\xi,\eta) d\tau d\xi d\eta$$

whenever both terms are well defined.

Set

$$\chi(\xi,\xi_1,\eta,\eta_1) := 3\xi\xi_1(\xi-\xi_1) - \frac{(\eta\xi_1 - \eta_1\xi)^2}{\xi\xi_1(\xi-\xi_1)}.$$

We note that

$$\chi(\xi,\xi_1,\eta,\eta_1) = (\tau_1 - \xi_1^3 - \frac{\eta_1^2}{\xi_1}) + (\tau - \tau_1 - (\xi-\xi_1)^3 - \frac{(\eta-\eta_1)^2}{\xi-\xi_1}) - (\tau - \xi^3 - \frac{\eta^2}{\xi}).$$

Therefore, taking into account Lemma 1, one obtains that $\chi(\xi,\xi_1,\eta,\eta_1)$ appears as a phase function in the representation of the second KP-I iteration, with data $\phi_{\alpha,N}$. In this context the large set of zeros of $\chi(\xi,\xi_1,\eta,\eta_1)$ is crucial for our analysis. We remark that the function corresponding to $\chi(\xi,\xi_1,\eta,\eta_1)$ in the KP-II context is zero only for zero ξ frequencies. Using Lemma 1 we can write the expression

$$\int_0^t U^-(t-t')\left[(U^-(t')\phi_{\alpha,N}(x,y))(U^-(t')\phi_{\alpha,N}(x,y))_x\right]dt'$$

as

$$c(f_1(t,x,y)+f_2(t,x,y)+f_3(t,x,y)),$$

where f_1, f_2, f_3 are defined by their Fourier transforms with respect to (x,y)

$$\mathcal{F}_{(x,y)\mapsto(\xi,\eta)}(f_1)(t,\xi,\eta) = \frac{c\,\xi e^{it(\xi^3+\frac{\eta^2}{\xi})}}{\alpha^3}\int_{\substack{(\xi_1,\eta_1)\in D_1\\(\xi-\xi_1,\eta-\eta_1)\in D_1}}\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}d\xi_1 d\eta_1$$

$$\mathcal{F}_{(x,y)\mapsto(\xi,\eta)}(f_2)(t,\xi,\eta) = \frac{c\,\xi e^{it(\xi^3+\frac{\eta^2}{\xi})}}{\alpha^3 N^{2(s_1+2s_2)}}\int_{\substack{(\xi_1,\eta_1)\in D_2\\(\xi-\xi_1,\eta-\eta_1)\in D_2}}\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}d\xi_1 d\eta_1$$

$$\begin{aligned}\mathcal{F}_{(x,y)\mapsto(\xi,\eta)}(f_3)(t,\xi,\eta) &= \frac{c\,\xi e^{it(\xi^3+\frac{\eta^2}{\xi})}}{\alpha^3 N^{s_1+2s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_1\\(\xi-\xi_1,\eta-\eta_1)\in D_2}}\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}d\xi_1 d\eta_1\\ &+ \frac{c\,\xi e^{it(\xi^3+\frac{\eta^2}{\xi})}}{\alpha^3 N^{s_1+2s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_2\\(\xi-\xi_1,\eta-\eta_1)\in D_1}}\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}d\xi_1 d\eta_1\end{aligned}$$

It is easy to see that the supports of $\mathcal{F}_{(x,y)\mapsto(\xi,\eta)}(f_1)(t,\xi,\eta)$, $\mathcal{F}_{(x,y)\mapsto(\xi,\eta)}(f_2)(t,\xi,\eta)$, $\mathcal{F}_{(x,y)\mapsto(\xi,\eta)}(f_3)(t,\xi,\eta)$ are disjoint. Therefore

$$\|u_2(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \geq \|f_3(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)}.$$

The essential part of the proof is to minorize $\|f_3(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)}$. For that purpose we use the next lemma.

Lemma 2 *(cf. [13]) Let*

$$(\xi_1,\eta_1)\in D_1,\quad (\xi-\xi_1,\eta-\eta_1)\in D_2$$

or

$$(\xi_1,\eta_1)\in D_2,\quad (\xi-\xi_1,\eta-\eta_1)\in D_1$$

Then

$$|\chi(\xi,\xi_1,\eta,\eta_1)| \lesssim \alpha^2 N.$$

Choose now α and N so that $\alpha^2 N = N^{-\epsilon}$, where $0 < \epsilon \ll 1$. Then using Lemma 2 we obtain

$$\left| \frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)} - 1}{\chi(\xi,\xi_1,\eta,\eta_1)} \right| = |t| + O(N^{-\epsilon})$$

for $(\xi_1, \eta_1) \in D_1,\ (\xi - \xi_1, \eta - \eta_1) \in D_2$ or $(\xi_1, \eta_1) \in D_2,\ (\xi - \xi_1, \eta - \eta_1) \in D_1$. Hence

$$\|f_3(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \gtrsim \alpha^{\frac{3}{2}} N \sim N^{\frac{1}{4}-\frac{3\epsilon}{4}}.$$

Therefore

$$1 \gtrsim \|\phi_{\alpha,N}\|^2_{H^{s_1,s_2}(\mathbb{R}^2)} \gtrsim \|u_2(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \geq \|f_3(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \gtrsim \alpha^{\frac{3}{2}} N \sim N^{\frac{1}{4}-\frac{3\epsilon}{4}}$$

Contradiction for $N \gg 1$, $0 < \epsilon \ll 1$.

4 Related results

In [13] the local well-posedness of the KP-I equation via compactness arguments for data in $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1 > 3/2$, $s_2 > 1/2$ is obtained. The proof does not make use of the dispersive nature of the equation. In particular it works in the absence of dispersion operator. However, it seems possible to extend the compactness argument by involving some dispersive estimates such as the Strichartz inequalities. The main point is to low as much as possible (s_1, s_2) in an estimate

$$\int_0^t \|u_x(t')\|_{L^\infty_{xy}} dt' \leq C(\|\phi\|_{H^{s_1,s_2}})$$

for a smooth solution of the KP-I equation.

The construction used in the proof of Theorem 3 implies some properties of the flow map data-solution for the KP-I equation with initial data in $H^{s_1,s_2}(\mathbb{R}^2)$. More precisely the flow map fails to be of class C^2 from $H^{s_1,s_2}(\mathbb{R}^2)$ to $H^{s_1,s_2}(\mathbb{R}^2)$ (cf. [13] Theorem 5.1)[2]. We note that, in general, an iterative method applied to the integral formulation of a nonlinear PDE with smooth nonlinearity provides the smoothness of the map data solution.

The failure of C^2 regularity of the flow map in the KP-I context implies the failure of the "natural" bilinear estimates in Bourgain spaces associated to the KP-I equation. Suppose that such estimates hold. Then the linear estimates (cf. [7]) would imply the local well-posedness of the KP-I equation via an iterative method in Bourgain spaces. This would imply in particular smoothness of the flow map which is a contradiction.

[2]In [5], [23] similar questions in the context of the KdV equation are considered.

In [13] counterexamples covering all possible parameters involved in the definition of Bourgain spaces associated to the KP-I equation are performed. The corresponding examples in the periodic setting are constructed in [19].

The idea of the proof of Theorem 3 can be employed in some other contexts. For instance, in [15] a semilinear heat equation is considered. It is proved that the flow map fails to be C^2 above the scaling Sobolev exponent. This seems to be the first known semilinear parabolic equation with such a property. Moreover the failure of C^2 regularity of the flow map of the Benjamin-Ono and related equations in 1D is established in [14].

A totally different approach to solve globally the KP equations is described in [25]. It is based on a generalization of Poincaré perturbation theory for finite dimensional Hamiltonian systems to infinitely many degrees of freedom. It applies to the case of periodic boundary conditions. In the KP-II context there exists a canonical transformation writing the KP-II equation into a Normal form. As a consequence the KP-II equation is solvable for smooth initial data which is small in $L^2(\mathbb{T}^2)$. In addition asymptotic states exist, they are solutions of the free problem and coincide at $+\infty$ and $-\infty$. In the KP-I case a similar canonical transformation does not exist because of small denominators problems. It is interesting to mention that the small denominators are related to the zeroes of the function $\chi(\xi, \xi_1, \eta, \eta_1)$ occurring in the proof of Theorem 3.

We finally note that an iterative method applied to a KP-I type equation can however be successful. This is for example the case of the fifth order KP-I equation (cf. [18]) or a dissipative KP-I equation (cf. [12]). In both cases one can develop the local well-posedness theory for rough data and in particular to obtain global finite energy solutions.

References

[1] B. Birnir, C. Kenig, G. Ponce, N. Svanstedt, L. Vega, *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations*, Annales I.H.P., Analyse non linéaire 13 (1996), 529-535.

[2] B. Birnir, G. Ponce, N. Svanstedt, *The local ill-posedness of the modified KdV equation*, J. London Math. Soc. 53 (1996), 551-559.

[3] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations I. Schrödinger equations*, GAFA 3 (1993), 107–156, *II. The KdV equation*, GAFA 3 (1993), 209–262.

[4] J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation*, GAFA 3 (1993), 315-341.

[5] J. Bourgain, *Periodic KdV equation with measures as initial data*, Sel. Math. New Ser. 3 (1997) 115-159.

[6] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, IMRN 5 (1998), 253-283.

[7] J. Ginibre, *Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain)*, Séminaire Bourbaki n. 796, Astérisque 237 (1996), 163-187.

[8] P. Isaza, J. Mejia, *Problemas de Cauchy local y global para la ecuacion de Kadomtsev-Petviashvili (KP-II) en espacios de Sobolev con indices negativos*, Preprint.

[9] C. E. Kenig, G. Ponce, L. Vega, *Well-posedness and scattering results for the generalized KdV equations via the contraction principle*, Comm. Pure Appl. Math. 46 (1993), 527-620.

[10] C. E. Kenig, G. Ponce, L. Vega, *A bilinear estimate with applications to the KdV equations*, J. AMS, 9 (1996), 573-603.

[11] C. E. Kenig, G. Ponce, L. Vega, *On the ill-posedness of some canonical dispersive equations*, Preprint 2000.

[12] L. Molinet, F. Ribaud, *The global Cauchy problem in Bourgain's type spaces for a dissipative semilinear equation*, Preprint 2000.

[13] L. Molinet, J.C. Saut, N. Tzvetkov, *Well-posedness and ill-posedness for the Kadomtsev-Petviashvili-I equation*, Preprint 2001.

[14] L. Molinet, J.C. Saut, N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, Preprint 2001.

[15] F. Ribaud, A. Youssfi, *Well-posedness and ill-posedness for a semilinear heat equation*, Preprint 2001.

[16] J. C. Saut, *Remarks on the generalized Kadomtsev-Petviashvili equations*, Indiana Univ. Math. J., 42 (1993), 1017-1029.

[17] J.C. Saut, N. Tzvetkov, *The Cauchy problem for higher order KP equations*, J. Diff. Eq. 153 (1999), 196-222.

[18] J.C. Saut, N. Tzvetkov, *The Cauchy problem for the fifth order KP equations*, J. Math. Pures Appl. 79 (2000), 307-338.

[19] J. C. Saut, N. Tzvetkov, *On periodic KP-I type equations* , Preprint 2000.

[20] H. Takaoka, *Global well-posedness for the Kadomtsev-Petviashvili II equation*, Discrete Contin. Dynam. Systems, 6 (2000), 483–499.

[21] H. Takaoka, N. Tzvetkov, *On the local regularity of Kadomtsev-Petviashvili-II equation*, IMRN 8 (2001), 77-114.

[22] T. Tao, *Multilinear weighted convolution of L^2 functions, and applications to nonlinear dispersive equations*, Amer. J. Math. (to appear).

[23] N. Tzvetkov, *Remark on the local ill-posedness for KdV equation*, C. R. Acad. Sci. Paris, 329 (1999), 1043-1047.

[24] N. Tzvetkov, *Global low regularity solutions for Kadomtsev-Petviashvili equation*, Diff. Int. Eq. 13 (2000), 1289-1320.

[25] V. Zakharov, *Weakly nonlinear waves on the surface of an ideal finite depth fluid*, Amer. Math. Soc. Transl. 182 (1998), 167-197.

[26] V. Zakharov, E. Schulman, *Degenerative dispersion laws, motion invariants and kinetic equations*, Physica 1D (1980), 192-202.