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CUBIC QUASILINEAR WAVE EQUATION AND BILINEAR ESTIMATES

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Introduction

In this paper, our interest is to prove local solvability for equations of the type

$$(EC)\begin{cases} \partial_t^2 u - \Delta u - \displaystyle\sum_{1\leq j,k\leq d} g^{j,k}\partial_j\partial_k u &= 0 \\ \Delta g^{j,k} &= Q_{j,k}(\partial u,\partial u) \\ (u,\partial_t u)_{|t=0} &= (u_0,u_1). \end{cases}$$

where $Q_{j,k}$ are quadratic forms on $\mathbb{R}^{d+1}$. In all this work, we shall state, for a function u on $[0,T]\times\mathbb{R}^d$,

$$\nabla u \stackrel{\rm def}{=} (\partial_1 u,\cdots,\partial_d u)\,,\ \partial u \stackrel{\rm def}{=} (\partial_t u,\partial_1 u,\cdots,\partial_d u) \quad\text{and}\quad g\cdot\nabla^2 u \stackrel{\rm def}{=} \sum_{1\leq j,k\leq d} g^{j,k}\partial_j\partial_k u.$$

When no confusion is possible, we shall also state $\gamma \stackrel{\rm def}{=} (\nabla u_0, u_1)$. This problem of course is a model one. The general problem consists in considering equations of the type

$$\begin{cases} \partial_t^2 u - \Delta u - \displaystyle\sum_{1\leq j,k\leq d} g^{j,k}\partial_j\partial_k u &= \displaystyle\sum_{1\leq j,k\leq d} \widetilde{Q}_{j,k}(\partial g^{j,k},\partial u) \\ \Delta g^{j,k} &= Q_{j,k}(\partial u,\partial u) \\ (u,\partial_t u)_{|t=0} &= (u_0,u_1). \end{cases}$$

where $\widetilde{Q}_{j,k}$ are quadratic form on $\mathbb{R}^{d+1}$ and where all the quadratic forms are supposed to be smooth functions of u. This simply complicates a little the estimates without any relevant new phenomenon. In the frame work of (EC), it makes sense to work with small data and this simplifies the proofs.

Energy methods allow to prove local wellposedness for initial data (u_0,u_1) in $H^{\frac d2+\frac12}\times H^{\frac d2-\frac12}$. More precisely, we have the following theorem.

Theorem 0.1. *If $d\geq 3$, let (u_0,u_1) be in $H^{\frac d2+\frac12}\times H^{\frac d2-\frac12}$ such that $\|\gamma\|_{\dot H^{\frac d2-1}}$ is small enough. Then, a positive times T exists such that a unique solution u of (EC) exists in $C([0,T];H^{\frac d2+\frac12})\cap C^1([0,T];H^{\frac d2-\frac12})$. Moreover, a constant C exists (which of course does not depend on the initial data) such that $T\geq C\|\gamma\|^{-2}_{\dot H^{\frac d2-\frac12}}$.*

Let us recall that H^s is the usual Sobolev space on $\mathbb{R}^d$ and that $\dot{H}^s$ is the homogeneous one and we shall state

$$\|f\|_s^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi.$$

This is an Hilbert space when $s < d/2$.

The goal of this paper is to go below the regularity $H^{d/2+1/2}$ for the initial data. Let us have a look to the scaling properties of equation (EC). If u is a solution of (EC), then $u_\lambda(t,x) \stackrel{\text{def}}{=} u(\lambda t, \lambda x)$ is also a solution of (EC). The space which is invariant under this scaling is $\dot{H}^{\frac{d}{2}}$. So the above theorem appears to require 1/2 derivative more than the scaling. The goal of this work is to try to go as closed as possible to the scaling invariant regularity.

Some results in that direction have been proved by the authors (see [1] and [2]) and also by D. Tataru (see [22]) for quasilinear wave equations of the following type

$$(E) \left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u - G(u) \cdot \nabla^2 u & = & F(u) Q(\partial u, \partial u) \\ (u, \partial_t u)_{|t=0} & = & (u_0, u_1) \end{array} \right.$$

where G is a smooth function vanishing at 0 with value in K such that $\mathrm{Id} + K$ is a convex compact subset of the set of positive symmetric matrices. Let us recall this results. Let us notice that the scaling of the two equations (E) and (EC) is the same.

Theorem 0.2. *If $d \geq 3$, let (u_0, u_1) be in $H^s \times H^{s-1}$ for $s > s_d$ with $s_d = \dfrac{d}{2} + \dfrac{1}{2} + \dfrac{1}{6}\cdot$ Then, a positive time T exists such that a unique solution u exists such that*

$$\partial u \in C([0,T]; H^{s-1}) \cap L^2([0,T]; L^\infty).$$

Moreover, a constant C exists such that $T^{\frac{2}{3}+(s-s_d)} \geq C \|\gamma\|_{\dot{H}^{s-1}}^{-1}$.

This theorem has been proved with 1/4 instead than 1/6 in [1] and then improved a little bit in [2] and proved with 1/6 by D. Tataru in [22]. Strichartz estimates for quasilinear equations are the key point of the proofs. Recently, S. Klainerman and S. Rodnianski have announced a better index. Their proof is based on very different methods. In this case, the energy methods give the classical index $s > d/2 + 1$ and $T \geq C \|\gamma\|_{H^{s-1}}^{-1+s-s_d}$.

The goal of this work is to do the analogous in the case of Equation (EC). The result will be the following.

Theorem 0.3. *If $d \geq 5$, let (u_0, u_1) be in $H^s \times H^{s-1}$ with $s > \dfrac{d}{2} + \dfrac{1}{6}$ such that $\|\gamma\|_{\dot{H}^{\frac{d}{2}-1}}$ is small enough. Then, a positive times T exists such that a unique solution u of (EC) exists such that $\partial u \in C([0,T]; H^{s-1}) \cap L^2_T(\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2})$ where $\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}$ is the Besov space defined in Definition 1.1. Moreover, for any positive α, we have that $T^{\frac{1}{6}+\alpha} \geq C_\alpha \|\gamma\|^{-1}_{\dot{H}^{\frac{d}{2}-\frac{5}{6}+\alpha}}$.*

The case of dimension 4 is a little bit different. The theorem is the following.

Theorem 0.4. *If $d = 4$, let (u_0, u_1) be in $H^s \times H^{s-1}$ with $s > 2 + \dfrac{1}{6}$ such that $\|\gamma\|_{\dot{H}^1}$ is small enough. Then, a positive times T exists such that a unique solution u of (EC) exists such that*

$$\partial u \in C([0,T]; H^{s-1}) \cap L^2_T(\dot{B}^{\frac{1}{6}}_{6,2}) \quad \text{and} \quad \partial g \in L^1_T(L^\infty).$$

where $\dot{B}^{\frac{d}{6}-\frac{1}{2}}_{6,2}$ denotes the Besov space defined in Definition 1.1. Moreover, for any positive α, a constant C_α exists such that $T^{\frac{1}{6}+\alpha} \geq C_\alpha \|\gamma\|^{-1}_{\dot{H}^{\frac{d}{2}-\frac{5}{6}+\alpha}}$.

Remarks

- If we think in term of small data (i.e. of initial data of the type $\varepsilon(u_0, u_1)$), then energy methods give a life span in ε^{-2}. The above theorem gives a life span of order $\varepsilon^{-6+\alpha}$ for any positive α.
- As we shall see, the case when $d \geq 5$ can be treated only with Strichartz estimates simply because if ∂u belongs to $L^2_T(\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2})$ then ∂g is in $L^1_T(L^\infty)$.
- The case when $d = 4$ requires bilinear estimates. This fact appears in the statement of Theorem 0.4 through the following phenomenon: the fact that ∂u is in $L^2_T(\dot{B}^{\frac{1}{6}}_{6,2})$ does not imply that the time derivative of g belongs to $L^1_T(L^\infty)$. Of course this condition is crucial in particular to get the basic energy estimate. But we have been unable to exibit a Banach space $\mathcal{B}$ which contains the solution u and such that if a function a is in $\mathcal{B}$, then $\partial\Delta^{-1}(a^2)$ belongs to $L^1_T(L^\infty)$.

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1. Method of the proof

1.1. **Some basic facts in Littlewood-Paley theory.** Let us begin by recalling the basis of Littlewood-Paley theory. Let us denote by $\mathcal{C}$ the ring of center 0, of small radius 3/4 and of big radius 8/3. Let us choose two non negative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0, 4/3))$ and $\mathcal{D}(\mathcal{C})$ such that

$$\chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = \sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \tag{1}$$

$$|p - q| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-q}\cdot) \cap \text{Supp } \varphi(2^{-p}\cdot) = \emptyset, \tag{2}$$

$$q \geq 1 \Rightarrow \text{Supp } \chi \cap \text{Supp } \varphi(2^{-q}\cdot) = \emptyset, \tag{3}$$

and if $\widetilde{\mathcal{C}} = B(0, 2/3) + \mathcal{C}$, then $\widetilde{\mathcal{C}}$ is a ring and we have

$$|p - q| \geq 5 \Rightarrow 2^p\widetilde{\mathcal{C}} \cap 2^q\mathcal{C} = \emptyset. \tag{4}$$

Notations

$$h = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \widetilde{h} = \mathcal{F}^{-1}\chi,$$

$$\Delta_q u = \varphi(2^{-q}D)u = 2^{qd} \int h(2^q y)u(x - y)dy,$$

$$S_q u = \sum_{p\leq q-1} \Delta_p u = \chi(2^{-q}D)u = 2^{qd} \int \widetilde{h}(2^q y)u(x - y)dy.$$

We shall often denote $\Delta_q u$ by u_q. Let us recall the definition of Besov spaces.

Definition 1.1. *Let s be a real number, and (p, r) in $[1, \infty]^2$. Let us state*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{def}{=} \left\| \left(2^{qs}\|\Delta_q u\|_{L^p}\right)_{q\in\mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}$$

If $s < d/p$ then the closure of the compactly smooth functions with respect to this norm is a Banach space and we have that $\dot{H}^s = \dot{B}^s_{2,2}$ and the norm $\|\cdot\|_{\dot{B}^s_{2,2}}$ is equivalent to $\|\cdot\|_{\dot{H}^s}$.

Notations We shall also state

$$\|a\|_s \overset{\text{def}}{=} \|a\|_{\dot B^s_{2,2}}\,,\ \|b\|_{L^p_I(E)} \overset{\text{def}}{=} \|b\|_{L^p(I;E)}\,,\ \|b\|_{L^p_T(E)} \overset{\text{def}}{=} \|b\|_{L^p([0,T];E)}$$

$$\text{and}\quad \|b\|_{T,s} \overset{\text{def}}{=} \|b\|_{L^\infty_T(\dot B^s_{2,2})}.$$

Here we want to explain the problems we have to solve to prove Theorem 0.4. As in the case of Equation (E), the basic fact is energy estimates. This implies the control of

$$\int_0^T \|\partial g(t,\cdot)\|_{L^\infty} dt.$$

In the case of Equation (E), it is obtained by Strichartz estimates. This will be the case here when $d \geq 5$ not when $d=4$. Let us follow now ideas of S. Klainerman and D. Tataru (see [17]). If u is the solution of the constant coefficient wave equation, let us estimate

$$\int_0^T \|\partial \Delta^{-1}\big(\partial_j u(t,\cdot)\partial_k u(t,\cdot)\big)\|_{L^\infty} dt.$$

As $\partial_t \Delta^{-1}\big(\partial_j u(t,\cdot)\partial_k u(t,\cdot)\big) = \Delta^{-1}\big(\partial_t\partial_j u \partial_k u(t,\cdot)\big) + \Delta^{-1}\big(\partial_j u \partial_t \partial_k u(t,\cdot)\big)$, we have to control expression of the type

$$\int_0^T \|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{L^\infty} dt.$$

When $d\geq 4$, we have that $\|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{B^{\frac d2}_{2,1}} \leq C\|\partial u(t,\cdot)\|^2_{\frac d2-\frac12}$. So we get that

$$\int_0^T \|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{L^\infty} dt \leq T\|\partial u\|^2_{T,\frac d2-\frac12}.$$

Then the proof of Theorem 0.1 is routine. If we want to go below this $H^{\frac d2+\frac12}$ regularity of the initial data, we shall use Strichartz estimates. Let us introduce Bony's decomposition which consists in writing

$$ab = \sum_q S_{q-1}a\Delta_q b + \sum_q S_{q-1}b\Delta_q a + \sum_{\substack{-1\leq j\leq 1\\ q}} \Delta_q a\Delta_{q-j} b.$$

When $d\geq 4$, we have $\|\partial^k u_q\|_{L^2_T(L^\infty)} \leq C2^{q\left(\frac d2-\frac12+k-1\right)}\|\gamma_q\|_{L^2}$. Then it is easy to prove that

$$\Big\|\Delta^{-1}\Big(\sum_q S_{q-1}\partial^2 u\partial u_q\Big)\Big\|_{L^1_T(L^\infty)} \leq C\|\gamma\|^2_{\frac d2-1}.$$

The symmetric term can be treated exactly along the same lines. The remainder term

$$\Delta^{-1}\Big(\sum_{\substack{-1\leq j\leq 1\\ q}} \partial^2 u_q\partial u_{q-j}\Big)$$

is much more difficult to treat particulary in dimension 4. The reason why is the following. When d is greater or equal to 5, the Strichartz estimates tells us that

$$\|\partial^k u_q\|_{L^2_T(L^4)} \leq 2^{q\left(\frac d4-\frac12+k-1\right)}\|\gamma_q\|_{L^2}.$$

So thanks to Bernstein inequality, we infer that

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leq j\leq 1\\ q\geq p-N_0}}\Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} \leq C2^{p(\frac{d}{2}-2)}\sum_{\substack{-1\leq j\leq 1\\ q\geq p-N_0}}2^{q\frac{d}{2}}\|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}$$

$$\leq C\sum_{\substack{-1\leq j\leq 1\\ q\geq p-N_0}}2^{-(q-p)(\frac{d}{2}-2)}2^{q(d-2)}\|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}.$$

Convolution and Cauchy-Schwarz inequalities implies that $\big\|\Delta^{-1}\big(\partial^2 u\partial u\big)\big\|_{L^1_T(L^\infty)}\leq C\|\gamma\|^2_{\frac{d}{2}-1}$. The case of dimension 4 is much more delicate. In dimension 4, the Strichartz estimate is

$$\|\partial^k u_q\|_{L^2_T(L^6)}\leq 2^{q(\frac{4}{3}-\frac{1}{2}+k-1)}\|\gamma_q\|_{L^2}.$$

So the series $\partial^2 u_q\partial u_{q-j}$ does not converge in $L^1_T(L^3)$ because the only estimate we have is

$$\|\partial^2 u_q\partial u_{q-j}\|_{L^1_T(L^3)}\leq C2^{q\frac{8}{3}}\|\gamma_q\|^2_{L^2}\leq C2^{q\frac{2}{3}}d_q\|\gamma\|^2_1 \quad\text{with}\quad \sum_q d_q=1.$$

To overcome this difficulty, we follow an idea of D. Tataru and S. Klainerman (see [17]).

1.2. **Bilinear estimates and precised Strichartz estimates.** To explain the basic ideas of bilinear estimates, let us consider the case of constant coefficient case.

Proposition 1.1. *Let u_1 and u_2 two solutions of $\partial_t^2 u_j-\Delta u_j=0$ and $(\partial u_j)_{|t=0}=\gamma_j$. Then, if $d\geq 4$,*

$$\|\partial\Delta^{-1}Q(\partial u_1,\partial u_2)\|_{L^1_T(L^\infty)}\leq C_{\varepsilon,T}\|\gamma_1\|_{\frac{d}{2}-1+\varepsilon}\|\gamma_2\|_{\frac{d}{2}-1+\varepsilon}.$$

The precised Strichartz estimates are described by the following proposition.

Proposition 1.2. *If $d\geq 3$, a constant C exists such that for any T and any $h\leq 1$, if* Supp $\widehat{u}_j$ *and* Supp $\mathcal{F}(\Box\, u(t,\cdot))$ *are included in a ball of radius h and in the ring $\mathcal{C}$, we have*

$$\|u\|_{L^2_T(L^\infty)}\leq C\big(h^{d-2}\log(e+T)\big)^{\frac{1}{2}}\big(\|u(0)\|_{L^2}+\|\partial_t u(0)\|_{L^2}+\|\Box\, u\|_{L^1_T(L^2)}\big).$$

As usual it is deduced with the $TT^\star$ argument from the following dispersive inequality.

Lemma 1.1. *Let $\mathcal{C}$ be a ring of $\mathbb{R}^d$. A constant C exists such that if u_0 and u_1 are functions in $L^1(\mathbb{R}^d)$ such that*

$$\text{Supp }(\widehat{u}_j)\subset\mathcal{C}\quad\textit{and}\quad\max\big\{\delta(\text{Supp }(\widehat{u}_0),\delta(\text{Supp }(\widehat{u}_1)\big\}\leq h,$$

then, for any $\widetilde{d}$ between 0 and $d-1$, we have

$$\|u(t,\cdot)\|_{L^\infty}\leq\frac{Ch^{d-\widetilde{d}}}{t^{\frac{\widetilde{d}}{2}}}\big(\|u_0\|_{L^1}+\|u_1\|_{L^1}\big),$$

where u denotes the solution of $\partial_t^2 u-\Delta u=0$ and $\partial_t^j u_{|t=0}=u_j$.

This inequality is proved in [17] in the case when $\widetilde{d}=d-1$. The general case is obtained by interpolation with the classical Sobolev embedding.

Let us recall that we want to estimate the

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leq j\leq 1\\ q\geq p-N_0}}\Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)}$$

With scaling, we can assume that $q=1$ and let us state $h=2^{p-q}$. Let us define $(\phi_\nu)_{1\leq\nu\leq N_h}$ a partition of unity of the ring $\mathcal{C}$ such that Supp $\phi_\nu\subset B(\xi_\nu,h)$. Then, using the fact that the support of the Fourier transform of the product of two functions is included in the sum of the support of their Fourier transform, a family of functions $(\widetilde{\phi}_\nu)_{1\leq\nu\leq N_h}$ exists such that Supp $\widetilde{\phi}_\nu \subset B(-\xi_\nu, 2h)$ and

$$\chi(h^{-1}D)(\partial^2 v\partial v)) = \sum_{\nu=1}^{N_h} \chi(h^{-1}D)\big(\partial^2\widetilde{\phi}_\nu(D)v\partial\phi_\nu(D)v)\big). \tag{5}$$

Applying Proposition 1.2 gives

$$\|\chi(h^{-1}D)(\partial^2 v\partial v))\|_{L^1_T(L^\infty)} \leq Ch^{d-2}\log(e+T)\sum_{\nu=1}^{N_h}\|\widetilde{\phi}_\nu(D)\gamma\|_{L^2}\|\phi_\nu(D)\gamma\|_{L^2}.$$

The Cauchy Schwarz inequality implies that

$$\|\chi(h^{-1}D)(\partial^2 v\partial v))\|_{L^1_T(L^\infty)} \leq Ch^{d-2}\log(e+T)\left(\sum_{\nu=1}^{N_h}\|\widetilde{\phi}_\nu(D)\gamma\|^2_{L^2}\right)^{\frac12}\left(\sum_{\nu=1}^{N_h}\|\phi_\nu(D)\gamma\|^2_{L^2}\right)^{\frac12}.$$

The almost orthogonality of $(\widetilde{\phi}_\nu(D)\gamma_1)_{1\leq\nu\leq N_h}$ and $(\phi_\nu(D)\gamma_2)_{1\leq\nu\leq N_h}$ implies that

$$\|\chi(h^{-1}D)(\partial^2 v\partial v))\|_{L^1_T(L^\infty)} \leq Ch^{d-2}\log(e+T)\|\gamma\|_{L^2}\|\gamma\|_{L^2}. \tag{6}$$

So after rescaling, we get that

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leq j\leq 1\\ q\geq p-N_0}}\Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} \leq 2^{p(d-4)}\sum_{\substack{-1\leq j\leq 1\\ q\geq p-N_0}}\log(e+2^qT)2^{2q}\|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}.$$

If $\gamma\in H^{\frac d2-1+\frac\varepsilon2}$ then we have

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leq j\leq 1\\ q\geq p-N_0}}\Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} \leq (2^pT)^{-\varepsilon}\sum_{\substack{-1\leq j\leq 1\\ q\geq p-N_0}}2^{-(q-p)(d-4+\varepsilon)}$$
$$\times\, 2^{q(\frac d2-1)}(2^qT)^{\frac\varepsilon2}\|\gamma_q\|_{L^2}2^{(q-j)(\frac d2-1)}(2^qT)^{\frac\varepsilon2}\|\gamma_{q-j}\|_{L^2}.$$

So the series convergences in $L^1_T(L^\infty)$ for large p. The case when p is small (low frequencies) is nothing but Sobolev embeddings.

The problem we have to solve in this work is to prove this bilinear estimate in the context of quasilinear wave equation. To do this, we follow the lines of [1] and [2]. As we shall use geometrical optics technics, we need to deal with smooth functions in time also. This leads to the following iterative scheme introduced in [2]. Let us define the sequence $(u^{(n)})_{n\in\mathbb{N}}$ by the first term $u^{(0)}$ satisfying

$$\left\{\begin{array}{rcl}\partial_t^2u^{(0)}-\Delta u^{(0)} & = & 0\\ (u^{(0)},\partial_tu^{(0)})_{|t=0} & = & (S_0u_0,S_0u_1),\end{array}\right.$$

and by the following induction

$$(\mathcal{R}_n)\left\{\begin{array}{rcl}\partial_t^2u^{(n+1)}-\Delta u^{(n+1)}-G_{n,T}\cdot\nabla^2u^{(n+1)} & = & 0\\ (u^{(n+1)},\partial_tu^{(n+1)})_{|t=0} & = & (S_{n+1}u_0,S_{n+1}u_1)\end{array}\right.$$

with $G_{n,T}\stackrel{\text{def}}{=}\theta(T^{-1})G_n$ with $G_n^{j,k}\stackrel{\text{def}}{=}\Delta^{-1}Q_{j,k}(\partial u^{(n)},\partial u^{(n)})$ where θ is a function of $\mathcal{D}(]-1,1[)$ whose value is 1 near 0. Let us point out that the sequence $(u^{(n)})_{n\in\mathbb{N}}$ does depend on T. We introduce some notations which will be used all along this work. If α is a (small) positive number, let us define

$$s_\alpha \stackrel{\text{def}}{=} \frac{d}{2}+\frac{1}{6}+\alpha \quad \text{and} \quad N_T^\alpha(\gamma) \stackrel{\text{def}}{=} T^{\frac{1}{6}+\alpha}\|\gamma\|_{s_\alpha-1}.$$

Let us introduce the assertions we are going to prove by induction.

- If $d \geq 5$,

$$(\mathcal{P}_n)\left\{\begin{array}{lll} \|\partial u^{(n)}\|_{L^2([0,T];\dot B^{\frac{d}{4}-\frac{1}{2}}_{4,2})} & \leq & C_0 N_T^\alpha(\gamma) \\ \|\partial u^{(n)}\|_{T,s-1} & \leq & e^3\|\gamma\|_{s-1} \quad \text{for any} \quad s\in\left[s_\alpha-1,\dfrac{d}{2}+\dfrac{1}{2}\right]\ ; \end{array}\right.$$

- if $d=4$,

$$(\mathcal{P}_n)\left\{\begin{array}{lll} \|\partial u^{(n)}\|_{L^2([0,T];\dot B^{\frac{d}{6}-\frac{1}{2}}_{6,2})} & \leq & C_0 N_T^\alpha(\gamma) \\ \|\partial G_{n,T}\|_{L^1([0,T];L^\infty)} & \leq & 2 \\ \|\partial u^{(n)}\|_{T,s-1} & \leq & e^3\|\gamma\|_{s-1} \quad \text{for any} \quad s\in\left[\dfrac{3}{2}+\alpha,\dfrac{d}{2}+\dfrac{1}{2}\right]. \end{array}\right.$$

This paper consists in proving that if $\|\gamma\|_{\dot H^{\frac{d}{2}-1}} + N_T^\alpha(\gamma)$ is small enough, $(\mathcal{P}_0)$ is true and $(\mathcal{P}_n)$ implies $(\mathcal{P}_{n+1})$. The remainder is routine of non linear partial differential equations.

2. Reduction to microlocalized estimates

By microlocalization of the estimates, we mean that we shall prove estimates that are valid on time intervals whose length depend on the frequency parameter. These techniques have been introduced in [1] and used in [2] and improved by D. Tataru in [22].

For $\Lambda_0>0$, we shall consider a family of smooth functions $\mathcal{G}=(G_\Lambda)_{\Lambda\geq\Lambda_0}$ defined on $I_\Lambda\times\mathbb{R}^d$ such that G_Λ is small enough in L^∞ norm and for any $k\geq 0$, the following quantities

$$\|\mathcal{G}\|_0 \stackrel{\text{def}}{=} \sup_{\Lambda\geq\Lambda_0}\|\nabla G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} + |I_\Lambda|\|\nabla^2 G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \quad \text{and} \tag{7}$$

$$\|\mathcal{G}\|_k \stackrel{\text{def}}{=} \sup_{\Lambda\geq\Lambda_0}|I_\Lambda|\Lambda^k\|\nabla^{k+2}G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \quad \text{for} \quad k\geq 1. \tag{8}$$

are finite. Let us denote by P_Λ the operator $P_\Lambda v \stackrel{\text{def}}{=} \partial_\tau^2 v - \Delta v - \sum_{j,k} G_\Lambda^{j,k}\partial_j\partial_k v.$

Theorem 2.1. *Let $\mathcal{C}$ be a ring of $\mathbb{R}^d$ and ε_0 a positive real number. Let us assume that $\|\mathcal{G}\|_0$ is small enough. For any positive real number $\varepsilon\leq\varepsilon_0$, a constant C_ε exists which satisfies the following properties. Let f_1 and f_2 two functions in $L^1_{I_\Lambda}(L^2)$ and γ_1 and γ_2 two functions of L^2; let us assume that all the spectrum of those functions is included in $\mathcal{C}$. Let us assume that $|I_\Lambda|\leq\Lambda^{2-\varepsilon}$. Then if $v_{1,\Lambda}$ and $v_{2,\Lambda}$ are solutions of*

$$(E_\Lambda)\left\{\begin{array}{lll} P_\Lambda v_{j,\Lambda} & = & f_j \\ \partial v_{j,\Lambda|\tau=0} & = & \gamma_j \end{array}\right.$$

we shall have the following properties:

- *if $d\geq 5$, we have $\|\partial v_{j,\Lambda}\|_{L^2_{I_\Lambda}(L^4)} \leq C(\|\gamma_j\|_{L^2}+\|f_j\|_{L^1_{I_\Lambda}(L^2)})$.*
- *if $d=4$, we have $\|\partial v_{j,\Lambda}\|_{L^2_{I_\Lambda}(L^6)} \leq C(\|\gamma_j\|_{L^2}+\|f_j\|_{L^1_{I_\Lambda}(L^2)})$.*

- *if $d \geq 3$, then we have, for any $h \leq 1$ and any $\varepsilon > 0$,*

$$\|\chi(h^{-1}D)(\partial v_{1,\Lambda}\partial v_{2,\Lambda})\|_{L^1_{I_\Lambda}(L^\infty)} \leq C_\varepsilon h^{d-2-\varepsilon}\log(e+|I_\Lambda|) \times \big(\|\gamma_1\|_{L^2}+\|f_1\|_{L^1_{I_\Lambda}(L^2)}\big)\big(\|\gamma_2\|_{L^2}+ \ |f_2\|_{L^1_{I_\Lambda}(L^2)}\big).$$

If h is small enough, this is Sobolev embedding. Using Bernstein inequality, we can write

$$\begin{aligned}\|\chi(h^{-1}D)(\partial v_{1,\Lambda}\partial v_{2,\Lambda})\|_{L^1_{I_\Lambda}(L^\infty)} &\leq h^d\|\partial v_{1,\Lambda}\partial v_{2,\Lambda}\|_{L^1_{I_\Lambda}(L^1)}\\ &\leq h^d|I_\Lambda|\,\|\partial v_{1,\Lambda}\|_{L^\infty_{I_\Lambda}(L^2)}\|\partial v_{2,\Lambda}\|_{L^\infty_{I_\Lambda}(L^2)}\\ &\leq h^d|I_\Lambda|\,\big(\|\gamma_1\|_{L^2}+\|f_1\|_{L^1_{I_\Lambda}(L^2)}\big)\big(\|\gamma_2\|_{L^2}+\|f_2\|_{L^1_{I_\Lambda}(L^2)}\big).\end{aligned}$$

So when $h^d|I_\Lambda| \leq h^{d-2-\varepsilon}$, the above bilinear estimate is proved. From now on, we assume that $|I_\Lambda| \geq h^{-2-\varepsilon}$.

3. Approximation of the solution and geometrical optics

3.1. **The approximation of the solution.** The key point is the use of Hamilton-Jacobi equation. The following proposition (and its proof) is a small modification of Proposition 6.1 of [1].

Proposition 3.1. *Let F be a real valued smooth function on $\mathbb{R}^d \times \mathbb{R}^N$ bounded as all its derivatives such that*

$$F(\zeta, G) = \pm\big(|\zeta|^2 + G(\zeta,\zeta)\big)^{\frac12} \quad \textit{for all} \quad \zeta \in \widetilde{\mathcal{C}}.$$

For any positive real number ε, a positive real number α exists such that, if $\|\mathcal{G}\|_0 \leq \alpha$, then, for any $\Lambda \geq \Lambda_0$, for any η, a solution Φ_Λ of the equation

$$(\widetilde{HJ}_\Lambda)\begin{cases}\partial_\tau\Phi_\Lambda(\tau,y,\eta) &= F_\Lambda(\tau,y,\partial_y\Phi_\Lambda(\tau,y,\eta))\\ \Phi_\Lambda(0,y,\eta) &= (y|\eta)\end{cases} \quad \textit{with} \quad F_\Lambda(\tau,z,\zeta) \stackrel{\text{def}}{=} F(\zeta, G_\Lambda(\tau,z)).$$

exists and is smooth on $I_\Lambda\times\mathbb{R}^d\times\mathbb{R}^d$. Moreover, the family defined by $\Phi \stackrel{\text{def}}{=} (\Phi_\Lambda)_{\Lambda\geq\Lambda_0}$ satisfies the following properties: For any couple of integer (k,ℓ), a constant $C_{k,\ell}$ (independant of ε) exists such that

$$\sup_{\Lambda\geq\Lambda_0}\|\partial_y\partial_\eta\Phi_\Lambda - \mathrm{Id}\,\|_{L^\infty(I_\Lambda\times\mathbb{R}^{2d})} \leq \varepsilon C\,, \tag{9}$$

$$\sup_{\Lambda\geq\Lambda_0}|I_\Lambda|\Lambda^k\|\partial_\eta^\ell\nabla^{2+k}\Phi_\Lambda\|_{L^\infty(I_\Lambda\times\mathbb{R}^{2d})} \leq \varepsilon C_{k,\ell} \quad \textit{and} \tag{10}$$

$$\sup_{\Lambda\geq\Lambda_0}\|\partial_\eta^{\ell+2}\Phi_\Lambda\|_{L^\infty(I_\Lambda\times\mathbb{R}^{2d})} \leq \varepsilon C_\ell|I_\Lambda|. \tag{11}$$

The link between the solution of the Hamilton-Jacobi equation and the hamiltonian flow is decribed by the two following lemmas.

Lemma 3.1. *Let Φ_Λ be the solution of the above Hamilton-Jacobi equation $(\widetilde{HJ}_\Lambda)$ and Ψ_Λ the Hamiltonian flow of $-F_\Lambda(\tau,Y)$ i.e. the solution of*

$$\begin{cases}\dfrac{d\Psi_\Lambda}{d\tau}(\tau,y,\eta) &= -H_{F_\Lambda}(\tau,\Psi_\Lambda(Y))\\ \Psi_\Lambda(0,y,\eta) &= (y,\eta).\end{cases}$$

Then we have

$$\begin{aligned}(\partial_\eta\Phi_\Lambda)(\tau,\Psi^y_\Lambda(\tau,y,\eta),\eta) &= y \quad \text{and}\\ (\partial_y\Phi_\Lambda)(\tau,\Psi^y_\Lambda(\tau,y,\eta),\eta) &= \Psi^\eta_\Lambda(\tau,y,\eta).\end{aligned}$$

Lemma 3.2. *A constant C_0 exists such that for any couple of positive number (r, h) such that $|I_\Lambda| \geq h^{-2}$ then if*

$$g_a(dy^2, d\eta^2) \stackrel{\text{def}}{=} \frac{dy^2}{K^2} + \frac{d\eta^2}{h^2} \quad \text{with} \quad K = C|I_\Lambda|h$$

then, we have the following two properties.

- *For any couple $(Y, Z)$and for any $\tau \in I_\Lambda$, we have*

$$\frac{1}{C_0}\, g_a(Y - Z) \leq g_a(\Psi_\Lambda(\tau, Y) - \Psi_\Lambda(\tau, Z)) \leq C_0\, g_a(Y - Z). \tag{12}$$

- *For any couple of points (Y_0, Z_τ) of $T^\star\mathbb{R}^d$ such that $g_a(Z_\tau, \Psi_\Lambda(\tau, Y_0))^{\frac{1}{2}} \geq C_0 r$, if we have $(z, \eta) \in B_{g_a}(Y_0, r)$ and $(y, \zeta) \in B_{g_a}(Z_\tau, r)$ then*

$$g_a(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta\eta) \geq \frac{1}{C_0} g_a(Z_\tau, \Psi_\Lambda(\tau, Y_0)).$$

Let us state the approximation theorem which tells us that the solution can be represented by a Fourier integral operator up to arbitrary small error term (i.e. terms which are smaller than any given power of Λ^{-1}).

Theorem 3.1. *Let us assume that $\|\mathcal{G}\|_0$ is small enough and that $|I_\Lambda| \leq \Lambda^{2-\varepsilon}$. Then, for any integer N, two families of functions $(\sigma_\Lambda^\pm)$ (with value in $\mathbb{R}^2$) on $I_\Lambda \times \mathbb{R}^{2d}$ and a constant C exist such that the following properties are satisfied.*

- *Let $(v_\Lambda)_{\Lambda \geq \Lambda_0}$ be the family of solutions of (E_Λ) with $f = 0$ and with initial data $\gamma = (\gamma^0, \gamma^1)$; if we state*

$$\mathcal{I}_\Lambda^\pm(\gamma) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} e^{i\Phi_\Lambda^\pm(\tau,y,\eta)} \sigma_\Lambda^\pm(\tau, y, \eta) \cdot \widehat{\gamma}_\pm(\eta) d\eta \quad \text{then} \tag{13}$$

$$\|\nabla(v_\Lambda - \mathcal{I}_\Lambda^+(\gamma) - \mathcal{I}_\Lambda^-(\gamma))\|_{L^\infty_{I_\Lambda}(L^2)} \leq C\Lambda^{-N}\|\gamma\|_{L^2}. \tag{14}$$

- *We have*

$$\|\partial_{\tau,y}^\alpha \partial_\eta^\beta \sigma^\pm\|_{L^\infty(\mathcal{I}_\Lambda \times \mathbb{R}^{2d})} \leq C_{\alpha,\beta}\Lambda^{-|\alpha|}.$$

The proof of this is done in [1] and [2].

3.2. **The precised Strichartz estimate.** The theorem is the following.

Theorem 3.2. *Let $\mathcal{C}$ be a ring of $\mathbb{R}^d$ and let us assume that $\mathcal{G}_0$ is small enough. For any positive real number ε, a constant C_ε exists which satisfies the following properties. Let f be a function in $L^1_{I_\Lambda}(L^2)$ and γ a function of L^2; let us assume that those two functions have their support included in $\mathcal{C}$ and of diameter less than h. Let us assume $|I_\Lambda| \leq \Lambda^{2-\varepsilon}$. Then if v_Λ is the solution of*

$$(E_\Lambda)\left\{\begin{array}{rcl} P_\Lambda v_\Lambda & = & f \\ \partial v_{\Lambda|\tau=0} & = & \gamma. \end{array}\right.$$

we have

$$\|v_\Lambda\|_{L^2_{I_\Lambda}(L^\infty)} \leq Ch^{\frac{d-2}{2}}\big(\log(e + |I_\Lambda|)\big)^{\frac{1}{2}}\big(\|\gamma\|_{L^2} + \|f\|_{L^1_{I_\Lambda}(L^2)}\big).$$

The proof of this theorem consits in an adaptation of [17] which leads to following dispersive inequality

$$\|v_\Lambda(\tau, \cdot)\|_{L^\infty} \leq C\frac{h^{d-2}}{t}\|\gamma\|_{L^1}.$$

Then the classical $TT^\star$ argument concludes the proof of Theorem 3.2.

4. The concept of microlocalized functions

In this section, we present the concept of microlocalized functions introduced by J.-M. Bony in [5]. This is related to the Weyl-Hörmander calculus (see [10], [7]).

4.1. **A simplified version of pseudo-differential calculus.** In this paragraph, we shall consider a positive quadratic form g on $T^\star\mathbb{R}^d$ such that the symplectic conjugate defined by

$$g^\sigma(T) \stackrel{\text{def}}{=} \sup_{W\neq 0} \frac{[T,W]^2}{g(W)}$$

satisfies the uncertainty principle $g^\sigma \geq g$. Here $[\cdot,\cdot]$ denotes the basic symplectic form on $T^\star\mathbb{R}^d$

$$[(x,\xi),(y,\eta)] = \sum_{j=1}^{d} \bigl(\xi^j y_j - \eta^j x_j\bigr).$$

In all this paper, we are going to be in the case when

$$g(dx,d\xi) = \frac{dx^2}{K^2} + \frac{d\xi^2}{h^2}\,\cdot$$

In this case, we have $g^\sigma = \lambda^2 g$ with $\lambda = Kh$. The uncertainty principle means that $\lambda \geq 1$.

We shall mesure the length of derivatives of smooth functions on $T^\star\mathbb{R}^d$ with respect to this metric g. More precisely, let us define, for any smooth function φ on $T^\star\mathbb{R}^d$,

$$\|\varphi\|_{j,g} \stackrel{\text{def}}{=} \sup_{\substack{k\leq j\\ X\in T^\star\mathbb{R}^d}} \ \sup_{\substack{(T_\ell)_{1\leq\ell\leq k}\\ g(T_\ell)\leq 1}} |D^k\varphi(X)(T_1,\cdot,T_k)|.$$

Now, to a function φ in $\mathcal{D}(T^\star\mathbb{R}^d)$, we associate the operator φ^D defined by

$$(\varphi^D u)(x) = (2\pi)^{-d} \int_{T^\star\mathbb{R}^d} e^{i(x-y|\xi)}\varphi(y,\xi)u(y)dyd\xi.$$

This choice of the quantization process makes the computation of section 5 simpler. If the function $\varphi(x,\xi)$ is equal to $\varphi_1(x)\varphi_2(\xi)$, then $\varphi^D u = \mathcal{F}^{-1}\bigl(\varphi_2(\mathcal{F}(\varphi_1 u)\bigr)$. Moreover we have

$$\mathcal{F}(\varphi^D u)(\xi) = \int_{\mathbb{R}^d} e^{-i(y|\xi)}\varphi(y,\xi)u(y)dy.$$

Later on in this paper we shall need to decompose L^2 functions whose Fourier transform is supported in the ring $\mathcal{C}$ using these operators φ^D. These two lemmas are proved in [6].

Lemma 4.1. *A sequence $(X_\nu)_{\nu\in\mathcal{Z}}$ exists such that two sequencies $(\varphi_\nu)_{\nu\in\mathcal{Z}}$ and $(\psi_\nu)_{\nu\in\mathcal{Z}}$ which satisfy the following properties.*

- *the support of φ_ν is included in a ball $B_\nu \stackrel{\text{def}}{=} B_g(X_\nu,r)$,*
- *A sequence $(C_j)_{j\in\mathbb{N}}$ exists (which depends only on r and not in K and h) such that*

$$\forall \nu\in\mathcal{Z}\,,\ \|\varphi_\nu\|_{j,g} \leq C_j,$$

- *the functions ψ_ν are not supported in B_ν but confined, i.e. a sequence $(C_N)_{N\in\mathbb{N}}$ exists such that*

$$\forall \nu\in\mathcal{Z}\,,\ \|\psi_\nu\|_{N,g,X} \stackrel{\text{def}}{=} \sup_{\substack{k\leq N\\ X\in T^\star\mathbb{R}^d}} (1+\lambda^2 g(X-B_\nu))^N \sup_{\substack{(T_\ell)_{1\leq\ell\leq k}\\ g(T_\ell)\leq 1}} |D^k\psi_\nu(X)(T_1,\cdot,T_k)| \leq C_N,$$

- *For any function u of L^2 whose Fourier transform has a support included in $\mathcal{C}$, we have*
$$\sum_{\nu\in\mathcal{Z}} \varphi_\nu^D \psi_\nu^D u = \sum_{\nu\in\mathcal{Z}} \varphi_\nu^D u = u.$$

Such partitions of unity are "compatible" with L^2 in the following sense.

Lemma 4.2. *A constant C exists such that*
$$C^{-1}\|u\|_{L^2}^2 \leq \sum_\nu \|\varphi_\nu^D u\|_{L^2}^2 \leq \ C\|u\|_{L^2}^2 \quad \text{and} \quad \sum_\nu \|\psi_\nu^D u\|_{L^2}^2 \leq C\|u\|_{L^2}^2.$$

Lemma 4.3. *The operator φ^D maps L^p into L^p for any $p \in [1,\infty]$. More precisely, a constant C and an integer N exists such that*
$$\forall X_0 \in T^\star\mathbb{R}^d\,,\ \|\varphi^D a\|_{L^p} \leq C\|\varphi\|_{N,g,X_0}\|a\|_{L^p}.$$

Now we can define the concept of microlocalized function.

Definition 4.1. *Let X_0 be a point of $T^\star\mathbb{R}^d$ and (C_0,r) a couple of positive real numbers. A function u in $L^2(\mathbb{R}^d)$ is said to be (C_0,r)-microlocalized in X_0 if a sequence of integer $(k_N)_{N\in\mathbb{N}}$ exists such that, for any N, the quantities*
$$\mathcal{M}^{C_0,r}_{X_0,N}(u) \stackrel{\text{def}}{=} \sup_{g(X-X_0)^{\frac12}\geq C_0 r} \lambda^{2N} g(X-X_0)^N \sup_{\substack{\varphi\in\mathcal{D}(B_g(X,r))\\ \|\varphi\|_{k_N,g}\leq 1}} \|\varphi^D u\|_{L^2}$$
are finite ($B_g(X,r)$ denotes the set of points of $T^\star\mathbb{R}^d$ such that $g(Y-X)^{\frac12}\leq r$).

A basic example of microlocalized functions is given by the following proposition which is a corollary of Theorem 2.2.1. of [7].

Proposition 4.1. *A sequence of integers $(k_N)_{N\in\mathbb{N}}$ and a sequence of positive real numbers $(C_N)_{N\in\mathbb{N}}$ exist such that the following properties are satisfied. Let X_0 be a point of $T^\star\mathbb{R}^d$, φ_0 a function in $\mathcal{D}(B_g(X_0,r))$ and u a function of $L^2(\mathbb{R}^d)$. Then the function $\varphi_0^D u$ is $(3,r)$-microlocalized in X_0 and, for any N, we have*
$$\mathcal{M}^{3,r}_{X_0,N,g}(u) \leq C_N\|\varphi_0\|_{k_N,g}\|u\|_{L^2}.$$

The concept of uniformly microlocalized families of functions will be a basic tool.

Definition 4.2. *Let $g \stackrel{\text{def}}{=} (g_a)_{a\in A}$ be a family of metrics, $\mathcal{X} \stackrel{\text{def}}{=} (X_a)_{a\in A}$ a family of points of $T^\star\mathbb{R}^d$ and (C_0,r) a pair of positive real numbers. A family of functions $U \stackrel{\text{def}}{=} (u_a)_{a\in A}$ of $L^2(\mathbb{R}^d)$ is said to be uniformly (C_0,r)-microlocalized in $\mathcal{X}$ with respect to g if, for any integer N,*
$$\mathcal{M}^{C_0,r}_{N,X,g}(U) \stackrel{\text{def}}{=} \sup_{a\in A} \mathcal{M}^{C_0,r}_{X_a,N,g_a}(u_a) < \infty.$$

4.2. **A lemma about the product.** Using suitable integrations by part, we prove in [3] the following lemma.

Lemma 4.4. *A constant C_0 exists such that, for any integer N, a constant C_N and an integer k_N exist which satisfy the following properties.*

If u_1 and u_2 are two L^2 functions on $\mathbb{R}^d$, if χ is a function of $\mathcal{D}(\mathbb{R}^d)$ supported in an euclidian ball of radius r, if φ_1 and φ_2 are two functions of $\mathcal{D}(T^\star\mathbb{R}^d)$ respectively supported in $B_g(Y_1,r)$ and in $B_g(Y_2,r)$, then if $g(\check{Y}_1-Y_2)^{\frac12}\geq C_0 r$, for any N, we have
$$\big\|\chi(h^{-1}D)(\varphi_1^D u_1 \varphi_2^D u_2)\big\|_{L^1} \leq C_N\|\varphi_1\|_{k_N,g}\|\varphi_2\|_{k_N,g}\big(1+\lambda^2 g(\check{Y}_1-Y_2)\big)^{-N}\|u_1\|_{L^2}\|u_2\|_{L^2}$$
where we are defined $\check{Y} \stackrel{\text{def}}{=} (y,-\eta)$ if $Y=(y,\eta)$.

5. The propagation theorem

Let us prove that microlocalization properties propagates along the flow of P_Λ.

Theorem 5.1. *A constant C_0 exists which satisfies the following property.*

Let us consider a point $Y_0=(y_0,\eta_0)$ of $T^\star\mathbb{R}^d$ such that η_0 belongs to $\mathcal{C}$, a smooth function ϕ supported in $B_{g_a}(Y_0,r)$ and a function γ of L^2. Then $\mathcal{I}^\pm_\Lambda(\phi^D\gamma)(\tau,\cdot)$ is (C_0,r)-microlocalized near $\Psi^\pm_\Lambda(\tau,Y_0)$. Moreover, for any integer N, a constant C and an integer k exist (which depend only on N) such that

$$\mathcal{M}^{C_0,r}_{\Psi^\pm_\Lambda(\tau,Y_0),N,g_a}\big(\mathcal{I}^\pm_\Lambda(\phi^D\gamma)(\tau,\cdot)\big)\leq C\,\|\phi\|_{k,g_a}\|\gamma\|_{L^2}.$$

In the following proof of this theorem, we shall drop the exponent $^\pm$ for sake a simplicity of the notations. By definition of the microlocalized functions, we have to estimate

$$\mathcal{J}\stackrel{\rm def}{=}\mathcal{F}\big(\varphi^D_{Z_\tau}\mathcal{I}_\Lambda(\phi^D\gamma)(\tau,\cdot)\big)$$

where Z_τ is a point of $T^\star\mathbb{R}^d$ such that $g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))^{\frac12}\geq C_0 r$. By definition, we have

$$\begin{aligned}\mathcal{J}(\zeta)&=\int_{\mathbb{R}^d}\mathcal{K}(\zeta,z)\gamma(z)dz\quad\text{with}\\ \mathcal{K}(\zeta,z)&\stackrel{\rm def}{=}\int_{\mathbb{R}^{2d}}e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)dyd\eta.\end{aligned}$$

The proof consists in integrations by parts in the above integral with respect to $\mathcal{L}$ defined by

$$\begin{aligned}\mathcal{L}f&\stackrel{\rm def}{=}\frac{1}{1+|\Theta|^2}\bigg(f-i|I_\Lambda|^{-\frac12}\Theta_y\partial_\eta f-i|I_\Lambda|^{\frac12}\Theta_\eta\partial_y f\bigg)\quad\text{with}\\ \Theta&\stackrel{\rm def}{=}(\Theta^y,\Theta^\eta)\stackrel{\rm def}{=}\Big(|I_\Lambda|^{-\frac12}\big((\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z\big),|I_\Lambda|^{\frac12}\big(\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)\Big).\end{aligned}$$

It is obvious that $\mathcal{L}\big(e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}\big)=e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}$. So as usual, we have

$$\mathcal{K}(\zeta,z)=\int_{\mathbb{R}^{2d}}e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}({}^t\mathcal{L})^N\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)dyd\eta$$

Let us state the following technical lemma.

Lemma 5.1. *For any integer N, a family of functions $(L_{\alpha,N})_{|\alpha|\leq N}$ exists such that $L_{\alpha,N}(Y,\mathcal{Y})$ is a smooth function from $T^\star\mathbb{R}^d\times(T^\star\mathbb{R}^d)^{M_N}$ and such that*

$$\|\partial^\beta_Y L_{\alpha,N}(Y,\cdot)\|_{L^\infty((T^\star\mathbb{R}^d)^{M_N})}\leq C_{N,|\beta|}(1+|Y|^2)^{-\frac{N+|\beta|}{2}}.\tag{15}$$

Moreover, they satisfy

$$({}^t\mathcal{L})^N f=\sum_{|\alpha|\leq N}L_{\alpha,N}(\Theta,(\partial^\beta\Theta)_{|\beta|\leq N})\widetilde{\partial}^\alpha f$$

where $\widetilde{\partial}$ denotes differentiation of length 1 for the metric $\widetilde{g}_a$ defined by

$$\widetilde{g}_a(dy^2,d\eta^2)\stackrel{\rm def}{=}|I_\Lambda|^{-1}dy^2+|I_\Lambda|d\eta^2=\lambda g_a(dy^2,d\eta^2).$$

As the metric $\widetilde{g}_a$ is greater than $g_\Lambda\stackrel{\rm def}{=}\Lambda^{-2}dx^2+d\eta^2$, we have thanks to Leibniz formula that derivatives of $\widetilde{g}_a$-length 1 of $\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)$ are bounded uniformly with respect to the involved parameters. So using Lemma 5.1, we have

$$\forall N\,,\ \exists C_N\ /\ \Big|({}^t\mathcal{L})^N\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)\Big|\leq C_N(1+|\Theta|^2)^{-\frac N2}.$$

So by definition of Θ and $\widetilde{g}_a$, we infer that, for any integer N, we have

$$|\mathcal{K}(\zeta,z)| \leq C_N \int_{\substack{|y-z_\tau|\leq rK\\ |\eta-\eta_0|\leq rh}} \Big(1+\lambda g_a\big(\nabla_\eta \Phi_\Lambda(\tau,y,\eta)-z, \nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)\Big)^{-N} dyd\eta.$$

Now let us apply Lemma 3.2. Using the fact that $g_a(Z_\tau,\Psi_\Lambda(\tau,Y_0))^{\frac12}$ is greater than $C_0 r$, that (z,η) belongs to $B_{g_a}(Y_0,r)$ and (y,ζ) to $B_{g_a}(Z_\tau,r)$ we infer that

$$|\mathcal{K}(\zeta,z)| \leq C_N\Big(1+\lambda g_a\big(Z_\tau-\Psi_\Lambda(\tau,Y_0)\big)\Big)^{-N}\Big(1+\lambda g_a\big((z,\zeta)-(y_0,\Psi^\eta_\Lambda(\tau,Y_0))\big)\Big)^{-N}$$
$$\times \int_{\substack{|y-z_\tau|\leq rK\\ |\eta-\eta_0|\leq rh}} \frac{dyd\eta}{\Big(1+\lambda g_a\big(\nabla_\eta \Phi_\Lambda(\tau,y,\eta)-z, \nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)\Big)^{N}}\,\cdot$$

Let us state the change of variables

$$y' = |I|^{-\frac12}\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z\big)\quad\text{and}\quad \eta'=|I|^{\frac12}\big(\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big).$$

As the jacobian of this change of variables is closed to 1, it turns out that

$$|\mathcal{K}(\zeta,z)| \leq C_N\Big(1+\lambda g_a\big(Z_\tau-\Psi_\Lambda(\tau,Y_0)\big)\Big)^{-N}\Big(1+\lambda g_a\big((z,\zeta)-(y_0,\Psi^\eta_\Lambda(\tau,Y_0))\big)\Big)^{-N}.$$

Immediate integrations imply that

$$|I_\Lambda|^{-\frac d2}\int|\mathcal{K}(\zeta,z)|dz + |I_\Lambda|^{\frac d2}\int|\mathcal{K}(\zeta,z)|d\zeta \leq C_N\big(1+\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))\big)^{-N}.$$

By Schur's lemma we get that, for any N

$$\|\mathcal{J}\|_{L^2}\leq C_N\big(1+\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))\big)^{-N}\|\gamma\|_{L^2}.$$

As $g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))^{\frac12}\geq C_0 r$, then $\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))\geq C_0 r\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))^{\frac12}$. So Theorem 5.1 is proved.

6. The conclusion of the proof

To conclude the proof of theorem 2.1 let us first apply Lemma 4.4 and Theorem 5.1 to concentrate on real interaction. Because variable coefficents do not respect the localization in frequency space, we shall need to decompose the interval I_Λ. In this section, we shall state

$$\mathcal{J}(\tau,y)\stackrel{\text{def}}{=}\chi(h^{-1}D)\big(\mathcal{I}_\Lambda(\gamma_1)(\tau,y)\mathcal{I}_\Lambda(\gamma_2)(\tau,y)\big).$$

The equivalent of Identity (5) of the constant coefficient case is the following lemma.

Lemma 6.1. *Let $J=(\tau_J,\tau_J^+)$ be a subinterval of I_Λ such that*

$$|J|\leq h|I_\Lambda|\quad\text{and}\quad \|\nabla G_\Lambda\|_{L^1_J(L^\infty)}\leq h\|\nabla G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)}.$$

Then two families (ϕ_μ) and (θ_μ) of confined symbols exists such that, for any integer N,

$$\forall\mu\,,\ \|\phi_\mu\|_{N,g_a,\Psi_\Lambda(\tau_J,Y_\mu)}+\|\theta_\mu\|_{N,g_a,\check\Psi_\Lambda(\tau_J,Y_\mu)}\leq C_N$$

and, for any N, a constant C_N exists such that

$$\begin{aligned}\|\mathcal{J}-\underline{\mathcal{J}}\|_{L^1_J(L^\infty)} &\leq C_N h\lambda^{-N}(|I_\Lambda|h^2)h^{d-2}\|\gamma_1\|_{L^2}\|\gamma_2\|_{L^2}\quad\textit{with}\\ \underline{\mathcal{J}}(\tau) &\stackrel{\text{def}}{=} \sum_{\substack{\mu\\ \mu'\in A_\mu}}\chi(h^{-1}D)\Big(\phi^D_\mu\mathcal{I}_\Lambda\big(\varphi^D_\mu\psi^D_\mu\gamma_1\big)(\tau,\cdot)\times\theta^D_\mu\mathcal{I}_\Lambda\big(\varphi^D_{\mu'}\psi^D_{\mu'}\gamma_2\big)(\tau,\cdot)\Big)\quad\textit{and}\\ A_\mu &\subset \big\{\mu'\,/\,g_a(Y_{\mu'}-\Psi_\Lambda(\tau_J,\check\Psi_\Lambda(\tau_J,Y_\mu))\leq Cr\big\}.\end{aligned}$$

Now we shall decompose the interval I_Λ on subintervals J such that the above lemma can be applied on J. Let us introduce the following function on the interval $\mathcal{I}_\Lambda$

$$H(\tau) \stackrel{\text{def}}{=} \Big(\sum_\mu \|\mathcal{I}_\Lambda(\varphi_\mu^D \psi_\mu^D \gamma_1)(\tau,\cdot)\|_{L^\infty}^2\Big)^{\frac{1}{2}} \Big(\sum_\mu \|\mathcal{I}_\Lambda(\varphi_\mu^D \psi_\mu^D \gamma_2)(\tau,\cdot)\|_{L^\infty}^2\Big)^{\frac{1}{2}}.$$

Using precised Strichartz estimates, we get that

$$\|\mathcal{I}_\Lambda(\varphi_\mu^D \psi_\mu^D \gamma_j)\|_{L^2_{I_\Lambda}(L^\infty)} \leq C\big(\log(e+|I_\Lambda|)\big)^{\frac{1}{2}} h^{\frac{d-2}{2}} \|\psi_\mu^D \gamma_j\|_{L^2}.$$

So, using Cauchy-Schwartz inequality and Lemma 4.2, we get that

$$\begin{aligned} \int_{I_\Lambda} H(\tau)d\tau &\leq C\big(\log(e+|I_\Lambda|)\big) h^{d-2} \Big(\sum_\mu \|\psi_\mu^D \gamma_1\|_{L^2}^2\Big)^{\frac{1}{2}} \Big(\sum_\mu \|\psi_\mu^D \gamma_2\|_{L^2}^2\Big)^{\frac{1}{2}} \\ &\leq C\big(\log(e+|I_\Lambda|)\big) h^{d-2} \|\gamma_1\|_{L^2}^2 \|\gamma_2\|_{L^2}^2. \end{aligned}$$

As in section 2, we decompose I_Λ in intervals J such that

$$|J| \leq h|I_\Lambda|\,,\ \|\nabla G_\Lambda\|_{L^1_J(L^\infty)} \leq h\|\nabla G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \quad \text{and} \quad \int_J H(\tau)d\tau \leq h\int_{I_\Lambda} H(\tau)d\tau.$$

Let us estimate $\|\underline{\mathcal{J}}\|_{L^1_J(L^\infty)}$. Lemma 4.3 implies that

$$\|\underline{\mathcal{J}}(\tau)\|_{L^\infty} \leq \sum_{\substack{\mu \\ \mu' \in A_\mu}} \big\|\mathcal{I}_\Lambda\big(\varphi_\mu^D \psi_\mu^D \gamma_1\big)(\tau,\cdot)\big\|_{L^\infty} \big\|\mathcal{I}_\Lambda\big(\varphi_{\mu'}^D \psi_{\mu'}^D \gamma_2\big)(\tau,\cdot)\beta igr\|_{L^\infty}.$$

Because the number of elements of A_μ is bounded uniformly with respect to μ, by Cauchy-Schwarz inequality, we infer that $\|\underline{\mathcal{J}}(\tau)\|_{L^\infty} \leq H(\tau)$. So by construction of J,

$$\|\underline{\mathcal{J}}\|_{L^1_J(L^\infty)} \leq Ch\big(\log(e+|I_\Lambda|)\big) h^{d-2} \|\gamma_1\|_{L^2}^2 \|\gamma_2\|_{L^2}^2.$$

As in section 2, the number of intervals J is less than Ch^{-1}. As $\lambda = |I_\Lambda|h^2$, the theorem is proved if we apply Lemma 6.1 with N large enough.

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