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Trace class pseudodifferential calculus with operator valued symbols and unusual index formulas

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Abstract

For several classes of pseudodifferential operators with operator-valued symbol analytic index formulas are found. The common feature is that usual index formulas are not valid for these operators. Applications are given to pseudodifferential operators on singular manifolds.

1. Introduction Analytical index formulas play an important part in the study of topological characteristics of elliptic operators. They complement index formulas expressed in topological and algebraical terms, and often enter in these formulas as an ingredient. For elliptic pseudodifferential operators (ΨDO) on compact manifolds, such formulas were found by Fedosov [4]; later, analytical index formulas for elliptic boundary value problems were obtained in [5]. These formulas involve an integral, with integrand containing analytical expressions for the classical characteristic classes entering into the co-homological formulas.

In 90-s a systematic study started of topological characteristics of operators on singular manifolds - [7], [8], [10], [12], [13], [18] etc. Even before these papers had appeared, it became clear that analysis of operators on singular manifolds must involve many-level symbolic structure with a hierarchy of operator-valued symbols responsible for the singularities (see [3],[9],[14-16], etc.). Each of these symbols contributes to the index formulas. In some, topologically simple, cases, such contributions can be separated, and thus the problem arises of calculation of the index for ΨDO with operator-valued symbol. However, even one-dimensional examples show that the usual formulas, originating from the scalar or matrix situation, may be unsuitable in the operator-valued case. Let A be the Toeplitz operator on the real line $\mathbb{R}$, with symbol $a(x)$, i.e. it acts in the Hardy space $H^2(\mathbb{R})$ by the formula

$$Au = Pau$$

where $P : L_2 \to H^2$ is the Riesz projection. Under the condition that the symbol $a(x)$ is smooth, invertible and stabilises to 1 at infinity, the operator A is Fredholm and its index equals $\operatorname{ind} A = -(2\pi i)^{-1} \int a(x)^{-1} a'(x) dx$. If we consider a Toeplitz operator in the space of *vector*-functions, so that the symbol is a matrix, the same formula for the index holds, with a natural modification:

$$\text{ind } A = -(2\pi i)^{-1} \int \text{tr}\,(a(x)^{-1} a'(x)) dx, \tag{1}$$

where tr is the usual matrix trace. However, when we move to an even more general case, of functions with values in an infinite-dimensional Hilbert space, so that $a(x)$ is an operator in this space, the formula (1) makes sense only under the condition that $a^{-1}a'$ belongs to the trace class. If this is not the case, (1) makes no sense, so even if the Toeplitz operator happens to be Fredholm, one needs another formula for the index to be found (and justified). We call such formulas *u*nusual.

In the present paper we consider a class of ΨDO with operator symbols and find analytical index formulas for elliptic operators in this class. The formulas depend on the quality of the symbol, and are derived by means of cyclic co-homology approach. Some operators arising in analysis on singular manifolds fit into our scheme. Thus, we find index formulas for Toeplitz operators with operator-valued symbols, for cone Mellin operators and for edge ΨDO arising in analysis on manifolds with edge-type singularities. Our abstract approach to the index formulas requires less structure from the operator symbols compared with the traditional one (see, e.g., [7], [13-16]), therefore we present here a new version of the edge calculus.

2. The algebraical scheme We recollect some constructions from the K-theory for operator algebras and cyclic cohomologies (see [1],[2].) Let $\mathfrak{S}$ be a Banach *-algebra with unit, $M(\mathfrak{S})$ be the set of matrices over $\mathfrak{S}$. The groups $\mathbf{K}_j(\mathfrak{S}), j = 0, 1$ are the usual K-groups in the theory of Banach *-algebras. In particular, $\mathbf{K}_1(\mathfrak{S})$ consists of equivalence classes of invertible matrices in $M(\mathfrak{S})$, i.e., elements in $GL(\mathfrak{S})$. If $\mathfrak{S}$ does not have a unit, one attaches it and thus replaces $M(\mathfrak{S})$ by $M(\mathfrak{S})^+$ in the latter definition. The $\mathbf{K}$-cohomological group $\mathbf{K}^1(\mathfrak{S})$ consists of equivalence classes of 'quantisations', i.e. unital homomorphisms of the algebra of matrices over $\mathfrak{S}$ to the Calkin algebra in some Hilbert space $\mathfrak{H}$, or, what is equivalent, *-linear mappings $\tau : M(\mathfrak{S}) \to B(\mathfrak{H})$, multiplicative up to a compact error. Each element $[t] \in \mathbf{K}^1(\mathfrak{S})$ defines the index homomorphism $\text{ind}_{[\tau]} : \mathbf{K}_1(\mathfrak{S}) \to \mathbb{Z}$, associating to the matrix $a \in GL(\mathfrak{S})$ the index of the operator $\tau(a)$. Thus we have the integer index coupling between $\mathbf{K}^1(\mathfrak{S})$ and $\mathbf{K}_1(\mathfrak{S})$: $[\tau] \times [a] = \text{ind } \tau(a)$. Again, if $\mathfrak{S}$ is non-unital, the unit is attached.

For a normed *-algebra $\mathfrak{S}$, the group $C^k_\lambda(\mathfrak{S})$ of cyclic cochains consists of $(k + 1)$- linear continuous functionals $\varphi(a_0, a_1, ..., a_k)$, cyclic in the sense $\varphi(a_0, a_1, ..., a_k) = (-1)^k \varphi(a_1, a_2,..., a_0)$. The Hochschild co-boundary operator $b : C^k_\lambda(\mathfrak{S}) \to C^{k+1}_\lambda(\mathfrak{S})$ generates, in a usual way, co-homology groups $HC^k_\lambda(\mathfrak{S})$.

There is also a coupling of $HC^{2k+1}_\lambda(\mathfrak{S})$ and $K_1(\mathfrak{S})$ (see [2], Ch.III.3):

$$[\varphi] \times_k [a] = \gamma_k (\varphi \otimes \text{tr}\,)(a^{-1} - 1, a - 1, a^{-1} - 1, a - 1, ..., a^{-1} - 1, a - 1), \tag{2}$$

where tr is the matrix trace and γ_k is the normalisation constant, chosen in [2] to be equal to $(2i)^{-1/2} 2^{-2k-1} \Gamma(k + \frac{3}{2})$. In this context, the problem of finding an analytic index formula for a given 'quantisation' $[\tau] \in \mathbf{K}^1(\mathfrak{S})$ consists in determining a proper element $[\varphi] = [\varphi^{[\tau]}]$ in the cohomology group of some order, $[\varphi] \in HC_\lambda^{2k+1}(\mathfrak{S})$ such that

$$[\tau] \times [a] = [\varphi] \times_k [a], \ [a] \in \mathbf{K}_1(\mathfrak{S}), \tag{3}$$

or even a cyclic cocycle $\varphi \in C_\lambda^{2k+1}(\mathfrak{S})$ such that (3) holds.

Unfortunately, for $*$-algebras arising in concrete analytical problems, the cyclic co-homology groups are often not rich enough to carry the index classes one needs. Therefore, one has to chose some 'natural' dense local subalgebra $\mathfrak{S}_0 \subset \mathfrak{S}$, equipped with a norm, stronger than the initial norm in $\mathfrak{S}$, having rich enough cyclic co-homologies. On the level of $\mathbf{K}$-groups this substitution is not felt, since the natural inclusion $\iota : \mathfrak{S}_0 \to \mathfrak{S}$ generates isomorphism $\iota^* : \mathbf{K}^1(\mathfrak{S}) \to \mathbf{K}^1(\mathfrak{S}_0)$, but in co-homologies this may produce analytical index formulas. Moreover, the choice of the dimension $2k+1$ of the target cyclic cohomology group may depend on the properties of the subalgebra $\mathfrak{S}_0$. An example of this can be found in [2], II.2.α, III.6.β. There, for the dense local subalgebra $\mathfrak{S}_1 = C_0^1(\mathbb{R}^1)$ in the C^*-algebra $\mathfrak{S} = C_0(\mathbb{R}^1)$, one associates to the Toeplitz quantisation $[\tau]$ the class $[\varphi_1^{[\tau]}] \in HC_\lambda^1(\mathfrak{S}_1)$,

$$\varphi_1^{[\tau]}(a_0, a_1) = -(2\pi i)^{-1} \int \operatorname{tr}(a_0 da_1). \tag{4}$$

Coupling with this class gives the standard formula (1) for the index of the Toeplitz operator. However, the cocycle (4) is not defined on larger subalgebras in $\mathfrak{S}$, for example on the Lipschitz class $\mathfrak{S}_\gamma = C_0^\gamma(\mathbb{R}^1)$, $0 < \gamma < 1$, so one can't use (1) for calculating the index. To deal with this situation, it is proposed in [2] to consider the image of $[\varphi_1^{[\tau]}]$ in $HC_\lambda^{2l+1}(\mathfrak{S}_1)$ under l times iterated suspension homomorphism $S : HC_\lambda^k(\mathfrak{S}) \to HC_\lambda^{k+2}(\mathfrak{S})$ which is consistent with the coupling (3):

$$[\varphi] \times_k [a] = S[\varphi] \times_{k+1} [a], \ [a] \in \mathbf{K}_1(\mathfrak{S}), [\varphi] \in H_\lambda^{2k+1}(\mathfrak{S}),$$

with properly chosen l. This produces cocycles $\varphi_{2l+1}^{[\tau]}$ on $\mathfrak{S}_1$, functionals in $2l+2$ variables, which give new analytical index formulas for the Toeplitz operators with differentiable symbols - see the formula on p. 209 in [2]. These cocycles, for $2l+1 > \gamma$, admit continuous extension to the algebra $\mathfrak{S}_\gamma$, thus defining elements in $HC_\lambda^{2l+1}(\mathfrak{S}_\gamma)$ and giving index formulas for less and less smooth functions.

3. Operator-valued symbols In this paper we deal only with operators acting on functions defined on the Euclidean space. For this situation, we describe here algebras of operator-valued symbols and develop the corresponding pseudodifferential calculus.

In the literature, there exist several versions of operator-valued pseudodifferential calculi, each adopted to some particular, more or less general, situation (see, e.g., [3],[14], [19]). Each time the problem arises, of finding a convenient description for the property of improvement of the symbol under the differentiation in co-variables.

Let us, in the most simple case, in $L_2(\mathbb{R}^n) = L_2(\mathbb{R}^m \times \mathbb{R}^k)$, consider the pseudodifferential operator $a(x, D_x)$ with a symbol $a(x, \xi) = a(y, z, \eta, \zeta)$, zero order homogeneous and smooth in ξ, $\xi \neq 0$, which we treat as an operator in $L_2(\mathbb{R}^m, L_2(\mathbb{R}^k))$ with operator valued symbol $\mathbf{a}(y,\eta) = a(y,z,\eta,D_z)$. Differentiation in $\eta, \eta \neq 0$, produces the operator symbol $\partial_\eta \mathbf{a}$ of order -1, one more η-differentiation gives the symbol $\partial_\eta^2 \mathbf{a}$ of order -2, etc. We refer to this effect by saying that the quality of the operator symbol is improved under η-differentiation. Usually, in concrete situations, this property is described by introducing proper scales of 'smooth' spaces, usually, the weighted Sobolev spaces in $\mathbb{R}^k$, and describing the spaces where the differentiated operator symbol acts. This, however, requires a rather detailed analysis of action of 'transversal operators' $a(y,z,\eta,D_z)$ in these scales and becomes fairly troublesome in singular cases. At the same time, these extra spaces are in no way reflected in index formulas and are superfluous in this context. Our approach is based on describing the property of improvement of operator valued symbol under differentiation not by improvement of smoothness but by improvement of *compactness.* So, in the above example, suppose that the symbol a has compact support in z variable. Then, if the differential order γ of the operator is negative, the operator symbol $\mathbf{a}(y,\eta)$ is a compact operator, and its singular numbers $s_j(\mathbf{a}(y,\eta))$ decay as $O(j^{\gamma/k})$. Each differentiation in η variable, lowering the differential order, leads to improvement of the decay rate of these s-numbers; after N differentiations, the s-numbers of the differentiated symbol decay as $O(j^{(\gamma-N)/k})$. The decay rate as $|\eta| \to \infty$ of the operator norm of the differentiated symbol also improves under the differentiation. This justifies the introduction of classes of symbols in the abstract situation.

So, let $\mathfrak{K}$ be a Hilbert space. By $\mathfrak{s}_p = \mathfrak{s}_p(\mathfrak{K})$, $0 < p < \infty$ we denote the Shatten class of operators T in $\mathfrak{K}$ for which the sequence of singular numbers (s-numbers) $s_j(T) = (\lambda_j(T^*T))^{1/2}$ belongs to l_p. In the definition below, as well as in the formulations, N is some sufficiently large integer. We do not specify the particular choice of N in each case, as long as it is of no importance.

Definition 3.1. *Let $\gamma \leq 0, q > 0$. The class $\mathcal{S}_q^\gamma = \mathcal{S}_q^\gamma(\mathbb{R}^m \times \mathbb{R}^{m'}, \mathfrak{K})$ consists of functions $\mathbf{a}(y,\eta)$, $(y,\eta) \in \mathbb{R}^m \times \mathbb{R}^{m'}$, such that for any (y,η), $\mathbf{a}(y,\eta)$ is a bounded operator in $\mathfrak{K}$ and, moreover,*

$$\|D_\eta^\alpha D_y^\beta \mathbf{a}(y,\eta)\| \leq C_{\alpha,\beta}(1+|\eta|)^{-|\alpha|+\gamma}, \tag{5}$$

$$|D_\eta^\alpha D_y^\beta \mathbf{a}(y,\eta)|_{\frac{q}{-\gamma+|\alpha|}} \leq C_{\alpha,\beta}. \tag{6}$$

for $|\alpha|, |\beta| \leq N$.

Note here that when M is a k-dimensional compact manifold and $a(y,z,\eta,\zeta)$ is a classical symbol of order less than γ on $\mathbb{R}^m \times M$, the operator symbol $\mathbf{a}(y,\eta) = a(y,z,\eta,D_z)$ acting in $\mathfrak{K} = L_2(M)$ belongs to $\mathcal{S}_k^\gamma$ for any N. For $-\gamma + |\alpha| - q > 0$ the derivatives in (5), (6) belong to trace class and for $-\gamma + |\alpha| - q > m$ the integral of its trace class norm with respect to η converges. Also the classes $\mathcal{S}_k^\gamma$ behave themselves in a natural way under multiplication.

We are going to sketch the operator-valued version of the usual pseudodifferential calculus. The main difference of this calculus from the usual one is the notion of 'negligible' operators. In the scalar case, one considers as negligible the infinitely smoothing operators. In our case, we take trace class operators as negligible, and it is up to a trace class error, that the classical relations of the pseudodifferential calculus will be shown to hold. This is sufficient for the needs of index theory. For $\mathbf{a}(y,y',\eta)\in\mathcal{S}_q^\gamma(\mathbb{R}^{2m}\times\mathbb{R}^m,\mathfrak{K})$, we define the ΨDO as

$$(OPS(\mathbf{a})u)(y)=(\mathbf{a}(y,y',D_y)u)(y)=(2\pi)^{-m}\int\int e^{i(y-y')\eta}\mathbf{a}(y,y',\eta)u(y')d\eta dy', \tag{7}$$

where $u(y)$ is a function on $\mathbb{R}^m$ with values in $\mathfrak{K}$.

The main technical tool in the calculus is the following proposition giving a sufficient condition for a ΨDO to belong to trace class.

Proposition 3.2. *Let the operator-valued symbol $\mathbf{a}(y,y',\eta)$ in $\mathbb{R}^{2m}\times\mathbb{R}^m$ be smooth with respect to y,y', let all y,y'-derivatives $D_y^\beta D_{y'}^{\beta'}\mathbf{a}$ up to some (sufficiently large) order N be trace class operators with trace class norm bounded uniformly in y,y'. Suppose that $g(y),h(y)=O((1+|y|)^{-2m})$. Then the operator $h\mathbf{a}(y,y',D_y)g$ belongs to $\mathfrak{s}_1(L_2(\mathbb{R}^m;\mathfrak{K}))$.*

Remark. Note that we do not impose on the symbol any smoothness conditions in η variable. This proves to be useful in concrete applications.

If symbols belong to the classes $\mathcal{S}_q^\gamma$, the usual properties and formulas in the pseudodifferential calculus hold, with our modification of the notion of negligible operators. The proofs are based on Proposition 3.2.

Theorem 3.3. (Pseudo-locality) *Let the symbol $\mathbf{a}(y,y',\eta)$ belong to $\mathcal{S}_q^\gamma(\mathbb{R}^{2n}\times\mathbb{R}^n)$ for some $q>0,\gamma\le 0$, let h,g be bounded functions with disjoint supports, at least one of them being compactly supported. Then $h\mathbf{a}(y,y',D)g\in\mathfrak{s}_1(L_2(\mathbb{R}^m;\mathfrak{K}))$.*

The usual formula expressing the symbol of the composition of operators via the symbols of the factors also holds in the operator-valued situation.

Theorem 3.4. *Let the symbols $\mathbf{a}(y,\eta),\mathbf{b}(y,\eta)$ belong to $\mathcal{S}_q^\gamma(\mathbb{R}^{2m}\times\mathbb{R}^m)$ for some $q>0,\gamma\le 0$ and $h(y)=O((1+|y|)^{-m-1})$. Then, for N large enough, the operator $hOPS(\mathbf{a})OPS(\mathbf{b})$ $-hOPS(\mathbf{c}_N)$ belongs to trace class, where*

$$\mathbf{c}_N=\mathbf{a}\circ_N\mathbf{b}=\sum_{|\alpha|\le N}(\alpha!)^{-1}\partial_\eta^\alpha\mathbf{a}D_y^\alpha\mathbf{b}. \tag{8}$$

A version of Theorem 3.4 will also be used, where the function h is not present, but instead of this, as $y\to\infty$, the symbol $\mathbf{a}$ tends, sufficiently fast, to a symbol $\mathbf{a}_0\in\mathcal{S}_q^\gamma$ not depending on η: there exists a (smooth) function $h(y)=O((1+|y|)^{-m-1})$ such that $h(y)^{-1}(\mathbf{a}(y,\eta)-\mathbf{a}_0(y))\in\mathcal{S}_q^\gamma$. In the course of the paper, it is in this sense we will mean that the symbol *stabilises at infinity.*

Now we introduce the notion of ellipticity for our operators.

Definition 3.5. *The symbol $\mathbf{a}(y,\eta)\in\mathcal{S}_q^0(\mathbb{R}^m\times\mathbb{R}^m)$ stabilizing in y at infinity is called* elliptic *if for $|y|+|\eta|$ large, $\mathbf{a}(y,\eta)$ is invertible and $\|\mathbf{a}^{-1}\|\le C$.*

For small $|\eta|+|y|$, the symbol $\mathbf{a}(y,\eta)^{-1}$ is not necessarily defined. As usual, one often needs a regularising symbol defined everywhere and coinciding with $\mathbf{a}^{-1}$ for large η. This can also be done in our calculus, however the cut-off and gluing operations, used freely in the standard situation, are not so harmless now: even the multiplication by a nice function of η variable may throw us out of the class $\mathcal{S}_q^0$. Therefore we have to be rather delicate when operating with cut-offs. We give the proof here just to show some tricks used in dealing with operator-valued symbols.

Proposition 3.4. *Suppose that the symbol* $\mathbf{a} \in \mathcal{S}_q^0(\mathbb{R}^m \times \mathbb{R}^m)$ *is elliptic. Then there exists a symbol* $\mathbf{r}_0(y,\eta) \in \mathcal{S}_q^0(\mathbb{R}^m \times \mathbb{R}^m)$*, such that* $\mathbf{r}_0(y,\eta) = \mathbf{a}(y,\eta)^{-1}$ *for large* $|y|^2 + |\eta|^2$ *and the symbols* $\mathbf{r}_0\mathbf{a} - 1$ *and* $\mathbf{a}\mathbf{r}_0 - 1$ *belong to* $\mathcal{S}_q^{-1}$.

Proof Suppose that $\mathbf{a}$ is invertible for $|y|^2 + |\eta|^2 \geq R^2$. The inequalities of the form (5), (6) hold for $\mathbf{a}(y,\eta)^{-1}$ for such η. Thus we have to take care of small $|y|^2 + |\eta|^2$ only. Fix some η_0, $|\eta_0| \geq R$. Due to (3.2), the symbol $\mathbf{s}(y,\eta) = 1 - \mathbf{a}(y,\eta_0)^{-1}\mathbf{a}(y,\eta)$ belongs to $\mathfrak{s}_q$ for $|\eta| \leq R$. Set

$$\mathbf{r}_0'(y,\eta) = \mathbf{a}(y,\eta_0)^{-1} \exp(\mathbf{s}(y,\eta) + \mathbf{s}(y,\eta)^2/2 + \cdots + \mathbf{s}(y,\eta)^N/N), \qquad (9)$$

where the expression under the exponent is the starting section of the Taylor series for $-\log(1-\mathbf{s})$. From (9) it follows that $\mathbf{r}_0'$ belongs to $\mathcal{S}_q^0$, is invertible, and, moreover, $\mathbf{r}_0'(y,\eta) - \mathbf{a}^{-1}(y,\eta) \in \mathfrak{s}_{\frac{q}{N}}$ for $|y|^2 + |\eta|^2 \geq R^2$. Now take a cut-off function $\chi \in C_0^\infty(\{|\rho| < 2R\})$ which equals 1 for $|\rho| \leq R$ and set

$$\mathbf{r}_0(y,\eta) = \chi((|y|^2 + |\eta|^2)^{1/2})\mathbf{r}_0'(y,\eta) + (1 - \chi((|y|^2 + |\eta|^2)^{1/2}))\mathbf{a}(y,\eta)^{-1}.$$

the symbols $\mathbf{r}_0\mathbf{a} - 1$ and $\mathbf{a}\mathbf{r}_0 - 1$ *do not* improve their properties under η–differentiation, since the cut-off function prevents this, but they already belong to $\mathfrak{s}_{\frac{q}{N}}$ for all (y,η), together with all derivatives, and therefore (6) holds, for given N.

Remark. Proposition 3.6 illustrates usefulness of introducing symbol classes with only a finite number of derivatives subject to estimates of the form (5), (6). Even if for the symbol $\mathbf{a}$ in the Definition 3.1, estimates (5), (6) hold for all α, β, they hold only for derivatives of order up to N for our regularizer $\mathbf{r}_0$.

The notion of ellipticity is justified by the following construction of a more exact regularizer, inverting the give ΨDO up to a trace class error.

Theorem 3.1. *Let the symbol* $\mathbf{a} \in \mathcal{S}_q^0$ *stabilise in* y *at infinity and be elliptic. Then there exists a symbol* $\mathbf{r}(y,\eta) \in \mathcal{S}_q^0$ *such that*

$$OPS(\mathbf{a})OPS(\mathbf{r}) - 1,\ OPS(\mathbf{r})OPS(\mathbf{a}) - 1 \in \mathfrak{s}_1(L_2(\mathbb{R}^m, \mathfrak{K})).$$

The proof follows the usual 'scalar' reasoning. However, in the scalar calculus, the number of terms retained in the composition formula depends only on the dimension m of the space while in our operator-valued calculus it depends also on the number q involved in the definition of the symbol class.

4. Preliminary index formulas and $\mathbf{K}_1$-theoretical invariants As it follows from Theorem 3.7, a ΨDO with elliptic symbol in the class $\mathcal{S}_q^0$ is Fredholm. In fact, it is already well known for a long time that this is the case even for a much wider class of operator-valued symbols. Under our conditions, we will be able to investigate what the index of such operators can depend on.

We start by establishing an analytical index formulas for elliptic symbols. The first formula is rather rough, preliminary, and it will be improved later. This is the abstract operator-valued version of the 'algebraic index formula' obtained for the matrix situation in [4] and later for some concrete operator symbols in [7]. In what follows, the symbols are supposed to belong to classes $\mathcal{S}_q^0$, with N large enough.

Proposition 4.1. *Let* $\mathbf{a}(y,\eta) \in \mathcal{S}_q^0(\mathbb{R}^m \times \mathbb{R}^m; \mathfrak{K})$ *be an elliptic operator symbol stabilizing in* y *at infinity,* $\mathbf{r}(y,\eta)$ *is the regularizer constructed in Theorem 3.7,* A, R *be the corresponding operators in* $L_2(\mathbb{R}^m, \mathfrak{K})$*. Then, for* M *large enough,*

$$\operatorname{ind} A = \frac{1}{(2\pi)^m} \int\limits_{\mathbb{R}^m \times \mathbb{R}^m} \operatorname{tr}\left[(\mathbf{a} \circ_M \mathbf{r} - \mathbf{r} \circ_M \mathbf{a})\right]_M dy d\eta, \tag{10}$$

where tr *denotes the trace in the Hilbert space* $\mathfrak{K}$.

The proof is based on the classical formula $\operatorname{ind} A = \operatorname{Tr}(AR - RA)$, where the right-hand side is calculated in the terms of the symbols $\mathbf{a}, \mathbf{r}$. It follows the reasoning of [5], with modifications caused by the operator valued specifics.

Analyzing this preliminary index formula, we find out, on which characteristics of the symbol the index actually depends.

Proposition 4.2. *Denote by* $\mathcal{E}_q = \mathcal{E}_q(S_R)$ *the class of norm-continuous invertible operator-valued functions on the sphere* $S_R = \{(y,\eta), |y|^2 + |\eta|^2 = R^2\}$ *having first order* η*-derivatives in* $\mathfrak{s}_q$*. Let* $\mathbf{a}, \mathbf{a}' \in \mathcal{S}_q^0$ *be elliptic symbols stabilizing at infinity and invertible for* $|y|^2 + |\eta|^2 \geq R^2$*. Suppose that the restrictions* $\mathbf{b}$ *and* $\mathbf{b}'$ *of these symbols to the sphere* S_R *are homotopic in* $\mathcal{E}_q$*. Then* $\operatorname{ind} OPS(\mathbf{a}) = \operatorname{ind} OPS(\mathbf{a}')$*.*

The proof is based on lifting the homotopy of the spherical restrictions of symbols to the homotopy in the whole space, and then applying (10).

As a result of our considerations, we can see that the index of the ΨDO with operator-valued symbol depends only on the class of the symbol in $\mathbf{K}_1(\mathcal{E}_q(S_R))$, thus defining a homomorphism

$$\text{IND} : \mathbf{K}_1(\mathcal{E}_q(S_R)) \to \mathbb{Z}. \tag{11}$$

5. Reduction of index formulas The analytical index formula (10) has a preliminary character; it involves higher order derivatives of the symbol and its regularizer. Moreover, it does not reflect the algebraic nature of the index. In fact, (10) contains integration over the ball, while we already know (see Proposition 4.2) that the index depends only on the homotopy class of the symbol on the (large enough) sphere. In other words, (10) does not describe the homomorphism (11) from $\mathbf{K}_1$ for the symbol algebra to $\mathbb{Z}$. Thus a reduction of the formula is needed.

The starting point in this reduction is the result of Fedosov [6] establishing the formula of the required nature for the case of the space $\mathfrak{K}$ of finite dimension:

$$\operatorname{ind} A = c_m \int_{S_R} \operatorname{tr}\left((\mathbf{a}^{-1} d\mathbf{a})^{2m-1}\right), c_m = -\frac{(m-1)!}{(2\pi i)^m (2m-1)!}; \tag{12}$$

in the integrand, taking to power is understood in the sense of exterior product.

Taking into account Theorem 4.2, we can consider (12) not as the expression for the index of operator but as a functional on symbols defined on the sphere. To use the strategy outlined in Sect.2, we represent (12) by means of a cyclic cocycle in a local C^*-algebra.

We define several algebras consisting of operator-valued functions on the sphere $S = S_R$, continuous in the norm operator topology on the (infinite-dimensional) Hilbert space $\mathfrak{K}$. For $1 \leq q < \infty$, the subalgebra $\mathfrak{S}_q$ consists of once differentiable functions with η-derivative belonging to the class $\mathfrak{s}_q(\mathfrak{K})$. An ideal $\mathfrak{S}_q^0$ in $\mathfrak{S}_q$ is formed by the functions having values in $\mathfrak{s}_q(\mathfrak{K})$. The smallest subalgebra $\mathfrak{S}_0^0$ consists of functions with finite rank values, with rank uniformly bounded over S.

We introduce our initial cyclic cocycle: for $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{2m-1} \in \mathfrak{S}_0^0$ we set

$$\tau_{2m-1}(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{2m-1}) = (-1)^{m-1} c_m \int_S \operatorname{tr}(\mathbf{a}_0 d\mathbf{a}_1 \ldots d\mathbf{a}_{2m-1}). \tag{13}$$

The trace in (13) always exists, since at least one factor is a finite rank operator.

In a natural way, the cocycle (13) extends to the algebra $\mathfrak{S}_0$ obtained by attaching a unit to $\mathfrak{S}_0^0$: for $\tilde{\mathbf{a}}_j = \lambda_j \mathbf{1} + \mathbf{a}_j, \lambda_j \in \mathbb{C}, \mathbf{a}_j \in \mathfrak{S}_0^0$, we have

$$\tau_m(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{2m-1}) = \tau_m(\mathbf{a}_0, \ldots, \mathbf{a}_{2m-1}). \tag{14}$$

Next, the cocycle τ_m in (14) extends by continuity to the algebra $\mathfrak{S}_q^0$, for any $q < 2m-1$. Moreover, for an element $\tilde{\mathbf{a}} \in GL(\mathfrak{S}_q^0)$,

$$\operatorname{IND}([\tilde{\mathbf{a}}]) = \tau_m(\tilde{\mathbf{a}}^{-1} - 1, \tilde{\mathbf{a}} - 1, \ldots, \tilde{\mathbf{a}} - 1). \tag{15}$$

To find the index formula for the algebra $\mathfrak{S}_q$, $q < 2m-1$, take some $\mathbf{a} \in GL(\mathfrak{S}_q)$. For any y, $|y| \leq 1$, fix some $\eta_0(y)$ so that $|y|^2 + |\eta_0(y)|^2 = 1$, so that $\eta_0(y)$ depends continuously on y. For the symbol $\mathbf{a}_0(y, \eta) = \mathbf{a}(y, \eta_0(y))$, the index vanishes, since, after the natural continuation to the whole of $\mathbb{R}^{2m}$, it defines an invertible multiplication operator. Thus, for

$$\tilde{\mathbf{a}}(y, \eta) = \mathbf{a}(y, \eta)\mathbf{a}_0^{-1}(y, \eta), \tag{16}$$

we have $\operatorname{IND}([\tilde{\mathbf{a}}]) = \operatorname{IND}([\mathbf{a}])$. This gives us

$$\operatorname{IND}([\mathbf{a}]) = \tau_m(\tilde{\mathbf{a}}^{-1} - 1, \tilde{\mathbf{a}} - 1, \ldots, \tilde{\mathbf{a}} - 1), \mathbf{a} \in GL(\mathfrak{S}_q), \tag{17}$$

with $\tilde{\mathbf{a}}$ defined in (16).

Now we apply the strategy depicted in Sect.2, to construct index formulas for even wider classes of symbols. For doing this, we will use a specific algebraical realisation of the homomorphism S in cyclic co-homologies, introduced in [2]. For an algebra $\mathfrak{S}$, the graded differential algebra $\Omega^*(\mathfrak{S})$ is defined in the following way. Denote by $\tilde{\mathfrak{S}}$ the algebra obtained by adjoining a unit $\mathbf{1}$ to $\mathfrak{S}$. For $n \in \mathbb{N}$, $n \leq 1$, let Ω^n be the linear space

$$\Omega^n = \Omega^n(\mathfrak{S}) = \tilde{\mathfrak{S}} \otimes_{\mathfrak{S}} \mathfrak{S}^{\otimes n};\ \Omega = \oplus \Omega^n.$$

The differential $d : \Omega^n \to \Omega^{n+1}$ is given by

$$d((\mathbf{a}_0 + \lambda \mathbf{1}) \otimes \mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_n) = \lambda \mathbf{1} \otimes \mathbf{a}_0 \otimes \ldots \otimes \mathbf{a}_n \in \Omega^{n+1}.$$

The product, defined in a natural way, satisfies $\tilde{\mathbf{a}}_0 d\mathbf{a}_1 \ldots d\mathbf{a}_n = \tilde{\mathbf{a}}_0 \otimes \mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_n$, $\mathbf{a}_j \in \mathfrak{S}$, and gives Ω the structure of a graded differential algebra. According to [2], any cyclic cocycle $\tau \in C_\lambda^n(\mathfrak{S})$ of dimension n can be represented as

$$\tau(\mathbf{a}_0, \ldots \mathbf{a}_n) = \hat{\tau}(\mathbf{a}_0 d\mathbf{a}_1 \ldots d\mathbf{a}_n),$$

where $\hat{\tau}$ is a closed graded trace of dimension n on $\Omega(\mathfrak{S})$. In our particular case, this representation is generated by

$$\hat{\tau}(\omega) = (-1)^{m-1} c_m \int\limits_S \mathrm{tr}\ \omega,\ \omega = \tilde{\mathbf{a}}_0 d\mathbf{a}_1 \dots d\mathbf{a}_{2m-1}.$$

For $q \le 2m$, $\hat{\tau}$ is a graded closed trace of dimension $2m-1$ on $\Omega(\mathfrak{S}_q^0)$; for $q \le 2m-1$ the trace $\hat{\tau}$ and the cocycle τ, extend to the unitalisation of $\mathfrak{S}_q^0$.

We consider the representation of the homomorphism S on the cocycle level, in the terms of the above model. Consider the graded differential algebra $\Omega(\mathfrak{S})\otimes\Omega(\mathbb{C})$. For a cyclic cocycle $\tau \in C_\lambda^n(\mathfrak{S})$ and cyclic cocycle $\sigma \in C_\lambda^p(\mathbb{C})$, following [2], we define the cup product $\tau\sharp\sigma \in C_\lambda^{n+p}(\mathfrak{S}\otimes\mathbb{C}) = C_\lambda^{n+p}(\mathfrak{S})$ by setting

$$\tau\sharp\sigma(\mathbf{a}_0,\dots,\mathbf{a}_{n+p}) = (\hat{\tau}\otimes\hat{\sigma})((\mathbf{a}_0\otimes e)d(\mathbf{a}_1\otimes e)\dots d(\mathbf{a}_{n+p}\otimes e)), \qquad (18)$$

Here, e is the unit in $\mathbb{C}$, i.e. the element $1+0\mathbf{1} \in \tilde{\mathbb{C}}$, $\hat{\tau},\hat{\sigma}$ are graded closed traces of degree, respectively, n and p in $\Omega(\mathfrak{S}),\Omega(\mathbb{C})$ representing τ,σ and thus only terms of bidegree (n,p) survive in (18). In particular, for $\sigma=\sigma_1 \in C_\lambda^2(\mathbb{C})$, $\sigma_1(e,e,e) = 1$, the cup product with σ_1 generates the homomorphism S in cyclic co-homologies. For an even integer $p = 2l$, we consider $\sigma_l = \sigma_1^{\sharp l}$ where $\sharp l$ denotes taking to the power l in the sense of $\sharp$ operation. Cup multiplication with σ_l generates the iterated homomorphism S^l in cyclic cohomologies of the algebra $\mathfrak{S}$. Combinatorial calculations give an explicit expression for $S^l\tau_n$, and therefore, the index formulas.

Theorem 5.1 *For $\mathbf{a}\in\mathcal{E}_q = GL(\mathfrak{S}_q)$, denote*

$$\alpha_{2l}(\mathbf{a}) = (l!)^{-1}\int_{S_R} \mathrm{tr}\left[(\frac{d}{dt})^l(\mathbf{b}^{-1}(1-t\mathbf{c})^{-1}d\mathbf{b})^{2m-1}|_{t=0}\right], \qquad (19)$$

$$\alpha'_{2l}(\mathbf{a}) = \mathrm{tr}\int_{S_R}(\mathbf{c}+\mathbf{b}^{-1}d\mathbf{b})^{2m-1+l}, \qquad (20)$$

where $\mathbf{b}(y,\eta) = \mathbf{a}(y,\eta)\mathbf{a}(y,\eta_0(y))^{-1}$, $\mathbf{c} = (\mathbf{b}-1)(\mathbf{b}^{-1}-1)$ and in (20) only the term of degree $2m-1$ is naturally preserved under integration. Then for $2l+2m-1>q$ the form in the integrand in (19) and the integral in (20) belong to trace class and

$$\mathrm{IND}\,[\mathbf{a}] = c_{m,l}\alpha_{2l}(\mathbf{a}) = c_{m,l}\alpha'_{2l}(\mathbf{a}),\ c_{m,l} = -(2\pi i)^{-m}\frac{l!(m+l-1)!}{(2m+2l-1)!}. \qquad (21)$$

6. Applications In this section we show how the results of Sect.5 enable one to derive, in an uniform way, index formulas for some concrete situations.

6.1 Toeplitz operators Toeplitz operators on the line (or, what is equivalent, on the circle) with operator-valued symbols form an important ingredient in the study of ΨDO on manifolds with cone- and edge-type singularities (see, e.g., [3], [10]). Let $\mathbf{b}(y)$ be a function on the real line $\mathbb{R}^1$, with values being operators in the Hilbert space $\mathfrak{K}$, differentiable and stabilising sufficiently fast at infinity:

$$\mathbf{b} = \mathbf{1}+\mathbf{k},\ \mathbf{k}(y),\mathbf{k}'(y)\in\mathfrak{s}_q(\mathfrak{K});\ \|\mathbf{k}(y)\| = O((1+|y|)^{-q});\ ||\mathbf{k}'(y)||,|\mathbf{k}(y)|_q = O(1). \qquad (22)$$

We consider the Toeplitz operator $T_{\mathbf{b}}$ in the Hardy space $H^2(\mathbb{R}^1, \mathfrak{K})$, $T_{\mathbf{b}}u = P\mathbf{b}u$, $P : L_2 \to H^2$. The general scheme applied to the cocycle $\tau(\mathbf{k}_0, \mathbf{k}_1) = -(2\pi i)^{-1} \int \mathrm{tr}\,(\mathbf{k}_0 d\mathbf{k}_1)$ gives new index formulas.

Theorem 6.1. *If $l > 2q$, $\mathbf{b} = 1 + \mathbf{k}$ is an invertible symbol in and (22) holds then the index of $T_{\mathbf{b}}$ equals*

$$\mathrm{ind}\ T_{\mathbf{b}} = (2l+1)c_{1,l} \int \mathrm{tr}\,((\mathbf{b}^{-1} - 1)^l (\mathbf{b} - 1)^l \mathbf{b}^{-1} d\mathbf{b}). \qquad (23)$$

In particular, when $\mathbf{b}(y) = \mathbf{1} + \mathbf{k}(y)$ is a parameter dependent elliptic ΨDO on a compact k-dimensional manifold M, with $\mathbf{k}$ being an operator of negative order $-s$, the conditions (22) are satisfied with any $q > k/s$. This was the situation considered in [11].

6.2 Cone Mellin operators Cone Mellin operators (CMO) are involved into the local representation for singular pseudodifferential operators near conical points and edges. In an abstract setting, CMO in $L_2(\mathbb{R}_+, \mathfrak{K})$ have the form

$$(Au)(t) = \frac{1}{2\pi i} \int_\Gamma dz \int_0^\infty (t/t_1)^z \mathbf{a}(t, z) u(t_1) \frac{dt_1}{t_1},$$

where Γ is any fixed vertical line $\Gamma = \Gamma_\beta = \{\Re z = \beta\}$, the choice of β determines the choice of the weighted L_2 space where the operator is considered. The Mellin symbol $\mathbf{a}(t, z)$ is supposed to be a bounded operator in $\mathfrak{K}$ for all $(t, z) \in \mathbb{R}_+ \times \Gamma$. We say that it belongs to the class $\mathfrak{M}_q^\mu$, $\mu \geq 0$, if $\|\partial_t^\alpha \partial_z^\nu \mathbf{a}(t, z)\| = O((1 + |z|)^{-\nu+\mu})$ and $|\partial_t^\alpha \partial_z^\nu \mathbf{a}(t, z)|_{\frac{q}{\nu-\mu}} = O(1)$ uniformly in t, moreover, for $t \in (0, c]$ and for $t \in [C, \infty)$ the symbol does not depend on t.

The change of variables $y = \log t$, $\eta = iz$, transforms CMO to a ΨDO considered in Sect.3 with symbol in the class $\mathcal{S}_q^\mu$. Thus, elliptic symbols, i.e., those for which, for (t, ζ) outside some compact in $\mathbb{R}_+ \times \Gamma$, $\mathbf{a}(t, z)$ is invertible, with uniformly bounded inverse, give Fredholm operators. The index for such operators can be found by (19), (20), $m = 1$, with proper l. In the co-ordinates (t, z) this produces, for the CMO with elliptic symbol $\mathbf{a}(t, z), 2l > q$,

$$\mathrm{ind}\ A = (2l+1)c_{1,l} \int_{\mathcal{L}} \mathrm{tr}\,[((\mathbf{b}(t, z)^{-1} - 1)(\mathbf{b}(t, z) - 1))^l \mathbf{b}(t, z)^{-1} d\mathbf{b}(t, z)],$$

where $\mathcal{L}$ is a contour such that on and outside it, $\mathbf{a}(t, z)$ is invertible, $\mathbf{b}(t, z) = \mathbf{a}(t, z)\mathbf{a}(t, z_0)^{-1}$, and z_0 is chosen, so that $\mathbf{a}(t, z_0)$ is invertible for all t.

A more topological index formula for CMO, see [6] ind $A = \frac{1}{2\pi i} \int_{\mathbb{R}_+ \times \Gamma} Ch(\mathrm{Ind}\ \mathbf{a})$, where $Ch(\mathrm{Ind}\ \mathbf{a}) = \mathrm{tr}\,((d\mathbf{r} + \mathbf{r}(d\mathbf{a})\mathbf{r})d\mathbf{a})$ is the Chern class of the symbol $\mathbf{a}$ and $\mathbf{r}(t, z)$ is a 'good' regularizer ($\mathbf{ar} - 1, \mathbf{ra} - 1$ belong to trace class for all $(t, z) \in \mathbb{R}_+ \times \Gamma$ and vanish outside some compact set) is easily deduced from the above formula by means of a homotopy.

7.Edge operators In this section we apply our abstract results to the case of edge pseudodifferential operators arising in the study of ΨDO on singular manifolds, see [3], [14-16], [19], [20], etc. Usually one introduces such operators by some particular explicit representation. We depict a new version of calculus of edge operators defining operator symbols not by explicit formulas but rather by their properties. The leading term in our calculus turns out to be the same as in the standard one.

In the leading term, our edge operators will be glued together from usual ΨDO in the Euclidean space, with symbols having discontinuities at a subspace - see [9], [20]. Let $a(x,\xi) = a(y,z,\eta,\zeta)$ be a (matrix) function in $\mathbb{R}^n \times \mathbb{R}^n = (\mathbb{R}^m \times \mathbb{R}^k) \times (\mathbb{R}^m \times \mathbb{R}^k)$, zero order positively homogeneous in $\xi = (\eta,\zeta)$ variables. Suppose that a has compact support in x and is smooth in all variables unless $\xi = 0$ or $z = 0$. At the subspace $z = 0$ the function a has a discontinuity: it has limits as z approaches 0, but these limits may depend on the direction of approach:

$$\Phi(y,\omega,\eta,\zeta) = \lim_{\rho\to 0} a(y,\rho\omega,\eta,\zeta). \tag{24}$$

To the symbol a we associate, in the usual way, the pseudodifferential operator in $\mathbb{R}^n$ acting as $\mathcal{A} = \mathcal{F}_0^{-1} a \mathcal{F}_0$, where $\mathcal{F}_0$ is the Fourier transform in $\mathbb{R}^n$. At the same time, one can represent $\mathcal{A}$ as a ΨDO with operator-valued symbol. Denote by $\mathfrak{K}_0$ the (weighted) space $L_2(\mathbb{R}^k)$ and set $\mathbf{a}(y,\eta) = \mathcal{F}^{-1} a(y,z,\eta,\zeta)\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform in $L_2(\mathbb{R}^k)$. This operator-valued symbol is differentiable in y,η for $\eta \neq 0$ and $\partial_y^\alpha \partial_\eta^\beta \mathbf{a}$ is a ΨDO of order $-|\beta|$ in $\mathfrak{K}_0$, therefore (6) holds. Homogeneity implies (5). The symbol $\mathbf{a}$, however, does not belong to our symbol class $\mathcal{S}_q^0$ since these estimates hold only for η outside some fixed neighbourhood of zero. At the point $\eta = 0$ the η-derivatives of a have singularities and thus (5), (6) fail. Therefore, near $\eta = 0$ we introduce some correction for the symbol by means of a certain delicate smoothening procedure, after which the corrected symbol belongs to $\mathcal{S}_q^0$, while the operator itself gets a trace class perturbation.

Operator symbols obtained by this construction possess, besides general properties of the class $\mathcal{S}_q^0$, the following *radial pseudo-locality* property. If $\psi(\tau)$ is a cut-off function on the semi-axis, $\psi(\tau) = 1$ near zero, then

$$[D_\eta^\beta \mathbf{a}, \psi(|z|)] \in \mathcal{S}_q^{-|\beta|-1}, q > k, \tag{25}$$

for all $|\beta| < N'$, where N' can be made arbitrarily large.

The mapping $OS : a \mapsto \mathbf{a} \in \mathcal{L}_q^0$ is additive. It is, however, not multiplicative, even if one neglects compact errors. We introduce here the class of operator symbols arising as the multiplicative error.

Definition 7.1. *Let $\mathfrak{K}$ be the Hilbert space $L_2(\mathbf{K})$, where $\mathbf{K}$ is a cone with a compact manifold as a base. We say that the operator symbol $\mathbf{g}(y,\eta) \in \mathcal{L}_q^0(\mathbb{R}^m, \mathfrak{K})$ belongs to $\mathcal{I}_q^0$ if for any cut-off function ψ, as above, for all $|\beta| \le N'$,*

$$(1 - \psi(|z|)) D_\eta^\beta \mathbf{g} \in \mathcal{S}_q^{-|\beta|-1}. \tag{26}$$

An example of operator symbol in $\mathcal{I}_q^0$, for $\mathbf{K} = \mathbb{R}^k$, is given by $\mathbf{g}(y,\eta) = \mathcal{F}_0^{-1} g(y,z,\xi)\mathcal{F}_0$, with the function $g(y,z,\eta,\zeta) = |z|^{-\mu}((|\xi|^2+1)^{-\mu/2}))$, $\mu > 0$. Symbols in this ideal present an abstract generalisation of singular Green operators in the traditional construction of the edge calculus (see, e.g., [14]).

Proposition 7.2. *Let a, b be discontinuous scalar symbols in $\mathbb{R}^m \times \mathbb{R}^k$, where $\mathbb{R}^k$ is considered as a cone with base S^{k-1}. Then $OS(a)OS(b) - OS(ab) \in \mathcal{I}_q^0$, $OS(a^*) - OS(a)^* \in \mathcal{I}_q^0$.*

Definition 7.3. *The class* $\mathfrak{S}^0 = \mathfrak{S}^0(\mathbb{R}^m,\ L_2(\mathbb{R}^k))$ *consists of elements* $\mathbf{a} \in \mathcal{L}_q^0$ *for which there exist a discontinuous scalar symbol* a *and a symbol* $\mathbf{c} \in \mathcal{I}_q^0$ *such that*

$$\mathbf{a} = OS(a) + \mathbf{c}. \tag{27}$$

In the sequel, we will refer to $OS(a)$ as the pseudodifferential part and $\mathbf{c}$ as the Green part of the symbol $\mathbf{a} \in \mathfrak{S}^0$.

In order to handle cones with an arbitrary base, we introduce *directional localisation.* Let κ, κ' be smooth functions on the sphere S^{k-1}. For a discontinuous symbol a and corresponding operator symbol $\mathbf{a}$, we introduce the localised symbol $\mathbf{a}_{\kappa\kappa'} = \kappa(\omega)\mathbf{a}\kappa'(\omega), \omega = z/|z|$. Such operator symbol, obviously, belongs to $\mathcal{L}_q^0$. Operators in our class possess the *directional pseudo-locality* property: if supports of κ, κ' are disjoint then $\mathbf{a}_{\kappa\kappa'} \in \mathcal{I}_q^0$. Calculations also show that the class $\mathfrak{S}^0$ is invariant under homogeneous changes of variables in $\mathbb{R}^k$. Moreover, the pseudodifferential part of the symbol is transformed according to the standard rule for transformation of the leading symbol under the change of variables.

For a compact $k-1$-dimensional manifold, $\mathbf{M}$, let $\mathbf{K}$ be the cone over $\mathbf{M}$ and $\mathfrak{K} = L_2(\mathbf{K})$. Take a covering of $\mathbf{M}$ by co-ordinate neighbourhoods U_j, but instead of usual co-ordinate mappings of U_j to domains in the Euclidean space, we consider mappings $\varkappa_j : U_j \to \Omega_j$, where Ω_j are domains on the unit $k-1$-dimensional sphere in $\mathbb{R}^k$.

Definition 7.4. *The operator-valued symbol* $\mathbf{a} \in \mathcal{S}_q^0(\mathbb{R}^m \times \mathbb{R}^m; \mathfrak{K})$ *with values being operators in* $\mathfrak{K}$*, belongs to* $\mathfrak{S}^0(\mathbb{R}^m, \mathfrak{K})$ *if images of conic localisation of* $\mathbf{a}$ *to all* U_j *under* $\varkappa_j$ *belong to* $\mathfrak{S}^0(\mathbb{R}^m, \mathfrak{K}_0)$. The class $\mathfrak{S}^0(\mathbb{R}^m, \mathfrak{K})$ is well-defined, i.e. its definition does not depend on the choices made in the construction. We denote by $\mathfrak{S} = \mathfrak{S}(\mathbb{R}^m, \mathfrak{K})$ the algebra obtained by attaching the unit to $\mathfrak{S}^0$. Thus $\mathfrak{S}$ consists of operator symbols of the form $\mathbf{1} + \mathbf{b}$, $\mathbf{b} \in \mathfrak{S}^0$.

Let us compare the algebra $\mathfrak{S}$ with the algebra of edge operator symbols considered, e.g. in [7], [14]-[16]. This latter algebra is constructed in such way that it is the smallest possible *- algebra containing Mellin symbols. Our algebra $\mathfrak{S}$ is constructed as the *largest reasonable* algebra containing Mellin symbols. It, surely, contains Green operators since the latter belong to $\mathcal{I}_q^0$. Now we can apply our index formulas to the operators with symbols in $\mathfrak{S}$, thus generalyzing index theorems from [13] and [7].

Theorem 7.5. *Let* $\mathbf{a}$ *be a symbol in* $\mathfrak{S}$*, elliptic in the sense of Sect. 3, i.e.* $\mathbf{a}(y, \eta)$ *is invertible for* $|\eta|$ *large enough and this inverse is uniformly bounded for such* η*. Then the pseudodifferential operator* $\mathcal{A}$ *with symbol* $\mathbf{a}$ *is Fredholm in* $L_2(\mathbb{R}^m; \mathfrak{K})$ *and*

$$\operatorname{ind} \mathcal{A} = c_{m,l}\alpha_{m,l}(\mathbf{a}) = c_{m,l}\alpha'_{m,l}(\mathbf{a}), \tag{28}$$

where $\alpha_{m,l}(\mathbf{a}), \alpha'_{m,l}(\mathbf{a})$ *are given in (19),(20), and* l *is any integer such that* $2m+2l-1 > k$*. Moreover, if* $\mathbf{r}(y, \eta)$ *is a regularizer for* $\mathbf{a}$ *such that* $\mathbf{ar}-\mathbf{1}, \mathbf{ra}-\mathbf{1}$ *belong to trace class and have a compact support in* η, η *then*

$$\operatorname{ind} \mathcal{A} = ((2\pi i)^m m!)^{-1} \int_{\mathbb{R}^m \times \mathbb{R}^m} ch(\operatorname{ind} \mathbf{a}), \tag{29}$$

where $ch(\operatorname{ind} \mathbf{a}) = \operatorname{tr}((d\mathbf{r}d\mathbf{a} + (\mathbf{r}d\mathbf{a})^2)^m)$.

Proof The formula (28) is a particular case of Theorem 5.1; (29) is obtained by a specially constructed homotopy which transforms **a** to another symbol, to which the formula (28) with $l = 0$ can be applied. For the latter symbol, (28) gives (29) by means of Stokes formula, say, like in [7] or [13].

In conclusion, we note that one can attach boundary and co-boundary operators to the above symbols, thus coming to results for Boutet de Monvel-type matrix symbol algebra.

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