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# Uniqueness and Nonuniqueness for Nonsmooth Divergence Free Transport*

F. Colombini, T. Luo, J. Rauch

**Abstract**

We present an example of a uniformly bounded divergence free vector field $\mathbf{a}(x).\partial_x$ on $\mathbf{R}^3$ which has the property that the linear transport equation

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \mathbf{a}_j(t,x) \frac{\partial u}{\partial x_j} \ = \ 0\,, \qquad \operatorname{div} \mathbf{a} = \sum_{i=1}^{d} \frac{\partial \mathbf{a}_j}{\partial x_j} = 0 \qquad (1)$$

has a nontrivial bounded solution with vanishing Cauchy data. The coefficients have the property that $x_3 \nabla \mathbf{a}$ is a bounded measure.

For the same equation we prove uniqueness in the Cauchy problem when the coefficients $\mathbf{a}$ and $u$ belong to $(H^{1/2} \cap L^\infty)([0,T] \times \mathbf{R}^d)$.

## 1 Introduction

We prove two results concerning the uniqueness and nonuniqueness in the Cauchy problem for the linear transport equation (1). If the coefficients are Lipschitz, the proof of existence and uniqueness of solutions which are continuous in time with values in $L^p(\mathbf{R}^d)$, $p \in [1, \infty[$, is classical.

In the case of coefficients less regular than Lipschitz, the notion of a solution $u \in L^\infty([0,T] \times \mathbf{R}^d)$ of (1) with initial data equal to $u_0 \in L^\infty(\mathbf{R}^d)$ is taken in the weak sense that $\forall \phi \in C_0^\infty(]-\infty, T[\times \mathbf{R}^d)$,

$$\int_{[0,T[\times \mathbf{R}^d} \Big( -\frac{\partial \phi}{\partial t} - \sum_{i=1}^{d} \frac{\partial(\mathbf{a}_j\, \phi)}{\partial x_j} \Big)\ u \ dx\, dt \ = \ \int_{\mathbf{R}^d} u_0(x)\ \phi(0,x)\ dx\,. \qquad (2)$$

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Di Perna and P.-L. Lions [DL] showed uniqueness of solutions $u \in L^\infty$ when the coefficients $\mathbf{a} \in W^{1,1}(\mathbf{R}^{1+d})$. This result is particularly striking because the characteristics, that is the integral curves of the vector field, do not have uniqueness. Lions [Li] proved uniqueness for a generic class of piecewice $W^{1,1}$ functions, and raised the question of uniqueness for $BV$ vector fields. That uniqueness has recently been proved in a sequence of successively finer results.

Uniqueness for $BV$ fields and continuous solutions was proved in [CL1]. In [CL3] the uniqueness of $L^\infty$ solutions is proved for coefficients that, possibly excluding a relatively closed domain of d-dimensional Hausdorff measure zero, belong to the class of functions "Conormal $BV$". Roughly speaking, locally the derivatives of the vector field are measure only along one direction, while they are $L^1$ functions in the others. Starting from this and from a result of geometric measure theory by Alberti [Al], Ambrosio [Am] has proved the uniqueness of $L^\infty$ solutions for the general case of $BV$ coefficients.

One can ask for uniqueness results for coefficients, and solutions, belonging to other spaces. With nonlinear equations in mind it is desireable that the coefficients be no smoother than the solutions. In this direction, [CL2] proved the uniqueness of solutions $u \in H^{1/2}$ when $a$ is in the Besov space $B^{1/2}_{\infty,2} \supset \cup_{\delta>0} C^{1/2+\delta}$.

In space-time dimension equal to 2, [CL2] prove uniqueness of $L^\infty$ solutions when the coefficients are merely $L^\infty$. The proof is elementary. In higher dimension the corresponding uniqueness result is proved false in Theorem 2 below and in an independently discovered example of Depauw [De]. Both constructions are motivated by an important article of Aizenman [Ai].

For uniqueness we offer a result which is close to the Besov uniqueness result cited above. The interest of our result is that the regularity of solution and coefficient are the same and the proof is elementary.

**Theorem 1 (Uniqueness)** *If $\mathbf{a}, u \in (H^{1/2} \cap L^\infty)([0,T] \times \mathbf{R}^d)$ satisfy (1), then $u$ is a strongly continuous function of time with values in $L^2(\mathbf{R}^d)$, and $\|u(t)\|_{L^2(\mathbf{R}^d)}$ is independent of time. In particular, there is uniqueness in the Cauchy problem in this regularity class.*

A key element in the proof is that $(H^{1/2} \cap L^\infty)([0,T] \times \mathbf{R}^d)$ is an algebra. As in the earlier work of [CL2], it is worth noting that while there are good existence theorems in the framework of Di Perna-Lions and Ambrosio, we do not know of an existence theorem with regularity corresponding to Theorem 1.

For nonuniqueness the key reference is the article of Aizenman [Ai] published a full ten years before Di Perna-Lions. Aizenman's motivation was a question of Nelson concerning the incompressible Euler equations of fluid dynamics. The natural conservation law for Euler's equation guarantees that solutions are square integrable in space. Nelson [Ne] asked about uniqueness for the transport equation (1) in case that $\mathbf{a}$ is independent of time and square integrable. Aizenman constructed an example of an $\mathbf{a} \in L^\infty$ which generates several measure preserving flows. However, the construction of Aizenman is difficult and maybe impossible to follow. We replace an essential step in his construction by a sequence of four explicit steps and thereby show that his goal is achievable. We present the example in the language of PDE.

Completely independently and very slightly after us, Depauw constructed another example with the same scaling properties as ours and that of Aizenman but whose basic geometric idea is somewhat simpler and even more elegant than that of Aizenman. We do not know whether Depauw's example achieves Aizenman's goals concerning flow generation.

**Theorem 2 (Nonuniqueness)** *There is a uniformly bounded divergence free field $\mathbf{a}(x)$ on $\mathbf{R}^3$ and a nontrivial $L^\infty([0,T] \times \mathbf{R}^3)$ solution of the Cauchy problem (1) with initial value equal to zero. The field has the additional regularity that $x_3 \nabla_x \mathbf{a}$ is a finite Borel measure.*

Depauw presents a nonautonomous example with $d = 2$. By suitably trading $x_3$ for $t$ his example can be made autonomous with $d = 3$ and ours can be presented as nonautonomous in dimension 2.

**Open question.** Is there uniqueness when $x_3 \nabla_x \mathbf{a}$ is an integrable function or even a smooth integrable function?

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## 2 Uniqueness

Leibniz' chain rule for derivatives implies that, for smooth $\mathbf{a}$, $u$ and $f$, from equation (1) follows

$$\partial_t f(u) + \mathbf{a}.\partial_x f(u) = 0\,.$$

The proof of Theorem 1 relies on a justification of this implication for the non smooth field and solution.

**Theorem 3 (Leibniz' rule)** *Suppose that* $\mathbf{a}, u$ *satisfy (1) in* $]0,T[\times\mathbf{R}^d$ *and that* $\mathbf{a}$ *and* $u$ *belong to* $(H^{1/2}\cap L^\infty)([0,T]\times\mathbf{R}^d)$*. Then for any* $f\in C^\infty(\mathbf{R})$ *the function* $f(u)$ *is also a solution, that is* $\partial_t f(u) + \mathbf{a}.\partial_x f(u) = 0$ *on* $]0,T[\times\mathbf{R}^d$*.*

**Proof.** Subtracting a linear function from $f$, it suffices to prove the result when $f(0) = f'(0) = 0$. The Theorem is proved by showing that in the sense of distributions on $]0,T[\times\mathbf{R}^d$ one has

$$f'(u)(u_t + \mathbf{a}.\partial_x u) \;=\; \partial_t f(u) + \mathbf{a}.\partial_x f(u)\,. \qquad (3)$$

Let $n := 1+d$ denote the dimension of space time. The proof relies on two basic facts about multiplication in Sobolev spaces (see [Be]). The first is that for $s \geq 0$, $X^s := (H^s\cap L^\infty)([0,T]\times\mathbf{R}^d)$ is invariant under smooth functions $F$ which vanish at the origin. That is, $(\mathbf{a},u)\to F(\mathbf{a},u)$ is continuous from $(X^s)^d\times X^s \to X^s$. The second is that for $0\leq s< n/2$ , multiplication is a continuous bilinear map from $H^s([0,T]\times\mathbf{R}^d)\times H^{-s}([0,T]\times\mathbf{R}^d)$ to $H^\sigma([0,T]\times\mathbf{R}^d)$ for all $\sigma < -n/2$.

The first step is to show that these facts imply that each of the four summands appearing in (3) is a well defined element of $H^{-\sigma}([0,T]\times\mathbf{R}^d)$ for all $\sigma<-n/2$ and that the map from $(\mathbf{a},u)$ to each summand is continuous from $(X^{1/2})^d\times X^{1/2}$ to $H^\sigma([0,T]\times\mathbf{R}^d)$.

For the first term, one has $f'(u)\in H^{1/2}\cap L^\infty$ and $u_t\in H^{-1/2}$, so the product is continuous because $H^{1/2}\times H^{-1/2}\subset H^\sigma$.

For the next term, $f'(u)\,\mathbf{a}\in H^{1/2}\cap L^\infty$ because of the smooth invariance and $\partial_x u\in H^{-1/2}$.

For the third and fourth term, $\partial_t f(u), \partial_x f(u)\in H^{-1/2}$ and $\mathbf{a}\in H^{1/2}$.

Thus both sides of (3) are continuous functions of $(\mathbf{a},u)\in H^{1/2}\cap L^\infty$. Since one has equality for $\mathbf{a}$ and $u$ in the dense set $C_0^\infty([0,T]\times\mathbf{R}^d)$, identity (3) follows in general.

Since $u$ satisfies (1), the left hand side vanishes and therefore the right hand side vanishes. ∎

**Proof of Theorem 1.** First establish weak continuity of $u(t)$ as a function of time by the following sequence of observations:

$$u \in H^{1/2}([0,T] \times \mathbf{R}^d) \ \subset\ L^2([0,T]\ ;\ L^2(\mathbf{R}^d))\,, \tag{4}$$

$$u \in H^{1/2}([0,T] \times \mathbf{R}^d) \ \subset\ L^2([0,T]\ ;\ H^{1/2}(\mathbf{R}^d))\,,$$

$$\partial_x u \in L^2([0,T]\ ;\ H^{-1/2}(\mathbf{R}^d))\,,$$

$$\mathbf{a} \in L^2([0,T]\,;\,H^{1/2}(\mathbf{R}^d))\,,$$

$$u_t = -\mathbf{a}.\partial_x u \in L^1([0,T]\ ;\ H^\sigma(\mathbf{R}^d)) \qquad \forall\, \sigma < -d/2\,. \tag{5}$$

Then, (4) and (5) imply that

$$u \in C([0,T]\ ;\ H^\sigma(\mathbf{R}^d)) \qquad \forall\, \sigma < -d/2\,. \tag{6}$$

Theorem 3 implies that $u^2$ is a solution of the transport equation. Test this equation against the function $\phi(t)\chi(\epsilon x)$ with $\phi \in C_0^\infty(]0,T[)$, $\chi \in C_0^\infty(\mathbf{R}^d)$, $\chi(x) = 1$ for $x$ near 0, to find

$$\int_0^T \int_{\mathbf{R}^d} \Big( -\phi'(t)\chi(\epsilon x) - \epsilon\phi(t)\mathbf{a}.\partial_x\chi(\epsilon x)\Big)\ u^2\ dx\,dt = 0\,.$$

Letting $\epsilon \to 0$ yields

$$\int_0^T \int_{\mathbf{R}^d} -\phi'\ u^2\ dx\,dt = 0\,.$$

Thus the square integrable function $\int u^2(t,x)\,dx$ has vanishing distribution derivative on $]0,T[$ proving that the $L^2(\mathbf{R}^d)$ norm of $u(t)$ is constant.

The fact that this $L^2(\mathbf{R}^d)$ norm is bounded, together with (6), implies that $t \to u(t)$ is a continuous function of time with values in $L^2(\mathbf{R}^d)$ endowed with the weak topology.

This weak $L^2$ continuity together with the fact that $\|u(t)\|_{L^2(\mathbf{R}^d)}$ is also continuous implies that $u(t)$ is strongly continuous with values in $L^2(\mathbf{R}^d)$ and the proof of Theorem 1 is complete. ∎

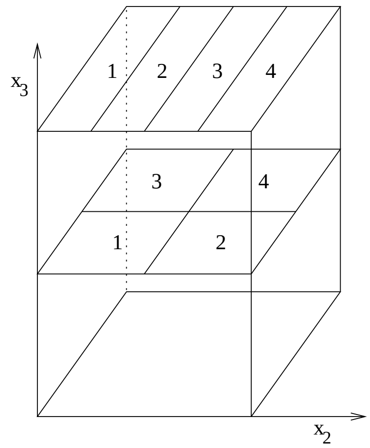


Figure 1: The basic region

# 3 The nonuniqueness example

The construction is inspired by Aizenman [Ai]. Our vector field is piecewise linear.

## 3.1 The main step

Denote by $\mathcal{C}$ the positive cube of side $L$,

$$\mathcal{C} \ := \ \{x \in \mathbf{R}^3 \ : \ 0 < x_i < L\}\,.$$

Denote by $\mathcal{B}$ the box $\{0 < x_1 < L\} \times \{0 < x_2 < L\} \subset \mathbf{R}^2$ which is the base of the cube $\mathcal{C}$. The size of $\mathcal{C}$ will be determined later.

Define $\mathbf{a}(x_1, x_2, x_3) = 0$ when $(x_1, x_2) \notin \mathcal{B}$. Points which do not project to $\mathcal{B}$ are stationary. The field will be divergence free for points which project to $\mathcal{B}$. To guarantee that $\mathbf{a}$ is divergence free in the sense of distributions on $\mathbf{R}^3$ the normal component of $a.\partial_x$ must be continuous at points which project to $\partial\mathcal{B}$ (see the Gluing Lemma below). This is guaranteed by requiring systematically that the orbits of points projecting to $\mathcal{B}$ remain always in the set of points which project to $\mathcal{B}$. Therefore the normal component on the inside vanishes which matches the vanishing normal component in the exterior.

The field $\mathbf{a}.\partial_x$ is equal to $-\partial_3$ for $(x_1, x_2) \in \mathcal{B}$ and $x_3 > L$. The solutions $u$ will be thought of as density of fluid. With this intuition, fluid above the cube descends steadily at speed 1. Fluid which does not project to $\mathcal{B}$ is stationary.

The vector field $\mathbf{a}.\partial_x$ is first constructed in the upper half $L/2 < x_3 < L$ of $\mathcal{C}$. In that region it has the following seven properties.

(1) $\mathbf{a}_3 = -1$ when $x_3 = L$ and when $x_3 = L/2$.
(2) $\mathbf{a} \in L^\infty(\mathcal{C} \cap \{L/2 < x_3 < L\})$.
(3) $\mathrm{div}(\mathbf{a}.\partial_x) := \sum_{i=1}^3 \partial_j \mathbf{a}_j = 0$ in the sense of distributions.
(4) $\mathbf{a}_3$ is either 0 or bounded above by a strictly negative number.
(5) Orbits of $\mathbf{a}.\partial_x$ beginning in the interior of each of the four rectangles on the top of $\mathcal{C}$ in Figure 1 are uniformly Lipschitz and remain in $\mathcal{C}$ till they reach $x_3 = L/2$ in the corresponding little square.
(6) Each of the lines $x_2 = const$ on the top which do not bound one of the thin rectangles is mapped to a line $x_2 = new\ const$ in the corresponding little square. The map from line segment to line segment is affine preserving the orientation $dx_1$.
(7) The integral curves of $\partial_t + \mathbf{a}.\partial_x$ which start at $x_3 = L$ descend in such a way that they cross each horizontal plane $\{x_3 = const\}$ simultaneously.

Compared to the conditions of Aizenman note that we make explicit the requirement (6) which is not mentioned by Aizenman but is a key element of his construction. It is not at all clear to us how this property can be respected if the field is constructed following Figure 4 in his article.

Property (7) is guaranteed by the property that $\mathbf{a}_3$ does not depend on $x_1$ and $x_2$.

The field for $x_3 > L/2$ is piecewise linear in the sense that the half cube is decomposed into a finite family of open sets each bounded by a finite number of analytic hypersurfaces. The domains overlap at most in portions of hypersurfaces. In each domain the coefficients of the vector field are affine functions of the coordinates.

Property (1) involves the restriction of the $L^\infty$ function $\mathbf{a}_3$ to a hypersurface $x_3 = const$. The existence of this trace depends on the divergence free character of $\mathbf{a}$. We need analogous traces associated with the divergence free field $u\partial_t + u\mathbf{a}.\partial_x$ on $\mathbf{R}^{1+3}$ when $u$ satisfies the transport equation. The next Lemma is a classical result covering both cases.

**Lemma 4 (Trace Lemma)** *Suppose that $R \subset \mathbf{R}^{d-1}$ is an open subset and $\mathbf{v} \in L^\infty(R \times \{a < y_d < b\})$ is divergence free in the sense of distributions. Then the restriction of $\mathbf{v}_d$ to $y_d = \delta$, $\delta \in ]a,b[$, is a continuous function of $\delta$ with values in $L^\infty(R)$ endowed with the weak-$*$ topology. The continuous function extends uniquely to $[a,b]$ setting at $\delta = b$*

$$\lim_{\delta \to b-} \ \mathbf{v}_d(y_1, y_2, \ldots, y_{d-1}, \delta) \ := \ \mathbf{v}_d(y_1, y_2, \ldots, y_{d-1}, b-)\,,$$

*that turns out to be a well defined element of $L^\infty(R)$, and doing the same at $\delta = a$.*

**Proof.** Denote $y = (y', y_d)$. For $\phi \in C_0^\infty(R)$, $g(y_d) := \int \phi(y')\,\mathbf{v}_d(y', y_d)\,dy'$ is a well defined distribution on $]a,b[$ which satisfies in the sense of distributions,

$$\partial_d\Big(\int \phi(y')\,\mathbf{v}_d(y', y_d)\,dy'\Big) \ = \ \sum_{j<d} \int \mathbf{v}_j(y', y_d)\,\partial_j \phi(y')\ dy'\,.$$

The right hand side is bounded on $]a,b[$ and it follows that $g(y_d)$ is uniformly Lipschitz on $[a,b]$.

Thus the map

$$[a,b] \ni \delta \to \mathbf{v}_d(\cdot, \delta)$$

is continuous on $[a,b]$ with values in the space of distributions $\mathcal{D}'(R)$.

Since $\mathbf{v} \in L^\infty$, the map is bounded with values in $L^\infty(R)$. The Lemma follows. ∎

The field $\mathbf{a}.\partial_x$ in the upper half cube is constructed in five steps. Each step yields values for $\mathbf{a}$ in a slab in the $x_3$ direction.

The construction uses four explicit divergence free vector fields, steady descent, a compressor, a decompressor, and a shunt or shear. These elements are combined in a method which is best viewed as the design of fluid carrying ducts. The fluid starting over $\mathcal{B}$ follows a path as if it were guided by a duct. The guiding is by $\mathbf{a}.\partial_x$ and not by physical boundaries.

The field $\mathbf{a}.\partial_x$ has jump discontinuities accross the surfaces which are to be thought of as ducts. The normal component is continuous.

The first step is a compression and follows Aizenman exactly. The field $\mathbf{a}$ compresses in the $x_1$ direction and speeds up in the $x_3$ direction. Consider

$$\mathbf{a}.\partial_x \ = \ -x_1\,\partial_1 + (x_3 - L - 1)\partial_3 \tag{7}$$

The orbits satisfy

$$\frac{dx_1}{ds} = -x_1\,, \qquad \frac{dx_2}{ds} = 0\,, \qquad \frac{dx_3}{ds} = x_3 - L - 1\,,$$

$$x_1(s) = x_1(0)\, e^{-s}\,, \quad x_2(s) = x_2(0)\,, \quad x_3(s) - L - 1 = (x_3(0) - L - 1)\, e^{s}\,.$$

We follow these orbits for $0 \le s \le \ln 4$. At the end, the $x_1$ dimensions are compressed by a factor of $1/4$, and the orbits starting at the top, $\{x_3(0) = L\}$, satisfy

$$x_3(\ln 4) = L - 3\,.$$

Note that the orbits starting at the top arrive at $x_3 = const$ at the same value of $s$, verifying property (7).

The orbits with largest $x_1$ start at $x_1(0) = L,\ x_3(0) = L$ and therefore lie on the hyperbola with equation

$$x_1\,(x_3 - L - 1) \;=\; -L\,.$$

Now we do a little duct work. The orbits starting at the top of $\mathcal{C}$ flow in the region

$$0 < x_2 < L\,, \qquad L - 3 < x_3 < L\,, \qquad 0 < x_1\,(L + 1 - x_3) < L\,. \qquad (8)$$

For $(x_1, x_2) \in \mathcal{B}$ and $L - 3 < x_3 < L$ the field $a$ is defined by equation (7) in the region (8) and as 0 outside the region (8). It is as if there were a duct surrounding the region (8). There is fluid flowing inside and stationary fluid outside. The region (8) is a tube of orbits. Therefore, the field $\mathbf{a}.\partial_x$ is tangent to the boundary so has normal component equal to zero so the field defined by this duct work is divergence free in $L - 3 < x_3 < L$. At the top of $\mathcal{C}$ the vertical component of $\mathbf{a}.\partial_x$ is equal to $-1$ at $x_3 = L+$ and $x_3 = L-$. The next lemma implies that the the field is divergence free in $L - 3 < x_3$.

**Lemma 5 (Gluing Lemma)** *Suppose that $R \subset \mathbf{R}^{d-1}$ is an open subset and $\mathbf{v} \in L^\infty(R \times \{a < y_d < c\})$ is divergence free in the sense of distributions in $\{a < y_d < b\}$ and in $\{b < y_d < c\}$. Then in the sense of distributions one has*

$$\operatorname{div} \mathbf{v} = [\mathbf{v}_d(y', b+) - \mathbf{v}_d(y', b-)] \otimes \delta(y_d)\,.$$

*The field is divergence free if and only if $\mathbf{v}_d$ is continuous across the interface $\{y_d = b\}$.*

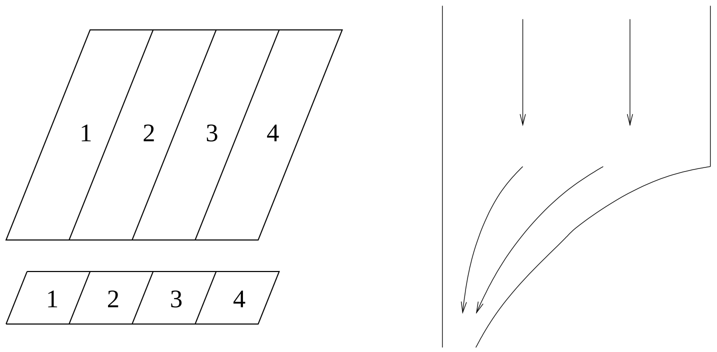


Figure 2: The compressor

**Proof.** Classic, follows the lines of the proof of the Trace Lemma, and is left to the reader. ∎

In the first slab, the flow is compressive in $x_1$, so $\mathbf{a}_3$ becomes more negative as $x_3$ decreases and at the exit of the duct

$$\mathbf{a}_3 = -4\,, \qquad \text{when} \quad x_3 = L - 3\,. \tag{9}$$

The flux through $x_3 = const$ is equal $4L^2$ for $L - 3 < x_3 < L$.

This first stage of duct work is summarized in the Figure 2 which indicates that the long rectangles are sqeezed by a factor $1/4$ in the $x_1$ direction when they arrive at $x_3 = L - 3$.

In the next slab the particles arriving in the squares 3 and 4 with vertical velocity equal to $-4$, will be redirected following the vector field

$$\mathbf{a}.\partial_x = L\partial_1 - 4\partial_3\,, \tag{10}$$

The integral curves satisfy

$$x_1(s) = x_1(0) + Ls\,, \qquad x_2(s) = x_2(0)\,, \qquad x_3(s) = x_3(0) - 4s. \tag{11}$$

The flux follows these curves for $0 \leq s \leq 1/4$ so that at the end the little squares 3 and 4 have been shunted as in Figure 3. The coordinate $x_3$ has decreased from $L - 3$ to $L - 4$.

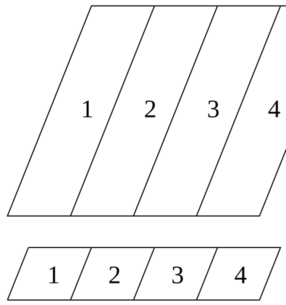


Figure 3: The shunt

Eliminating $s$ from two equations in (11) we obtain

$$x_1 = x_1(0) + (x_3(0) - x_3)L/4\,.$$

The field is given by formula (10) in the tube of orbits starting in little boxes 3 and 4, precisely, $\mathbf{a}.\partial_x$ is given by (10) when

$$L(L-3-x_3) < 4x_1 < L(L-2-x_3)\,, \quad L/2 < x_2 < L\,, \quad L-4 < x_3 < L-3\,. \tag{12}$$

This is the duct conducting the fluid starting in 3 and 4. The fluid starting in 1 and 2 descends steadily, that is

$$\mathbf{a}.\partial_x = -4\partial_3\,,$$

in the vertical duct

$$0 < x_1 < L/4\,, \qquad 0 < x_2 < L/2\,, \qquad L-4 < x_3 < L-3\,. \tag{13}$$

At all points in the slab $L-4 < x_3 < L-3$ not contained in the ducts (12), (13), the field is set equal to zero. The resulting field is divergence free in $L-4 < x_3$ and verifies also the other properties.

By an entirely analogous construction for $L-5 < x_3 < L-4$ the squares 1 and 2 continue their steady descent while the squares 3 and 4 are shunted left.

When the orbits starting at the top of $\mathcal{C}$ cross $x_3 = L-5$ one has achieved the transformation indicated in Figure 4.

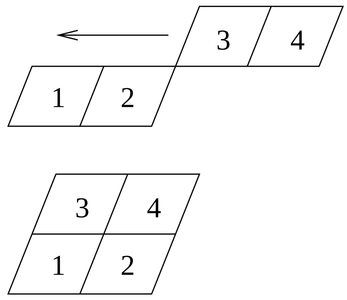


Figure 4: The second shunt

In the next stage the points in the four little squares at $x_3 = L - 5$ will be stretched in the $x_1$ direction till the $x_1$ equals $L$. This is indicated on the left hand side of Figure 5.

The stretching is performed with the vector field

$$\mathbf{a}.\partial_x = x_1\partial_1 - (x_3 - (L-5))\partial_3 - 4\partial_3 \,. \qquad (14)$$

This is essentially the opposite of the compressive step. The orbits satisfy

$$\frac{dx_1}{ds} = x_1\,, \qquad \frac{dx_2}{ds} = 0\,, \qquad \frac{dx_3}{ds} = -x_3 + (L-5) - 4\,.$$

Thus

$$x_1(s) = x_1(0)\,e^s\,, \quad x_2(s) = x_2(0)\,, \quad -x_3(s) + L - 9 = (-x_3(0) + L - 9)e^{-s}\,.$$

Orbits starting with $x_3(0) = L-5$, satisfy $x_3(\ln 2) = L-7$. In the hyperbolic duct

$$0 < x_1(x_3 - L + 9) < 2L\,, \quad 0 < x_2 < L/2\,, \quad L-7 < x_3 < L-5\,,$$

the field is defined by (14) and for other points in this slab the field is taken equal to zero. At the height $x_3 = L - 7$ the four rectangles occupy the left half of the $L \times L$ square as indicated on the left of Figure 5.

Finally inserting one final duct where the field is decompressive in $x_2$ one makes the final transformation achieving the result announced in Figure

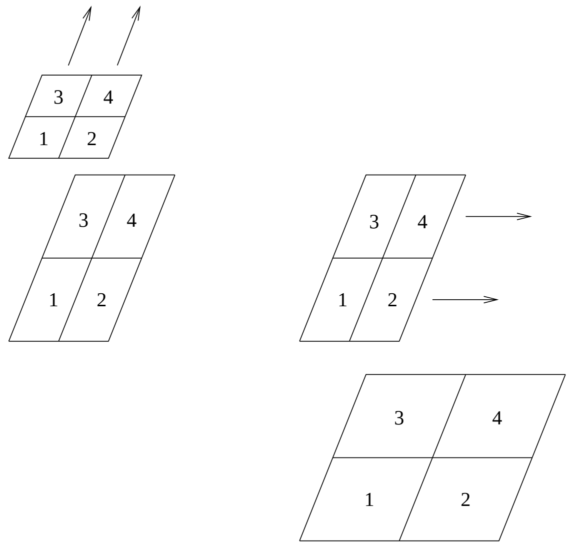


Figure 5: Two decompressions

1. We write the field in the hyperbolic duct starting from the rectangle $0 < x_1 < L,\ 0 < x_2 < L/2,\ x_3 = L - 7$:

$$\mathbf{a}.\partial_x = x_2 \partial_2 + (-x_3 + L - 9)\partial_3 \,,$$

and we follow the integral curves for $0 \leq s \leq \ln 2$, and so for $L - 8 \leq x_3 \leq L - 7$.

This last decompression in $x_2$ is sketched in the right hand side of Figure 5. This is obtained when the particles starting at the top arrive at $x_3 = L-8$. At this point we make the choice of $L = 16$ and the key step of the construction is complete.

The total time required for the descent from $x_3 = L$ to $x_3 = L/2$ is the sum of the times for crossing each of the five slabs, so it is equal to

$$T \ := \ \ln 4 + L/4 + L/4 + \ln 2 + \ln 2 \,. \tag{15}$$

We remark that $a_3$ is a continuous function across $x_3 = L-3\,, L-4\,, L-5$, where it takes the value $-4$, and across $x_3 = L - 7$, where its value is $-2$; moreover $x_3((L-8)+) = x_3((L/2)+) = -1$, as requested by property (1).

### 3.2 The iterative step.

Below each of the little boxes at height $L/2$, one places a scale $1/2$ model of the field in the upper half. By this we mean that only the geometry is scaled. The length of the vector field is not changed. Thus the vertical components of the vector fields are equal to $-1$ on both sides of $x_3 = L/2$ which is required for the gluing.

Fluid flowing through the ducts starting in each of the squares at height $L/2$ will be cut into four narrow rectangles which upon flowing downward a distance $L/4$ (since the vector field has the same length, we need to follow it for half as long) will be reassembled as four smaller squares.

One continues like this. In this way one obtains a divergence free bounded field in $x_3 > 0$ which has properties (1)-(7) suitably reproduced at each scale.

The time of descent from $x_3 = L$ to $x_3 = 0$ is equal to $2T$ with $T$ from equation (15).

The field in $x_3 < 0$ will be obtained from that in $x_3 > 0$ by a reflection argument. Before doing that we perform the heart of the analysis which takes place entirely in $x_3 > 0$.

### 3.3 Transport in $x_3 > 0$

Consider the solution $u$ satisfying

$$u(t, x_1, x_2, x_3) = \chi(x_1, x_2, x_3 + t)\,, \qquad \text{for} \quad t < 0$$

where $\chi$ is the characteristic function of the $L/2 \times L \times 1$ - box

$$R \;:=\; \{0 < x_1 < L/2\} \times \{0 < x_2 < L\} \times \{L < x_3 < L + 1\}\,. \tag{16}$$

For $t < 0$ the fluid is a block which descends vertically towards the half of the upper face of the cube $\mathcal{C}$ on which $x_1 < L/2$. The descent is at speed 1.

For $0 \le t < 2T$ ($T$ from (15)), fluid passes through the ducts we have constructed and at time $2T$ the first fluid particles arrive at $x_3 = 0$. The last fluid arriving from the top of $R$ arrives at time $t = 2T + 1$.

On $\{t < 2T + 1\} \cap \{x_3 > 0\}$, the solution is defined as being constantly 1 on the orbits with feet in the far past lying in $R$ and 0 otherwise. For $t > 2T + 1$ the function $u$ is taken equal to zero in the whole $\{x_3 > 0\}$.

The function $u$ so defined on $\{x_3 > 0\}$ takes only values 0 and 1. It is the characteristic function of the set of points belonging to the orbits launched from $R$ in the distant past. The next lemma shows that $u$ is a weak solution of the transport equation.

**Lemma 6** *In order that $u$ defined in $x_3 > 0$ satisfy the transport equation (1) it is sufficient that $u$ be piecewise smooth, satisfy the equation in the classical sense on each of the open sets bounded by the ducts, and have normal velocity continuous accross all two dimensional parts of the ducts.*

**Outline of proof.** For a test function $\phi$ in $x_3 > 0$ the integral $\int(-\phi_t - \mathbf{a}.\partial_x\phi)\, u\, dx\, dt$ is written as a sum over the finite number of open ducts which meet the support of $\phi$. In each open set one integrates by parts using the fact that $u$, $\mathbf{a}$, and $\phi$ are smooth.

Since $u$ satisfies the transport equation, only the boundary terms remain. The boundary terms from the two sides of each open piece of boundary cancel. ∎

**Lemma 7 (Flux at $x_3 = 0$)** *The flux arriving at $x_3 = 0$ from above is given by the formula*

$$\lim_{\delta \to 0+} \ (\mathbf{a}_3 u)(t, x_1, x_2, \delta) \ = \ \begin{cases} -1/2 & \text{on } \ \{2\mathrm{T} < \mathrm{t} < 2\mathrm{T}+1\} \times \mathcal{B} \\ 0 & \text{otherwise} \end{cases} \ . \quad (17)$$

**Proof.** The limit on the left exists thanks to the Trace Lemma applied to the divergence free field $u\partial_t + u\mathbf{a}.\partial_x$. Thus, it suffices to show that the distribution limit as $n \to \infty$ of $(\mathbf{a}_3 u)(t, x_1, x_2, L/2^n)$ is equal to the right hand side of (17).

Particles starting at the top of the cube $\mathcal{C}$ reach $x_3 = L/2$ at time $T$. Those starting at height $x_3 = L + \rho$ arrive at $x_3 = L/2$ at time $T + \rho$. The restriction of $u$ to $x_3 = L/2$ is therefore supported in $\{T \le t \le T + 1\} \times \overline{\mathcal{B}}$.

The first of the seven properties of $\mathbf{a}$ shows that $\mathbf{a}_3(x_1, x_2, L/2) = -1$ for $(x_1, x_2) \in \mathcal{B}$ and $x_3 = L/2$.

The values $u = 1$ of our solution are transported by the particles starting in $L \le x_3 \le L + 1$. They pass the top of $\mathcal{C}$ in the lower half, $\{x_1 < L/2\}$, of the four thin rectangles on the top of $\mathcal{C}$ in Figure 1. From property (6) of the vector field one sees that at $x_3 = L/2$ they pass through the lower half of the four little squares sketched in Figure 1.

Thus for $T < t < T+1$, $(\mathbf{a}_3 u)(t, x_1, x_2, L/2) = -1$ if and only if $(x_1, x_2)$ belongs to one of the four bottom halves of the four small squares in $x_3 = L/2$ in Figure 1. Otherwise, $\mathbf{a}_3 u(t, x_1, x_2, L/2) = 0$.

This computation repeats rescaled in the next interval $L/2 > x_3 > L/2^2$. At $x_3 = L/2^2$ the square $\mathcal{B}$ is decomposed into $4^2$ little squares and fluid passes through the bottom half of each of these squares during the interval of time $3(2T)/4 < t < 3(2T)/4 + 1$.

Continuing one sees that at $x_3 = L/2^n$ the square $\mathcal{B}$ is decomposed into $4^n$ squares and that fluid flows accross the bottom halves of these squares for $(2^n - 1)2T/2^n < t < (2^n - 1)2T/2^n + 1$.

For large $n$ these half squares become equidistributed in $\mathcal{B}$, with density equal to 1/2. And the time interval for the passages converges to $2T < t < 2T + 1$. Formula (17) follows. ∎

## 3.4 Three solutions in $\mathbf{x_3 > 0}$ with the same flux

Denote by $u_1$ the solution in $x_3 > 0$ which has just been constructed.

Denote by $u_2$ the entirely analogous solution which for $t < 0$ consists of a block of fluid descending vertically with speed one towards the half of the upper face of the cube $\mathcal{C}$ on which $x_1 > L/2$. Precisely for $t < 0$,

$$u_2(t, x_1, x_2, x_3) \ := \ \chi_{\{L/2<x_1<L\}\times\{0<x_2<L\}\times\{L<x_3<L+1\}}(x_1, x_2, x_3 + t)\,. \quad (18)$$

Denote by $u_3$ the solution in $x_3 > 0$ which for $t < 0$ represents a density $u = 1/2$ descending in a slab of thickness equal to 1 over $\mathcal{B}$, precisely for $t < 0$,

$$u_3(t, x_1, x_2, x_3) \ := \ \frac{1}{2}\,\chi_{\{0<x_1<L\}\times\{0<x_2<L\}\times\{L<x_3<L+1\}}(x_1, x_2, x_3 + t)\,. \quad (19)$$

These solutions are then extended to be defined in all of $\mathbf{R}_t \times \{x \,:\, x_3 > 0\}$ as being constant on orbits of $\partial_t + \mathbf{a}.\partial_x$.

Computations nearly identical to the proof of the preceding lemma prove the following.

**Lemma 8** *For $j = 1, 2, 3$*

$$\lim_{\delta\to 0+} \ (\mathbf{a}_3 u_j)(t, x_1, x_2, \delta) \ = \ \begin{cases} -1/2 & \text{on } \{2\mathrm{T} < \mathrm{t} < 2\mathrm{T}+1\} \times \mathcal{B} \\ 0 & \text{otherwise} \end{cases}\,. \quad (20)$$

This is the heart of the nonuniqueness. We have three solutions in $x_3 > 0$ which have exactly the same flux at $x_3 = 0+$.

## 3.5 Reflection

For $x_3 < 0$ define the field $\mathbf{a}$ by symmetry

$$\Big(\mathbf{a}_1(x_1\,,\,x_2\,,\,x_3)\,,\,\mathbf{a}_2(x_1\,,\,x_2\,,\,x_3)\,,\,\mathbf{a}_3(x_1\,,\,x_2\,,\,x_3)\Big) \\ \quad = \Big(-\mathbf{a}_1(x_1\,,\,x_2\,,\,-x_3)\,,\,-\mathbf{a}_2(x_1\,,\,x_2\,,\,-x_3)\,,\,\mathbf{a}_3(x_1\,,\,x_2\,,\,-x_3)\Big)\,.$$

**Lemma 9** *The field* $\mathbf{a}.\partial_x$ *is divergence free on* $\mathbf{R}^3$.

**Proof.** By the Gluing Lemma, it suffices to show that

$$\lim_{\epsilon\to 0+}\ \Big(\mathbf{a}_3(x_1,x_2,\epsilon) - \mathbf{a}_3(x_1,x_2,-\epsilon)\Big) = 0\,.$$

The definition of $\mathbf{a}$ guarantees that the quantity in parentheses is identically equal to zero. ∎

The next lemma is a straight forward computation.

**Lemma 10 (Reflection Lemma) i.** *The $x_3$ reflection of orbits in $x_3 > 0$ are time reversed orbits in $x_3 < 0$. Precisely, $x(s) := (x_1(s), x_2(s), x_3(s))$ with $x_3 > 0$ satisfies $dx/ds = \mathbf{a}(x(s))$, if and only if for the reflected set of $s$, $X(s) := (x_1(-s), x_2(-s), -x_3(-s))$ satisfies $dX/ds = \mathbf{a}(X(s))$.*

**ii.** *The function $u \in L^\infty(\{x_3 > 0\})$ is a weak solution of $\partial_t u + \mathbf{a}.\partial_x u = 0$, if and only if*

$$U(t, x_1, x_2, x_3) \ := \ u(-t, x_1, x_2, -x_3)$$

*satisfies $\partial_t U + \mathbf{a}.\partial_x U = 0$ in $\{x_3 < 0\}$.*

Starting with our three solutions $u_j$ in $x_3 > 0$, this reflection generates three solutions $U_j$ in $x_3 < 0$.

The normal component at $x_3 = 0$ of the divergence free field $U_j\partial_t + U_j\mathbf{a}.\partial_x$ defined in $\{x_3 < 0\}$ is given by

$$(u_j\,\mathbf{a}_3)(-t, x_1, x_2, 0+) \ = \ -\frac{1}{2}\,\chi_{[-2T-1,-2T]}(t)\ \chi_{\mathcal{B}}(x_1, x_2)\,.$$

Thus, for $j = 1, 2, 3$, the functions

$$v_j(t, x_1, x_2, x_3) \ := \ U_j(t - (4T + 1), x_1, x_2, x_3)$$

are solutions of $\partial_t v + \mathbf{a}.\partial_x v = 0$ in $x_3 < 0$ and their fluxes at $x_3 = 0-$ exactly match the fluxes at $x_3 = 0+$ of the $u_j$.

Gluing each solution $u_i$ for $x_3 > 0$ with any $v_j$ for $x_3 < 0$ we get nine solutions of the transport equation in $\mathbf{R}^{1+3}$. Two particular choices yield the following, which proves Theorem 2.

**Proof of Theorem 2.** The solutions which are equal to $u_1$ in $x_3 > 0$ and equal to $v_1$ (resp. $v_2$) in $x_3 < 0$ are both nonnegative bounded entropy solutions. They are identically equal for $t < 0$ and have disjoint supports for $t > 4T + 2$. Nonlinear functions of these solutions are solutions. ∎

**Remark.** If $w$ is the difference of these two solutions it follows that $w^2$ does NOT satisfy $(w^2)_t + \mathbf{a}.\partial_x(w^2) = 0$. If it were, it would follow that $w = 0$ as in the proof of Theorem 1. Thus the nonuniqueness example is also an example of the failure of Leibniz' rule. The failure of Leibniz' rule shows that there is more to being a solution than constancy on integral curves of the vector field, since that property is invariant under nonlinear functions.

## 3.6 Appendix. The space filling curve

This section is not needed to understand the examples, but too interesting to be omitted. It is also important for Aizenman's examples of the generation distinct measure preserving flows by $\mathbf{a}\partial_x$.

Consider the line segments

$$\ell(x_2) \ := \ \{0 < x_1 < L\,,\ x_2 \neq 0, \frac{L}{4}, \frac{2L}{4}\,\frac{3L}{4}, L\,,\ x_3 = L\}$$

on the top of $\mathcal{C}$. These are the segments parallel to those defining the long rectangles in Figure 1, with the bounding segments of those rectangles excluded.

The flow starting at any one of these segments arrives at $x_3 = L/2$ in a segment parallel to the original, half as long, i.e. of length $L/2$, and spanning one of the four small squares at the bottom of Figure 1.

Provided that $x_2$ is not an integer multiple of $L/4^2$, this argument can be repeated showing that the segment arrives at $x_3 = L/4$ in a parallel segment again halved in length, and so on.

Thus, if $x_2$ is not an integer multiple of $L/4^p$ for any integer power $p$, the segment $\ell(x_2)$ is compressed to a segment of length $L/2^p$ when it reaches $x_3 = L/2^p$. Passing to the limit, it is compressed to a single point $\alpha(x_2) = (\alpha_1(x_2), \alpha_2(x_2)) \in \mathcal{B}$ when the flow arrives at the bottom of the cube $\mathcal{C}$.

Denote by $\mathcal{E}$ the countable set of exceptional $x_2$, so

$$\alpha \;:\; [0, L] \setminus \mathcal{E} \;\to\; \mathcal{B}\,.$$

The binary expansion of $\alpha(x_2)/L$ has a simple description in terms of the expansions base 4 of $x_2/L$. The exceptional set of points $x_2$ are exactly those whose expansion in base 4 of $x_2/L$,

$$x_2/L \;=\; .q_1\, q_2\, q_3\, q_4\, \ldots \qquad q_j \in \{0, 1, 2, 3\}$$

end in an infinite string of solely 0's or of solely 3's.

The first digit, $q_1$, determines the membership in the rectangles on the top of $\mathcal{C}$ and therefore the image square at $x_3 = L/2$. The first binary digits satisfy

$$b_1(\alpha_1(x_2)) \;=\; \begin{cases} 0 & \text{if} \quad q_1 \in \{0, 2\} \\ 1 & \text{if} \quad q_1 \in \{1, 3\} \end{cases} \,,$$

$$b_1(\alpha_2(x_2)) \;=\; \begin{cases} 0 & \text{if} \quad q_1 \in \{0, 1\} \\ 1 & \text{if} \quad q_1 \in \{2, 3\} \end{cases} \,.$$

The next binary digits $b_j(\alpha)$ are determined from the base four digits $q_j(x_2)$ following the same rule.

Reading backwards, one sees that $\alpha$ is a bijection from $[0, L[\setminus \mathcal{E}$ to $\mathcal{B} \setminus \mathcal{F}$ where $\mathcal{F}$ is a countable set.

Considering two sequences of points of the type

$$x_2/L \;=\; .1000\ldots 0000111111\ldots \quad \text{and} \quad x_2'/L \;=\; .0333\ldots 3333111111\ldots\,,$$

one sees that $\alpha_2$ is discontinuous at $x_2 = L/4$. Similarly the map $\alpha$ is discontinuous at all points of $\mathcal{E}$. It is continuous from $[0, L[\setminus \mathcal{E}$ to $\mathcal{B}$.

The image by $\alpha$ of any base four interval

$$\Big\{ \frac{q}{4^p} < x_2 < \frac{q+1}{4^p} \Big\} \setminus \mathcal{E}$$

maps essentially one to one and onto a square whose area is equal to the length of the interval. Thus $\alpha$ is a measure preserving map of $[0, L]$ to $\mathcal{B}$.

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