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1. Introduction and Main Result

The nonlinear Schrödinger equations with a real valued potential $V(x)$:

$$i\partial_t u = -\Delta u + V(x)u - |u|^{p-1}u, \qquad (t,x) \in \mathbb{R}^{1+n} \tag{1.1}$$

arises in various physical contexts. When $V(x) \equiv 0$, equation (1.1) appears in such as nonlinear optics and plasma physics (see, e.g., [**5, 26, 29**]). The nonlinearity enters due to the effect of changes in the field intensity on the wave propagation characteristics of the medium. The potential $V(x)$ can be thought of as modeling inhomogeneities in the medium. In [**23**], equation (1.1) with a bounded potential $V(x)$ is studied as a model proposed to describe the local dynamics at a nucleation site. Equation (1.1) with a harmonic potential $V(x) = |x|^2$ is known as a model to describe the Bose-Einstein condensate with attractive inter-particle interactions under a magnetic trap (see, e.g., [**1, 12, 27**]).

We always assume $1 < p < 2^* - 1$. Here, we put $2^* = \infty$ if $n = 1, 2$, and $2^* = 2n/(n-2)$ if $n \geq 3$. In this talk, we particularly discuss the critical case $p = 1 + 4/n$ and treat the case $V(x) = |x|^2$ for the sake of simplicity. Our main purpose of this talk is to prove the stability of standing wave solution in such case.

By a standing wave, we mean a solution of (1.1) of the form

$$u_\omega(t,x) = e^{i\omega t}\phi_\omega(x),$$

where $\omega \in \mathbb{R}$ is a frequency, and $\phi_\omega(x)$ is a ground state of

$$-\Delta\phi + |x|^2\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n. \tag{1.2}$$

Indeed, there exists a unique positive radial solution $\phi_\omega(x)$ of the stationary problem (1.2) for any $\omega > -\lambda_1$ in the energy space, which is a ground state solution, where λ_1 is the first eigenvalue of the operator $-\Delta + |x|^2$ (see the author [**8**] for the existence, Li and Ni [**20**] for the radial symmetry of positive solutions, Kabeya and Tanaka [**17**], Hirose and Ohta [**15, 16**] for the uniqueness).

Many authors have been studying the problem of stability and instability of standing waves for nonlinear Schrödinger equations (see, e.g., [**2, 4, 6, 7, 8, 9, 10, 13, 14, 18, 21, 23, 24, 25, 28, 30, 31, 32**]). We recall some known results. First, we consider the case $V(x) \equiv 0$. For any $\omega > 0$, there exists a unique positive radial solution $\psi_\omega(x)$ of

$$-\Delta\psi + \omega\psi - |\psi|^{p-1}\psi = 0, \quad x \in \mathbb{R}^n \tag{1.3}$$

in $H^1(\mathbb{R}^n)$ (see Kwong [**19**] for the uniqueness), and the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (1.1) with $V(x) \equiv 0$ is stable for any $\omega > 0$ if $p < 1+4/n$ (see Cazenave and Lions [**4**]), and unstable for any $\omega > 0$ if $p \geq 1+4/n$ (see Berestycki and Cazenave [**2**], Weinstein [**28**]).

For the case where $V(x) = |x|^2$, Ohta and the author [**10**] showed that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable for ω such that $\omega > -\lambda_1$ and sufficiently close to $-\lambda_1$ (see also Kunze et al. [**18**]). Moreover, we proved in [**9, 10**] that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is unstable for sufficiently large $\omega > 0$ if $p > 1+4/n$ and that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable for sufficiently large $\omega > 0$ if $p < 1+4/n$.

Here, we define a real Hilbert space Σ by

$$\Sigma := \{v \in H^1(\mathbb{R}^n, \mathbb{C}) \ ; |x|^2|v(x)|^2 \in L^1(\mathbb{R}^n)\}$$

with the inner product

$$(v,w)_\Sigma := \mathrm{Re} \int_{\mathbb{R}^n} (v(x)\overline{w(x)} + \nabla v(x) \cdot \overline{\nabla w(x)} + |x|^2 v(x)\overline{w(x)})dx.$$

The norm of Σ is denoted by $\|\cdot\|_\Sigma$. Moreover, we define the energy functional E and the charge Q on Σ by

$$E(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|xv\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \quad Q(v) := \frac{1}{2}\|v\|_2^2.$$

The time local well-posedness for the Cauchy problem to (1.1) in Σ and the conservation of the energy $E(v)$ and the charge $Q(v)$ have been established (see Oh [**22**] and Theorem 9.2.5 of Cazenave [**3**]). Namely, we have the following proposition.

Proposition 1.1. *For any $u_0 \in \Sigma$, there exist $T = T(\|u_0\|_\Sigma) > 0$ and a unique solution $u(t) \in C([0,T],\Sigma)$ of* (1.1) *with $u(0) = u_0$ satisfying*

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0), \quad t \in [0,T].$$

The stability and instability of standing wave solutions are formulated as follows.

Definition 1. we put

$$U_\delta(\phi_\omega) := \left\{ v \in \Sigma : \inf_{\theta\in\mathbb{R}} \|v - e^{i\theta}\phi_\omega\|_\Sigma < \delta \right\}.$$

We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable in Σ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in U_\delta(\phi_\omega)$, the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in Σ.

Our main result is the following.

Theorem 1.1. *Let $p = 1+4/n$. Then there exists $\omega_* \in (0,\infty)$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of* (1.1) *is stable in Σ for any $\omega \in (\omega_*, \infty)$.*

REMARK 1.1. When $V(x) \equiv 0$ and $p = 1 + 4/n$, Weinstein [**28**] proved that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ is strongly unstable for $\omega > 0$ (see also Berestycki and Cazenave [**2**]). However, the argument in [**2**] and [**28**] cannot be applied to the case $V(x) \not\equiv 0$. The standing wave solution of (1.1) with $V(x) \equiv 0$ corresponds to 0 energy, but, the standing wave solution with $V(x) = |x|^2$ always corresponds to positive energy. In [**32**], Zhang discussed the instability of the standing wave solution for (1.1) with $V(x) = |x|^2$ and $p \geq 1 + 4/n$. He constructed a kind of cross-constrained minimization problem following [**2**], but it is not easy to verify his sufficient condition for the strong instability. To our knowledge, the problem whether the standing wave solution of (1.1) with $V(x) \not\equiv 0$ and $p = 1 + 4/n$ is stable or unstable is still open for $\omega > 0$. Therefore, by Theorem 1.1, we may answer that the standing wave solution of (1.1) with $p = 1+4/n$ and $V(x) = |x|^2$ is stable for sufficiently large $\omega > 0$. Furthermore, in Appendix, we give a sufficient condition for $V(x) \not\equiv 0$ to prove the same statement of Theorem 1.1. Here, we remark that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable for ω such that $\omega > -\lambda_1$ and sufficiently close to $-\lambda_1$ even if $p = 1 + 4/n$ (see [**10**]).

REMARK 1.2. For a bounded potential $V(x)$, Rose and Weinstein [**23**] studied by numerical simulations that if $p = 1 + 4/n$, $\|\phi_\omega\|_2^2$ would increase for large ω, so that $e^{i\omega t}\phi_\omega(x)$ would be stable. We can affirm that this numerical result is correct by Theorem 1.1 for large $\omega > 0$ since our result is also valid for a bounded potential (see Appendix).

Put $d(\omega) = S_\omega(\phi_\omega)$, where S_ω is the action functional, i.e.,

$$S_\omega(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|xv\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}.$$

To prove Theorem 1.1, we verify the following sufficient condition for stability which was obtained by Shatah [**24**].

PROPOSITION 1.2. *Let $1 < p < 2^* - 1$. If $d''(\omega) > 0$ at $\omega = \omega_0$, then the standing wave solution $e^{i\omega_0 t}\phi_{\omega_0}(x)$ of (1.1) is stable in Σ.*

In the case where $V(x) \equiv 0$, by the scaling $\psi_\omega(x) = \omega^{1/(p-1)}\psi_1(\sqrt{\omega}x)$, we have $d_0(\omega) = \omega^{2/(p-1)-n/2+1}d_0(1)$, where we put $d_0(\omega) = S_\omega(\psi_\omega)$ with $V(x) \equiv 0$. Therefore, it is easy to check the increase and decrease of $d_0'(\omega)$. However, it seems difficult to check this property of $d(\omega)$ for general $V(x)$.

2. Properties of a ground state

First we remark that $d(\omega)$ is simply rewritten by

$$d(\omega) = S_\omega(\phi_\omega) = \frac{p-1}{2(p+1)}\|\phi_\omega\|_{p+1}^{p+1} \tag{2.1}$$

(see [**9**]). In this section, we present the properties of a rescaled function of $\phi_\omega(x)$ to check the stability condition $d''(\omega) > 0$ in Proposition 1.2. Namely, we define the rescaled function $\tilde{\phi}_\omega(x)$ by

$$\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi}_\omega(\sqrt{\omega}x), \qquad \omega \in (0, \infty).$$

Then $\tilde{\phi_\omega}(x)$ satisfies

$$-\Delta\tilde{\phi_\omega} + \omega^{-2}|x|^2\tilde{\phi_\omega} + \tilde{\phi_\omega} - |\tilde{\phi_\omega}|^{p-1}\tilde{\phi_\omega} = 0 \tag{2.2}$$

and we have

$$(2.1) = \frac{p-1}{2(p+1)}\omega^\alpha \|\tilde{\phi_\omega}\|_{p+1}^{p+1}, \tag{2.3}$$

where $\alpha = (p+1)/(p-1) - n/2$.

REMARK 2.1. We note that $\alpha > 1$ if $p < 1+4/n$, $\alpha = 1$ if $p = 1+4/n$ and that $\alpha < 1$ if $p > 1+4/n$.

Define the linearized operator $\tilde{L_\omega}$ by

$$\tilde{L_\omega} := -\Delta + 1 + \omega^{-2}|x|^2 - p\tilde{\phi_\omega}^{\,p-1}(x), \qquad \omega \in (0,\infty).$$

PROPOSITION 2.1. *Let* $1 < p < 2^*-1$ *and* $\psi_1(x)$ *be the unique positive radial solution of* (1.3) *with* $\omega = 1$. *Then the followings hold.*

(i) $\lim_{\omega\to\infty} \|\tilde{\phi_\omega} - \psi_1\|_{H^1} = 0.$

(ii) $\tilde{\phi_\omega}(r) \to 0$ *as* $r \to \infty$. (*independent of* ω)

(iii) *There exist* $C_0(n) > 0$, $r_0(n,p) > 0$ *and* $\omega_1(n,p) > 0$ *such that*

$$|\tilde{\phi_\omega}(r)| \le C_0 r^{-(n-1)/2} e^{-r/2}$$

for any $r \ge r_0$ *and* $\omega \ge \omega_1$.

(iv) $\tilde{L_\omega}$ *is invertible and* $\tilde{L_\omega}^{-1}$ *is bounded for sufficiently large* ω, *i.e., there exist* $\omega_2 > 0$ *and* $C_2 > 0$ *such that for any* $\omega \ge \omega_2$

$$\|\tilde{L_\omega}v\|_2 \ge C_2\|v\|_2$$

for any $v \in H^2_{\mathrm{rad}}(\mathbb{R}^n)$ *and* $|x|^2 v \in L^2(\mathbb{R}^2)$.

(v) $\omega \mapsto \tilde{\phi_\omega}$ *is a* C^1 *mapping from* $(0,\infty)$ *to* Σ *for sufficiently large* ω.

REMARK 2.2. In order for the constant C_2 not to depend on the frequency ω, we show (iv) by considering $\tilde{L_\omega}$ as a perturbation of L_0, where $L_0 := -\Delta + 1 - p\psi_1^{p-1}$. It is known that there exists $C_1 > 0$ such that $\|L_0 v\|_2 \ge C_1\|v\|_2$ for any $v \in H^2_{\mathrm{rad}}(\mathbb{R}^n)$.

3. Proof of Theorem 1

In this section, we verify the sufficient condition for stability for large ω. First, we need the following lemma.

LEMMA 3.1. *Let* $1 < p < 2^*-1$. *Then we have*

$$\text{(i) } \tilde{L_\omega}\left(\frac{\partial\tilde{\phi_\omega}}{\partial\omega}\right) = 2\omega^{-3}|x|^2\tilde{\phi_\omega},$$

$$\text{(ii) } \int_{\mathbb{R}^n} \tilde{\phi_\omega}^{\,p}(x)\frac{\partial\tilde{\phi_\omega}}{\partial\omega}(x)dx = -\frac{2\omega^{-3}}{p-1}\int_{\mathbb{R}^n}|x|^2\tilde{\phi_\omega}^{\,2}(x)dx.$$

In the following Lemma, we check the sufficient condition for stability $d''(\omega) > 0$ for sufficiently large ω. Combining the following Lemma 3.2 with Proposition 1.2, we have Theorem 1.1.

LEMMA 3.2. *Let $p = 1 + 4/n$. Then there exists $\omega_* > 0$ such that $d''(\omega) > 0$ for any $\omega \in (\omega_*, \infty)$.*

Outline of the proof of Lemma 3.2. We directly differentiate $d(\omega)$ with respect to ω. Using Lemma 3.1, we have

$$\begin{aligned} d''(\omega) &= \omega^{-3} \int_{\mathbb{R}^n} |x|^2 \tilde{\phi_\omega}^2(x)dx - 4\omega^{-5} \int_{\mathbb{R}^n} |x|^2 \tilde{\phi_\omega} \tilde{L_\omega}^{-1}(|x|^2 \tilde{\phi_\omega})dx \\ &\geq \omega^{-3} \int_{\mathbb{R}^n} |x|^2 \tilde{\phi_\omega}^2(x)dx - C\omega^{-5} \int_{\mathbb{R}^n} |x|^4 \tilde{\phi_\omega}^2(x)dx \end{aligned} \tag{3.1}$$

for sufficiently large ω. We have used the boundedness of the linearized operator (Proposition 2.1 (iv)) in the last inequality. We divide (3.1) into three parts:

$$\begin{aligned} &(3.1) = (\mathrm{I}) - (\mathrm{II}) + (\mathrm{III}), \\ (\mathrm{I}) &:= \omega^{n/2-2} \int_{|y|\leq 1} |y|^2 \tilde{\phi_\omega}^2(\sqrt{\omega}y)dy, \\ (\mathrm{II}) &:= C\omega^{n/2-3} \int_{|y|\leq 1} |y|^4 \tilde{\phi_\omega}^2(\sqrt{\omega}y)dy, \\ (\mathrm{III}) &:= \omega^{n/2-2} \int_{|y|\geq 1} |y|^2 \tilde{\phi_\omega}^2(\sqrt{\omega}y)dy \\ &\qquad -C\omega^{n/2-3} \int_{|y|\geq 1} |y|^4 \tilde{\phi_\omega}^2(\sqrt{\omega}y)dy. \end{aligned}$$

Then it follows from Proposition 2.1 (i) and (iii) that

$$\begin{aligned} (\mathrm{I}) &= \omega^{-3} \int_{0\leq|x|\leq\sqrt{\omega}} |x|^2 \tilde{\phi_\omega}^2(x)dx \geq \omega^{-3} \int_{1\leq|x|\leq\sqrt{\omega}} \tilde{\phi_\omega}^2(x)dx \\ &\geq \frac{\omega^{-3}}{2} \int_{1\leq|x|} \psi_1^2(x)dx, \\ |(\mathrm{II})| &\leq C\omega^{-5} \int_{0\leq|x|\leq\sqrt{\omega}} |x|^4 \tilde{\phi_\omega}^2(x)dx \\ &\leq C\omega^{-5} \left\{ \int_{0\leq|x|\leq r_0} |x|^4 \tilde{\phi_\omega}^2(x)dx + \int_{r_0\leq|x|\leq\sqrt{\omega}} |x|^4 \tilde{\phi_\omega}^2(x)dx \right\} \\ &\leq C\omega^{-5} \left\{ r_0^4 \int_{\mathbb{R}^n} \psi_1^2(x)dx + \int_{r_0\leq|x|} |x|^{4-(n-1)} e^{-|x|}dx \right\}, \\ |(\mathrm{III})| &\leq C\omega^{-2} e^{-\sqrt{\omega}}, \end{aligned}$$

for sufficiently large ω, where r_0 is as in Proposition 2.1 (iii). Thus, we have consequencely that $d''(\omega)$ is strictly positive for sufficiently large ω.

4. Appendix

In this section, we give a sufficient condition for more general potentials $V(x)$ which are valid for Theorem 1.1. However, we need to consider this case in radially symmetric space and we have to assume the time local well-posedness for the Cauchy problem to (1.1).

Assumptions for $V(x)$. There exist real valued, radially symmetric functions $V_1(x) = V_1(|x|)$ and $V_2(x) = V_2(|x|)$ such that $V(x) = V_1(x) + V_2(x)$.

(V0) $V_j(x) \geq 0$ in $\mathbb{R}^n$ and $V_j(x) \in C^2(\mathbb{R}^n, \mathbb{R})$, for $j = 1, 2$.

(V1-1) For α with $|\alpha| \leq 2$, there exist $C_\alpha > 0$ and $m_\alpha > 0$ such that $|x^\alpha \partial_x^\alpha V_1(x)| \leq C_\alpha(1 + |x|^{m_\alpha})$ for $|x| \geq 1$.

(V1-2) $\Delta V_1(x) \in L^\infty(\{|x| \geq 1\})$.

(V2) $x^\alpha \partial_x^\alpha V_2(x) \in L^\infty(\{|x| \geq 1\})$ for $|\alpha| \leq 2$.

(V3-1) There exist $\delta_1 > 0$ and $\beta > 0$ such that $3x \cdot \nabla V(x) + \Sigma_{k,l} x_k x_l \partial_k \partial_l V(x) \geq \delta_1 |x|^\beta$ for $|x| \leq 1$.

(V3-2) There exist $\delta_2 > 0$ and $\varepsilon > 0$ with $0 < \beta < 2(1 + \varepsilon)$ such that $|V(x) + (1/2)x \cdot \nabla V(x)| \leq \delta_2 |x|^\varepsilon$ for $|x| \leq 1$.

Remark 4.1. The conditions (V3-1) and (V3-2) derive from the twice differentiation of $\omega^{-1} V(x/\sqrt{\omega})$ with respect to ω for the verification of the sufficient condition for stability.

Examples. (i) (Harmonic potentials) For $c_1, \cdots, c_n \in \mathbb{R}$, $\sum_{j=1}^n c_j^2 x_j^2$ satisfies (V0) (V1-1) (V1-2) (V3-1) and (V3-2) with $V_2(x) \equiv 0$.
(ii) Let $n \geq 2$ and $U(x) \in C^2(\mathbb{R}^n)$ be a nonnegative, radially symmetric function which satisfies $|\partial_x^\alpha U(x)| \leq C_\alpha < x >^{-|\alpha|}$ for $|\alpha| \leq 2$ and there exist $\theta \geq 2$ and $C > 0$ such that $U(x) = C|x|^\theta$ for $|x| \leq 1$. Then, $U(x)$ verifies (V0) (V2) (V3-1) (V3-2) with $V_1(x) \equiv 0$.
(iii) $V(x) \equiv 1$ satisfies (V0) (V1-1) (V1-2) and (V2), but does not satisfy (V3-1) and (V3-2) which bring out the difference from the pure power case.

Details shall be published in [**11**].

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