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Johannes Sjöstrand

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U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Asymptotics for Bergman kernels for high powers of complex line bundles, based on joint works with B.Berndtsson and R.Berman

Johannes Sjöstrand CMLS, Ecole Polytechnique, FR-91128 Palaiseau Cédex

Résumé: Nous discutons l'asymptotique des noyaux de Bergman pour des puissances élevées de fibrés de droites, d'après deux travaux récents avec B.Berndtsson et R. Berman*

0. Introduction.

We present some new proofs and results around the so called Tian–Yau–Zelditch–Catlin asymptotics for the orthogonal projections onto the spaces of harmonic forms with coefficients in a high power of a complex line-bundle:

1) For (0,0)-forms: Here we give a new proof (joint work with B.Berndtsson and R.Berman [BeBeSj].)

2) For (0,q)-forms: New result and proof (joint work with R.Berman [BeSj]).

The subject has gained new interest recently through the work of geometers. M. Shubin suggested closely related problems to me 11 years ago, and later I got more stimulation mainly through the works of Shiffman, Zelditch and coworkers and from dicussions with Berndtsson and Berman around the work [Be], as well as with X.Ma. The plan of the talk is:

1) Statement of the result.
2) Some historical remarks.
3) Quick outline of a new proof for (0,0)-forms.
4) Outline of the proof for (0,q)-forms.

1. The result

Let L be a holomorphic line bundle over a complex compact manifold X of dimension n. Assume the fibers L_x and $\wedge^{1,0}T_xX$ carry Hermitian metrics that depend smoothly on $x \in X$.

If s is a non-vanishing holomorphic section of L on the open subset $\widetilde{X} \subset X$, write $|s(x)| = e^{-\phi(x)}$ with $\phi(x)$ real and smooth. The curvature form of the line bundle is then determined by

$$\partial\overline{\partial}\phi = \sum \frac{\partial^2\phi}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k,$$

where the right hand side is written in local holomorphic coordinates. Assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_+, n_-) on X.

We shall replace L by L^k and consider the $\overline{\partial}$-complex:

$$\begin{aligned} &C^\infty(X; L^k \otimes \wedge^{0,0}T^*X) \to C^\infty(X; L^k \otimes \wedge^{0,1}T^*X) \\ &\to \ldots \to C^\infty(X; L^k \otimes \wedge^{0,n}T^*X). \end{aligned} \tag{1.1}$$

* Key words: complex, line, bundle, MSC 2000: 32L05, 35S30

If we also fix a positive smooth integration density $m(dx)$, we have the adjoint $\overline{\partial}^*$-complex

$$\begin{aligned} &C^\infty(X; L^k \otimes \wedge^{0,0} T^*X) \leftarrow C^\infty(X; L^k \otimes \wedge^{0,1} T^*X) \\ &\leftarrow ... \leftarrow C^\infty(X; L^k \otimes \wedge^{0,n} T^*X). \end{aligned} \tag{1.1*}$$

We introduce

$$h = 1/k \ll 1, \text{ for } k \gg 1 \tag{1.2}$$

and the Hodge Laplacian for $(0,q)$-forms:

$$\Delta_q = \Delta_{q,k} = h\overline{\partial} h\overline{\partial}^* + h\overline{\partial}^* h\overline{\partial}. \tag{1.3}$$

X being compact, Δ_q is essentially self-adjoint with discrete spectrum contained in $[0, +\infty[$. Let $\mathcal{N}(\Delta_q)$ be the kernel (i.e. the 0-eigenspace) and let

$$\Pi_q : L^2(X, L^k \otimes \wedge^{0,q} T^*X) \to \mathcal{N}(\Delta_q)$$

be the orthogonal (Bergman) projection. With $\widetilde{X}$, s, ϕ as above, we have the unitary identifications

$$\begin{aligned} L^2(\widetilde{X}; \wedge^{0,q} T^*X) &\leftrightarrow L^2(\widetilde{X}; L^k \otimes \wedge^{0,q} T^*X) \\ u &\leftrightarrow (e^\phi s)^k u \\ Z_\phi &\leftrightarrow h\overline{\partial} \\ \Delta_{q,\mathrm{loc}} &\leftrightarrow \Delta_q \\ \Pi_{q,\mathrm{loc}} &\leftrightarrow \Pi_q, \end{aligned}$$

with

$$\begin{aligned} Z_\phi &= (e^\phi s)^{-k} \circ h\overline{\partial} \circ (e^\phi s)^k = h\overline{\partial} + (\overline{\partial}\phi)^\wedge, \\ \Delta_{q,\mathrm{loc}} &= Z_\phi^* Z_\phi + Z_\phi Z_\phi^*, \\ \Pi_{q,\mathrm{loc}} &= (e^\phi s)^{-k} \Pi_q (e^\phi s)^k. \end{aligned}$$

For the proof in the case of (0,0)-forms we shall also use the unitary identification

$$\begin{aligned} L^2(\widetilde{X}; \wedge^{0,q} T^*X, e^{-\frac{2\phi}{h}} m) &\leftrightarrow L^2(\widetilde{X}; L^k \otimes \wedge^{0,q} T^*X) \\ e^{\phi/h} u &\leftrightarrow (e^\phi s)^k u. \end{aligned}$$

$\Delta_{q,\mathrm{loc}}$ has a scalar principal symbol $p \geq 0$ (times the identity matrix) vanishing precisely to the second order on the symplectic submanifold $\Sigma \subset T^*X$, given by

$$\zeta = \frac{2}{i} \frac{\partial \phi}{\partial z}, \ z = x + iy, \ \zeta = \xi - i\eta,$$

with $(x, y; \xi, \eta)$ as standard canonical coordinates on T^*X (and $z = (z_1, ..., z_n)$ denoting local holomorphic coordinates).

In [MeSj1] and later in [BoGu] it was established that there exist almost analytic manifolds (in the sense of [MelSj] and we shall from now on use the term almost holomorphic)

$$J_+, J_- \subset T^*X^{\mathbf{C}},\ J_- = \overline{J}_+,$$

such that $J_+ \cap J_- = \Sigma^{\mathbf{C}}$ with transversal intersection, such that locally

$$J_+ : f_1 = ... = f_n = 0,\ \{f_j, f_k\}_{|J_+} = 0,$$

$$(\frac{1}{i}\{f_j, \overline{f}_k\})_{j,k} > 0 \text{ on } \Sigma,\ p_{|J_+} = 0.$$

When $n_- = 0$, we can take f_j to be the semi-classical symbol of $h\frac{\partial}{\partial \overline{z}_j} + \frac{\partial \phi}{\partial \overline{z}_j}$ that will be given more explicitly below. The following theorem is mainly due to S.Zelditch and D.Catlin when $q = n_- = 0$ and to R.Berman and Sjöstrand in the general case.

Theorem 1.1. *For $k = 1/h$ sufficiently large, we have $\Pi_q = 0$, $q \neq n_-$ * and for $q = n_-$:*

$$\Pi_{q,\mathrm{loc}} u(x) = h^{-n} \int e^{\frac{1}{h}\psi(x,y)} b(x,y;h) u(y) m(dy) + Ru, \tag{1.4}$$

for $x \in \widetilde{X}$, $u \in L^2(\widetilde{X}, L^k \otimes \wedge^{0,q} T^*X)$, where $b \sim \sum_0^\infty b_j(x,y) h^j$ in
$C^\infty(\widetilde{X} \times \widetilde{X}; \mathcal{L}(\wedge^{0,q}T_y^*X, \wedge^{0,q}T_x^*X))$,
$Ru = \int r(x,y;h)u(y)m(dy)$, $\partial_{x,y}^\alpha r = \mathcal{O}(h^\infty)$. Further, $\psi(x,x) = 0$, $\mathrm{Re}\,\psi(x,y) \sim -\mathrm{dist}(x,y)^2$,

$$\begin{cases} (x, d_x \frac{1}{i}\psi(x,y)) \in J_+ \\ (y, -d_y \frac{1}{i}\psi(x,y)) \in J_- \end{cases} \mathrm{mod}\, \mathcal{O}(\mathrm{dist}(x,y)^\infty).$$

For $x = y$:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x},\ \frac{\partial \psi}{\partial \overline{x}} = -\frac{\partial \phi}{\partial \overline{x}},\ \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x},\ \frac{\partial \psi}{\partial \overline{y}} = \frac{\partial \phi}{\partial \overline{x}}.$$

2. Historical remarks.

Most of the earlier results concern the positively curved case $n_- = 0$. G.Tian [Ti], followed by W.Ruan [Ru] and Z.Lu [Lu], computed increasingly many terms of the asymptotic expansion on the diagonal, using Tian's method of peak solutions. T. Bouche [Bou] also got the leading term using heat kernels.

S.Zelditch [Ze], D.Catlin [Ca] established the complete asymptotic expansion at $x = y$ by using a result of Boutet de Monvel, Sjöstrand [BoSj] for the asymptotics of the Szegö kernel on a strictly pseudoconvex boundary (after the pioneering work of C.Fefferman [Fe]), here on the boundary of the unit disc bundle, and a reduction idea of Boutet de Monvel, Guillemin [BoSj]. Scaling asymptotics away from the diagonal was obtained later

* as follows from Hörmander's $L^2 - \overline{\partial}$ estimates [Hö].

by P.Bleher, B.Shiffman, Zelditch [BlShZe] and the full asymptotics by L. Charles [Ch], using again the reduction method.

In more general situations, full asymptotic expansions on the diagonal and in some sense away from the diagonal were obtained by X.Dai, K.Liu, X.Ma [DaLiMa] (see also [MaMar] for related spectral results).

Without a positive curvature assumption there are fewer results. J.M.Bismut [Bi] used the heat kernel method in his approach to Demailly's holomorphic Morse inequalities. X. Ma has pointed out to us that the method and results of [DaLiMa] can be extended to the case of non-positive holomorphic line bundles by using a spectral gap estimate from [MaMar].

3. Quick outline of a proof when $q = n_- = 0$ ([BeBeSj])

Locally, the problem is essentially to find the orthogonal projection from $L^2(\mathbf{C}^n, e^{-2\phi/h}m(dx))$ to its subspace of holomorphic functions. That projection was recently constructed in [MeSj3], and the method we present here is similar but differs on one essential point: A square root procedure is replaced by a simpler algorithm. Write for $u \in L^2_\phi \cap \mathrm{Hol}$:

$$1u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\theta} u(y)dyd\theta \tag{3.1}$$
$$\equiv \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\theta} a(x,y,\theta;h)u(y)dyd\theta$$

modulo an error $\mathcal{O}(h^\infty)$, provided that the symbol $a \sim \sum_0^\infty a_j h^j$ (is almost holomorphic at a suitable set and) satisfies

$$\sum_{\alpha\in\mathbf{N}^n} \frac{h^{|\alpha|}}{\alpha!} (\partial_\theta^\alpha D_y^\alpha a(x,y,\theta;h))_{y=x} \sim 1. \tag{3.2}$$

Let $\Psi(x,y)$, $M(x,y)$ be almost holomorphic with $\Psi(x,\overline{x}) = \phi(x)$, $M(x,\overline{x}) = m(x)$. Recall that in the case $n_- = 0$, ϕ is strictly plurisubharmonic and we have

$$-\phi(x) + 2\mathrm{Re}\,\Psi(x,\overline{y}) - \phi(y) \sim -|x-y|^2.$$

Consider

$$Ju(x) = \tag{3.3}$$
$$\iint e^{\frac{2}{h}(\Psi(x,w)-\Psi(y,w))} c(x,w;h) M(y,w) u(y) \frac{dydw}{h^n}$$
$$= \iint e^{\frac{2}{h}\Psi(x,\overline{y})} c(x,\overline{y};h) u(y) e^{-\frac{2}{h}\phi(y)} m(y) \frac{dyd\overline{y}}{h^n}$$

where we integrate over $w = \overline{y}$ in the first integral.

Use the Kuranishi trick:

$$2(\Psi(x,w) - \Psi(y,w)) = i(x-y)\cdot\theta(x,y,w),$$

$$Ju(x) = \iint e^{\frac{i}{h}(x-y)\cdot\theta} a(x,y,\theta;h)u(y)\frac{dyd\theta}{(2\pi h)^n},$$

$$a(x,y,\theta;h) = (2\pi)^n \frac{c(x,w(x,y,\theta);h)M(y,w(x,y,\theta))}{\det(\frac{\partial\theta}{\partial w})}.$$

Here the coefficients c_0, c_1, ... in the asymptotic expansion of c can be determined successively so that (3.2) holds.

4. Outline of the proof for general n_- ([BeSj])

We shall use the heat equation approach of [MeSj] with a Witten complex trick. Work locally with

$$\Delta_{q,\mathrm{loc}} = Z_\phi^* Z_\phi + Z_\phi Z_\phi^*.$$

Let $x_1, ..., x_{2n}$ be local coordinates. Construct a parametrix $U_q(t;h)$ for

$$(h\partial_t + \Delta_{q,\mathrm{loc}})U_q(t) = \mathcal{O}(h^\infty),\ U_q(0) = \mathrm{id}, \tag{4.1}$$

$$U_q(t)u(x) = \iint e^{\frac{i}{h}(\psi(t,x,\eta)-y\eta)} a(t,x,\eta;h)u(y)\frac{dyd\eta}{(2\pi h)^{2n}}. \tag{4.2}$$

Here we can solve

$$i\partial_t\psi + p(x,\psi_x') = \mathcal{O}((\mathrm{Im}\,\psi)^\infty),$$

locally with $\psi(0,x,\eta) = x\cdot\eta$ and with $\mathrm{Im}\,\psi \geq 0$, and more precisely

$$\mathrm{Im}\,\psi \sim \mathrm{dist}(x,\eta;\Sigma)^2,\ t \geq t_0 > 0,$$

$$\psi(t,x,\eta) = x\cdot\eta + \mathcal{O}(\mathrm{dist}(x,\eta;\Sigma)^2)$$

(See [MelSj2] and references given there to work of Kucherenko and others.) In [MeSj] a more detailed study was given, using that Σ is symplectic, and we showed that there exists a limiting function $\psi(\infty,x,\eta)$ such that

$$\partial^\alpha_{t,x,\eta}(\psi(t,x,\eta) - \psi(\infty,x,\eta)) = \mathcal{O}_\alpha(1)e^{-\frac{t}{C}}, \tag{4.3}$$

for $t \geq 0$, $(x,\eta) \in \Sigma$. As used in [MeSj1,2], $J_\pm$ can be viewed as the stable outgoing and incoming manifolds for the $i^{-1}H_p$ flow around the fixed point variety $\Sigma^{\mathbf{C}}$, and the canonical transformation κ_t generated by $\psi(t,\cdot,\cdot\cdot)$ converges to the limiting canonical relation κ_∞ characterized by saying that $(\rho,\mu) \in \mathrm{graph}\,(\kappa_\infty)$ if $\rho \in J_+$, $\mu \in J_-$ belong to bicharacteristics leaves of J_+, J_- respectively, containing the same point of $\Sigma^{\mathbf{C}}$.

The symbol

$$a(t,x,\eta;h) \sim \sum_0^\infty a_j(t,x,\eta)h^j,$$

is determined by a sequence of transport equations, and adapting the approach of [MeSj1] to the case of matrix-valued symbols, we get on Σ:

$$\partial^\alpha_{t,x,\eta}a_j = \begin{cases} \mathcal{O}_{\alpha,j}(1)e^{-t/C},\ q \neq n_- \\ \mathcal{O}_{\epsilon,\alpha,j}(1)e^{\epsilon t},\ \forall\epsilon > 0,\ q = n_-. \end{cases} \tag{4.4}$$

Now, let $q = n_-$ and apply a Witten trick: From

$$\Delta_{q+1,\mathrm{loc}} Z_\phi = Z_\phi \Delta_{q,\mathrm{loc}}, \quad \Delta_{q-1,\mathrm{loc}} Z_\phi^* = Z_\phi^* \Delta_{q,\mathrm{loc}},$$

we get

$$(h\partial_t + \Delta_{q-1,\mathrm{loc}}) Z_\phi^* U_q(t) = \mathcal{O}(h^\infty),$$
$$(h\partial_t + \Delta_{q+1,\mathrm{loc}}) Z_\phi U_q(t) = \mathcal{O}(h^\infty).$$

Here $Z_\phi U_q$, $Z_\phi^* U_q$ have the general form (4.2) and since $q-1 \neq n_- \neq q+1$, one can show that the symbols satisfy the same decay estimate as in the first case in (4.4).

This also applies to

$$\Delta_{q,\mathrm{loc}} U_q = Z_\phi (Z_\phi^* U_q) + Z_\phi^* (Z_\phi U_q),$$

and by (4.1) to

$$h \frac{\partial U_q(t)}{\partial t} u = \iint e^{\frac{i}{h}(\psi(t,x,\eta) - y\eta)} (i \frac{\partial \psi}{\partial t} a + h \frac{\partial a}{\partial t}) u(y) \frac{dy d\eta}{(2\pi h)^{2n}}.$$

This and (4.3) imply

$$\partial^\alpha_{t,x,\eta} \partial_t a_j = \mathcal{O}(1) e^{-t/C}, \ (x,\eta) \in \Sigma.$$

Hence, there exists a symbol $a_j(\infty, x, \eta)$ such that on Σ:

$$\partial^\alpha_{t,x,\eta} (a_j(t,x,\eta) - a_j(\infty,x,\eta)) = \mathcal{O}(e^{-t/C}).$$

We get the approximate null-projection:

$$\Pi^{\approx}_{q,\mathrm{loc}} u(x) = \iint e^{\frac{i}{h}(\psi(\infty,x,\eta) - y\eta)} a(\infty,x,\eta;h) u(y) \frac{dy d\eta}{(2\pi h)^{2n}}$$
$$= \int e^{\frac{1}{h}\psi_{\mathrm{new}}(x,y)} b(x,y;h) u(y) \frac{m(dy)}{h^n} + Ru,$$

where the last equality follows from complex stationary phase ([MelSj]) and the last expression is as in the theorem.

The remaining part is more routine. We get:

$$U_q(t) = \Pi^{\approx}_{q,\mathrm{loc}} + V_q(t),$$

$$V_q(t) = \mathcal{O}(e^{-t/C}) : H^{-\infty}_{\mathrm{comp}} \to H^\infty_{\mathrm{loc}}, \ t \geq t_0 > 0.$$

Modulo $\mathcal{O}(h^\infty)$:

$$\Delta_{q,\mathrm{loc}} \Pi^{\approx}_{q,\mathrm{loc}} \equiv \Pi^{\approx}_{q,\mathrm{loc}} \Delta_{q,\mathrm{loc}} \equiv 0,$$
$$(\Pi^{\approx}_{q,\mathrm{loc}})^* \equiv \Pi^{\approx}_{q,\mathrm{loc}},$$
$$[\Pi^{\approx}_{q,\mathrm{loc}}, V(t)] = \mathcal{O}(e^{-t/C} h^\infty).$$

Approximate resolvent for $\mathrm{Re}\, z < (2C)^{-1}$, $|z| \geq h^{N_0}$:

$$R^{\approx}_{\mathrm{loc}}(hz) = \frac{1}{hz}\Pi^{\approx}_{q,\mathrm{loc}} - \frac{1}{h}\int_0^{\infty} e^{tz}V_q(t)dt.$$

(When $q \neq n_-$, we have the simpler formula for $\mathrm{Re}\, z < (2C)^{-1}$:

$$R^{\approx}_{\mathrm{loc}}(hz) = -\frac{1}{h}\int_0^{\infty} e^{tz}U_q(t)dt.)$$

Notice that,

$$\Pi^{\approx}_{q,\mathrm{loc}} = \frac{-1}{2\pi i}\int_{|z|=r} R^{\approx}_{\mathrm{loc}}(z)dz,\ h^{N_0} \leq r \leq \frac{h}{2C}.$$

Back to the global situation, we glue the different $R^{\approx}_{\mathrm{loc}}$ together and get $R^{\approx}(z) : H^s(X) \to H^s(X)$ such that for $\mathrm{Re}\, z < (2C)^{-1}$, $|z| \geq h^{N_0}$:

$$(\Delta_q - hz)R^{\approx}(hz) \equiv R^{\approx}(hz)(\Delta_q - hz) \equiv \mathrm{id}, \tag{4.5}$$

which implies that

$$(\Delta_q - hz)^{-1} \equiv R^{\approx}(hz),$$

$$\Pi_q = \frac{1}{2\pi i}\int_{|z|=r} (z - \Delta_q)^{-1}dz \equiv \frac{-1}{2\pi i}\int_{|z|=r} R^{\approx}(z)dz$$

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