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COMPLEX EXTENSIONS

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A simple case of a complex extension in the complexification of the real number space R^n : if R^n is embedded in C^n as real part, C^n can be called a complex extension of R^n. In what follows, this notion is generalized to the class of analytic manifolds. Given a real analytic manifold M which is enumerable at infinity, we construct a complex manifold N containing M as its «real part» with respect to a certain subatlas β of N. The pair (N,β) is called a complex extension of M.

This exposition is complementary to the author's published work on this subject [9]. For other proofs of the existence theorem the reader is referred to Haefliger [6], Morrey [7] and Bruhat and Whitney [2]. Haefliger's proof, which appears on page 296 of his thesis, is similar to the author's original proof.

This existence theorem is the first lemma in the theorem that a real-analytic manifold (enumerable at infinity) can be analytically embedded in a number space R^{2n+1}($n = \dim M$). The compact case was proved by Morrey (loc. cit.), and Grauert proved the general result [5]. It is shown at the end of the exposition that the results of Grauert and Morrey can be used to strengthen the complex extension existence theorem.

1. Extension of local automorphisms. If f maps a subset of a set A onto a subset of B, it is convenient to denote the subsets by $U(f)$ and $V(f)$, respectively, so f is the ontomapping $f : U(f) \to V(f)$. We compose in the class of such mappings by the law $(f, g) \to fg = f \circ g \mid \bar{g}^{1}(U(f) \cap V(g))$, which is defined whenever $U(f)$ and $V(g)$ are subsets of the same set (the conventional mappings between empty subsets are included in the class). Note that $f(gh) = (fg)h$ but the class does not form a category for the units are not unique.

Let Λ_n^ω be the pseudogroup of analytic, regular, automorphisms between open sets of R^n and let Λ_n^c be the corresponding complex-analytic pseudogroup of C^n (4 p. 139). We define :

$$\Lambda_n^e = \{ g : g \in \Lambda_n^c \,;\, g \mid R^n,\ \bar{g}^1 \mid R^n \in \Lambda_n^\omega \}$$

Each member of Λ_n^e is called a *complex extension* of its restriction to R^n (which may be the conventional mapping defined on the empty subset).

PROPERTY 1. *Λ_n^e is a pseudogroup of automorphisms of C^n.* This follows immediately from the definition and the fact that Λ_n^e is a pseudogroup.

PROPERTY 2. *The germ of complex extension is unique : that is, if g_1 and g_2 are complex extensions of $f \in \Lambda_n^\omega$, then g_1 and g_2 coincide on some neighbourhood of $U(f)$.*

They in fact, coincide on those components of $U(g_1) \cap U(g_2)$ which are connected to R^n, because analytic functions defined of connected open sets are uniquely determined by their power series at one point([1] p. 33).

PROPERTY 3. *Any f of Λ_n^ω has a complex extension (and therefore infinitely many).*

This can be proved directly by putting complex values in the power series which define f locally and by using property 2. However, it will be seen that this property is a corollary to proposition 2 (b) which is proved below.

2. Extensions of manifolds : Definition.

Let M be a real analytic manifold defined by a complete atlas a of M on R^n compatible with Λ_n^ω ([4] p. 139). A *complex extension* of M is a pair (N,β), where N is a complex manifold which contains M as a subset and where β is a preferred subatlas of N compatible and complete with respect to Λ_n^e such that the set $\mathfrak{A}_1$ of restrictions of members of β to R^n is a subatlas of $\mathfrak{A}$. It will be shown later that $\mathfrak{A}_1 = \mathfrak{A}$.

The submanifold M can be called the *real part* of (N, β) and it is characterized by properties that it is an analytic, proper, closed, real, submanifold and that dim. M = complex dim. N ([9] p. 193).

3. Extensions of mappings in manifolds. From now on the manifold M is assumed to be enumerable at infinity.

THEOREM 1. *If G is a sheaf* (espace étalé, [6]. p. 255) *defined on a manifold A and if $G(M)$ is the restriction of G to a proper submanifold M of A, then any section γ of $G(M)$ can be extended to a section μ of G over some neighbourhood of M in A.*

PROOF. The existence of γ implies that each point x of M has an open neighbourhood U_x in A and a section μ_x over U_x in G such that the germ of μ_x at each point of $U_x \cap M$ is the value of γ at that point. This implies that μ_x and μ_y for x, y of M coincide on some neighbourhood of $U_x \cap U_y \cap M$. We seek a family $\{U_j, \mu_j\}_{j\in J}$ *of open sets* U_j of A and sections over them μ_j such that :

(a) $\{U_j \cap M\}_{j\in J}$ covers M.

(b) There is a mapping $r : J \to M$ such that $U_j \subset U_{r(j)}$ and μ_j is the restriction of $\mu_{r(j)}$.

(c) Each U_j only intersects a finite number of others.

(d) $\bar{U}_j \cap M = \overline{(U_j \cap M)}$ for all j of J.

We confine attention to some neighbourhood of M in $\bigcup_{x \in M} M$ which is enumerable at infinity. This neighbourhood can be covered by a refinement $\{U_j\}$ of the covering $\{U_x\}$ such that *(a)*, *(b)* and *(c)* are satisfied when μ_j is defined by *(b)* ([8] p. 4, theorem 1). Also, $U_j \cap M$ has an open neighbourhood in U_j which satisfies *(d)* so we can replace U_j by this neighbourhood.

For each pair j, k of J, let U_{jk} be the subset of $U_j \cap U_k$ on which μ_j and μ_k coincide and let $V_{jk} = (U_j - U_j \cap U_k) \cup U_{jk}$. We show that V_{jk} is a neighbourhood of $U_j \cap M$ in A. By *(d)*, it follows that $U_j - U_j \cap \bar{U}_k$ is a neighbourhood of $U_j \cap M - (U_j \cap M \cap \bar{U}_k)$ in A. Also, since μ_j and μ_k are restrictions of $\mu_{r(j)}$ and $\mu_{r(k)}$, the set on which $\mu_{r(j)}$ and $\mu_{r(k)}$ coincide is a neighbourhood if $\overline{U_j \cap U_k \cap M}$. This implies that there are no points in this neighbourhood on which μ_j and μ_k are both defined but unequal, so V_{jk} is a neighbourhood of $U_j \cap \overline{U_k \cap M}$. It follows that V_{jk} is a neighbourhood of $U_j \cap M$ in A.

Finally, we define

$$V_j = \bigcap_{\substack{k \in J \\ U_j \cap U_k \neq \emptyset}} \mathring{V}_{jk}$$

Since the intersection is finite, V is an open neighbourhood of $U_j \cap M$. The restrictions of μ_j define a section of G over $\bigcup_{j \in J} V$ and the theorem is proved.

An example of the use ot theorem 1, is the following proposition which generalizes the fundamental theorem of Differential Geometry (the implicit function theorem).

It is valid for any differentiability class including c^{∞}, c^{ω}, and the complex-analytic case.

PROPOSITION 1. *If A and A' are manifolds of class c^r and dimension n and if g is a c^r mapping of A inot A' which maps a proper submanifold M of A regularly and properly and which has Jacobian of rank n at points of M, then there is a neighbourhood N of M in A mapped regularly by g.*

PROOF. The set of points of A on which the Jacobian has rank n is an open neighbourhood B of M in A. The space of germs of g on B is isomorphic to B under the projection α and forms a sheaf G over $g(B)$ by the projection β. The fundamental theorem gives that g is locally regular on B and thus that $g(B)$ is an open neighbourhood of $g(M)$. Since $g|M$ is regular, the germs of g form a unique section γ over $g(M)$. By theorem 1, this gives a section μ over some open neighbourhood N' of $g(M)$ in $g(B)$. The mapping $\overset{-1}{\alpha}\mu$ is an inverse of g on $\overset{-1}{\alpha}\mu(N') = N$ so the proposition is proved.

PROPOSITION 2. *Complex extension of analytic mappings.*

Suppose (N, β) and (N', β') are complex extensions of M and M' respectively and suppose f is an analytic mapping of M into M'. Then f can be extended to give a complex-analytic mapping g of some neighbourhood of M into N' and further :

(a) if f is locally-regular, the extension g can be chosen locally-regular,

(b) if f is a regular isomorphism between M and M', $g$can be chosen regular.

PROOF. Let G be the sheaf of germs of complex-analytic mappings of N into N'. Since f can be extended locally to a neighbourhood of any point of M by putting complex values in the power series which define f referred to coordinate systems of β and β' it follows that f defines a section γ over M in $G(M)$. Theorem 1 gives an extension μ of γ which defines a complex extension g of f.

(a) Let G be the sheaf of germs of locally-regular complex-analytic mappings, the section γ is defined by f because the complex Jacobian of a local extension at a point of M is the same as the real Jacobian of f.

(b) This is an immediate consequence of (a) and proposition 1.

Two important corollaries to proposition 2 (b) are :

COROLLARY. *In the definition of a complex extension* $\mathfrak{A}_1 = \mathfrak{A}$ *: that is, any member of a can be extended to one of* β.

THEOREM 2. *Uniquences. If* N *and* N' *are complex extensions of the same manifold* M*, then there exist neighbourhoods of* M *in* N *and* N' *which are isomorphic, the manifold* M *being left pointwise invariant by the isomorphism.*

4. C^∞ Extensions. Let $r : C^n \to R^{2n}$ be the mapping defined by $r(\underset{\sim}{x} + \sqrt{-1}\,\underset{\sim}{y}) = (\underset{\sim}{x} > \underset{\sim}{y})$ for $\underset{\sim}{x}, \underset{\sim}{y} \in R^n (R^{2n} = R^n \times R^n)$. As a generalization of the pseudogroup of complex extensions, we define the pseudogroup Γ of C^∞ extensions of Λ_n^ω as the subset of Λ_{2n}^∞ which is classified by :

(a) each g of Γ satisfies $g|\, R^n, \overset{-1}{g}|\, R^n \in \Lambda_n^\omega$ $\qquad (R^n = R^n \times \{O\}$

(b) the derivative of g at points of R^n preserves the isomorphism $(\lambda^1, \lambda^2, \ldots, \lambda^n, 0, 0, \ldots 0) \to (0, 0, \ldots, 0, \lambda^1, \lambda^2, \ldots \lambda^n)$ of tangent vectors of R^n onto a transversal plane of tangent vectors.

The verification that Γ is a pseudogroup is straight-forward. Also, if g is any c^∞ mapping defined on a neighbourhood of $U(f)$ in R^{2n} where $f \in \Lambda_n^\omega$, if $g|\, R^n = f$, and, if g satisfies (b), then the Jacobian determinant of g is the square of the Jacobian determinant of f, so proposition 1 gives some restriction of g which belongs toΓ.

Any member f of λ can be extended to give a member Γ. Consider, in fact, the mapping g_o of $U(f) \times R^n$ onto $V(f) \times R^n$ given by the $2n$ functions

$$g_o^i = f^i \qquad\qquad \begin{pmatrix} i, j = 1, 2, \ldots n \\ i' = i + n,\ j' = j + n \end{pmatrix}$$
$$g_o^{i'} = \frac{\partial f^i}{\partial x^j}\, x^{j'}$$

Then g_o is a member of Γ which extends f. It will be noted that the extensions of the type $f \to g_o$ are exactly those which define the tangent bundle structure on an analytic manifold. The tangent bundle $T(M)$ of M admits a complete atlas β_Γ compatible with Γ and the restrictions of members of β_Γ to R^n form the analytic atlas $\mathfrak{A}$ of M.

We have the following proposition which is akin to saying that the sheaf of germs of c^∞ extensions of Λ_n^ω is fine.

PROPOSITION 3. *If $f \in \Lambda_n^\omega$, if $g_1 \in \Gamma$ is an extension of $f \mid U(g_1) \cap R^n$, and, if C is a relatively-compact open set of $U(g_1) \cap R^n$, then there is C^∞ extension g of f which coincides with g_1 in some neighbourhood of C in R^{2n}.*

PROOF. Let C_1 be a relatively-compact open neighbourhood of $\bar{C}$ in $U(g_1) \cap R^n$. Then there exits a C^∞ function ρ on $U(f)$ which takes its values in the closed unit interval taking the value 1 in C and the value 0 outside C_1 (8. p. 4 Lemma to theorem 1). We define ρ on $U(f) \times R^n$ by letting it be independent of the last n variables. Consider the mapping defined by

$$g = \rho\, g_1 + (1-\rho)\, g_o$$

where g_o is the extension of f defined int the preceding section. Then g is defined and of class c^∞ on some neighbourhood of $U(f)$ in R^{2n}. Both g_o and g_1 give f when restricted to R^n so g does also. Differentiate g,

$$\frac{\partial g^a}{\partial x^\beta} = \frac{\partial \rho}{\partial x^\beta} g_1^a + \rho \frac{\partial g_1^a}{\partial x^\beta} - \frac{\partial \rho}{\partial x^\beta} g_o + (1-\rho)\frac{\partial g_o^a}{\partial x^\beta};(a, \beta = 1, 2, .., 2n),$$

so,

$$\left(\frac{\partial g^a}{\partial x^\beta}\right)_{R^n} = \rho\left(\frac{\partial g_1^a}{\partial x^\beta}\right)_{R^n} + (1-\rho)\left(\frac{\partial g_o^a}{\partial x^\beta}\right)_{R^n}$$

Both g_1 and g_o satisfy the derivative condition (b) on R^n, so g does also. It follows from proposition 1 and the remarks of the last section that a restriction of g is a C^∞ extension of f. This proves proposition 3.

5. Existence Theorem.

THEOREM 3 : EXISTENCE THEOREM. *Any real-analytic manifold M which is enumerable at infinity admits a complex extension which is C^∞ isomorphic to a neighbourhood of M in the tangent bundle.*

Consider the pseudogroup $r(\Lambda_n^e)$ of R^{2n} defined as the image of Λ_n^e under the mapping $r : C^n \to R^{2n}$ defined above. The r image of the isomorphism obtained by multiplying real tangent vectors to R^n by $\sqrt{-1}$, is exactly the one in condition (b) for Γ, and, since (a) is also satisfied, $r(\Lambda_n^e)$ is a sub-pseudogroup of Γ. Hence, for the proof of the theorem, if $T(M)$ is the tangent bundle space of M with its complete atlas β_Γ compatible with Γ, we seek a neighbourhood N of M in $T(M)$ and an atlas β' of N in R^{2n} compatible with $r(\Lambda_b^e)$ and subordinate to β_Γ. Let G be the sheaf of germs of structures on $T(M)$ compatible with $r(\Lambda_n^e)$ and subordinate to β_Γ. By theorem 1, the proof of theorem 3 reduces to showing that there is a section over M in the restricted sheaf $G(M)$.

LEMMA. *If α is a section of $G(M)$ over an open set $U(\alpha)$ of M, if f is a coordinate mapping of M, and, if U' and V' are relatively compact open sets of $U(\alpha)$ and $V(f)$ respectively, then there is a section β of $G(M)$ over $U' \cup V'$ which coincides with α on U'.*

PROOF OF LEMMA. Theorem 1, implies that there exists a complex extension of $U(\alpha)$ defined by the germs of α. If $f' = f|\overset{-1}{f}(V(f) \cap U(\alpha))$ then f' is an analytic coordinate mapping for $U(\alpha)$. It follows from the corollary to theorem 2 that there exists a member g' of the complex extension atlas which extends f'. Let t be the tangent bundle coordinate mapping which extends f and let g_1 be the change of coordinates $\overset{-1}{g'}t$. Then, g_1 is a member of Γ defined on some neighbourhood of $U(f')$ in R^{2n} and g_1 is a c^∞ extension of the identity mapping of $U(f')$. Let $C = \overset{-1}{f}(U' \cap V')$ which is relatively compact in $U(f')$. By proposition 3, there exists a C^∞ extension g of the identity mapping of $U(f)$ which coincides with g in some neighbourhood of C. The coordinate mapping $t\overset{-1}{g}$ defines a complex extension on some neighbourhood of $V(f)$ in $T(M)$ and a section δ over $V(f)$ in $G(M)$. Then two sections α and δ coincide on $U' \cap V'$, hence $\alpha|U'$ and $\delta|V'$ define a section β on $U' \cup V'$ as required.

PROOF OF THEOREM 3. Since M is enumerable at infinity it admits a countable atlas $\{f_i^1\}_{i\in Z}$ such that each $V(f_i^1)$ is relatively compact and only intersects a finite number of others (8, page 4, theorem 1). This atlas then admits a refinement $\{f_i^\infty\}_{i\in Z^+}$ such that $\overline{V(f_i^\infty)} \subset V(f_i^1)$ for all i, and, in fact, we can find an atlas $\{f^\infty\}_{i\in Z^+}$ for each α of $Z^+ + \{\infty\}$ such that $\alpha > \alpha'$ implies $\overline{V(f_i^\alpha)} \subset V(f_i^{\alpha'})$ for all i. A section γ_i of $G(M)$ over $V(f_i^1)$ is given by the germ of complex extension structure defined by the tangent bundle extension of f_i^1. By the lemma, this can be extended to a section γ_2 over $V(f_1^2) \cup V(f_2^2)$. By repeated use of the lemma, we obtain a section γ over $\bigcup_{i\in Z^+} V(f_i^\infty) = M$. The required result is given by theorem 1.

Remarks. The tangent bundle is not indispensible in this proof of the existence theorem. If A is an analytic manifold containing M as closed, proper, analytic, sub-manifold and if the normal bundle, realised by an analytic field of transversal vector planes, is analytically isomorphic to the tangent bundle, then the atlas of A has a subatlas compatible with Γ and the proof works as before. In particular the product $M \times M$ with the diagonal $\Delta \cong M$ as submanifold satisfies the conditions (compare with 3 p. 417).

What is not apparent from the above proof is whether a manifold A with the above conditions admits a neighbourhood of M carrying a complex extension structure subordinate to its analytic structure. This is, in fact, the case, but it would seem to be a much deeper result for it is a consequence of the Grauert-Morrey embedding theorem. We can prove the following proposition which is more general :

PROPOSITION 4. *If M is an analytic proper sub-manifold of an analytic manifold A and is enumerable at infinity, then a neighbourhood of M in A is analytically isomorphic to a neighbourhood of M in the normal bundle $N_A(M)$.*

PROOF. There is a neighbourhood of M in A which is enumerable at infinity and which can be analytically embedded in a euclidean space. Hence, this neighbourhood carries an analytic Riemannian metric. The tangent vectors of A which are normal to M give a metric realisation ot the normal bundle $N_A(M)$. To such a vector, associate a point of A on the geodesic arc commencing at the origin of the vector in the direction of the vector, the distance of the point from the origin being given by the magnitude of the vector. This gives an analytic mapping g of a neighbourhood of M in $N_A(M)$ into A. Proposition 1 gives that the restriction of g to some neighbourhood of M is an isomorphism.

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