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Paul H. Rabinowitz

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NONLINEAR EIGENVALUE PROBLEMS
FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

by Paul H. RABINOWITZ

This talk represents in part joint work with A. Ambrosetti of the Scuola Normale Superiore, Pisa. For more details of the results below see [1] and [2].

Consider the nonlinear elliptic partial differential equations

(1) $$-\Delta u = \lambda(a(x)u + p(x,u)) \ , \ x \in \Omega \ ; \ u = 0 \ , \ x \in \delta\Omega$$

(2) $$-\Delta u = \lambda \ p(x,u) \ , \ x \in \Omega \ ; \ u = 0 \ , \ x \in \delta\Omega$$

where Ω is a smooth bounded domain in R^n, $a(x)$ is Holder continuous and positive in $\bar{\Omega}$, and $p(x,z)$ is smooth in its arguments. A solution of (1) ((2)) is a pair $(\lambda,u) \in R^+ \times C^2(\Omega)$ satisfying (1) ((2)). Our main interest is in obtaining lower bounds for the number of solutions of (1) and (2) as a function of λ.

Assume

(p_1) $$p(x,z) = o(|z|) \ \text{ at } \ z = 0$$

Then (1) and (2) possess the line of trivial solutions $\{(\lambda,0) | \lambda \in R^+\}$. Further assume

(p_2) $$p(x,z) = -p(-x,z)$$

Hence nontrivial solutions occur in antipodal pairs.

The structure of the set of solutions is governed by the behaviour of $p(x,z)$ at $z = 0$ and ∞. Indeed if p satisfies (p_1) - (p_2) and

(p_3) $$p(x,z)z^{-1} \to \infty \ \text{ as } \ z \to \infty \ ,$$

then modulo some additional conditions, for each $\lambda > 0$, (1) ((2)) possesses infinitely many distinct pairs of solutions. However if (p_3) is replaced by

(p_3^-) $$p(x,z)z^{-1} \to -\infty \ \text{ as } \ z \to \infty \ ,$$

then (1) ((2)) generally possesses only finitely many pairs of solutions, the number depending on the magnitude of λ.

To be more precise, some precise, some additional hypotheses are required.

(p_4) $$|p(x,z) \leq a_1 + a_2 |z|^s \ \text{ where } \ 1 < s < \frac{n+2}{n-2}$$

(p_5) $$P(x,z) \leq \theta \ z \ p(x,z) \ \text{ for } \ |z| \geq a_3 \ \text{ where } \ 0 \in [0,\tfrac{1}{2}] \ \text{ and}$$

$$P(x,z) = \int_0^z p(x,t)dt$$

In (p_4), (p_5) and below a_1, a_2 etc. denote positive constants. Condition (p_4) can be weakened if $n = 2$ and eliminated if $n = 1$. In fact our results can be considerably unproved for $n = 1$ by using other arguments. However we shall not discuss this case here. (p_5) is satisfied if e.g. $p(x,z)$ is the sum of a pure power and a lower order term at ∞.

THEOREM 3. <u>If</u> p <u>satisfies</u> $(p_1) - (p_5)$, <u>then for each</u> $\lambda > 0$, (1) ((2)) <u>possesses infinitely many distinct pairs of solutions.</u>

<u>Remarks on the proof.</u> This theorem and our later results are proved by variational arguments. Let

(4) $$I(u) = \frac{1}{2} \int_\Omega |u|^2 dx - \lambda \int_\Omega P(x,u(x))dx$$

and $J(u) = I(u) - \frac{1}{2}\lambda \int_\Omega a(x)u^2 dx$. Solution of (2) (resp. (1)) are obtained as critical points of I (resp. J). These critical points are characterized as the minimax of I over appropriate subsets of $E \equiv W_0^{1,2}(\Omega)$. (Note that (p_4) implies I is defined on E). In this sense our Theorem is of Ljusternik-Schnirelman type. (See e.g. [3] for a bibliography of such work). However unlike the usual such results, we do not extremize an even functional on a sphere-like submanifold of E but on all of E .

The details of the proof can be found in [2]. Actually a general theorem on the existence of critical points for an even functional on a Banach space is proved then and applied to (1) and (2). See also [1] for an alternate existence proof for (1), (2) using a Galerkin argument.

<u>Remarks</u>

1. (1) and (2) differ in that bifurcation occurs for (1) but not for (2).

2. Note that $I(u)$ satisfying $(p_1) - (p_5)$ is matter bounded from above or below : $I(\alpha u) \to -\infty$ for fixed $u \in E$, $u \not\equiv 0$ as $\alpha \to \infty$ and it is easy to construct a sequence (u_m) along which $\frac{1}{2} a u_m^2 + P(u_m)$ is bounded but $\int_\Omega |\nabla u_m|^2 dx \to \infty$. In fact we show ([1], [2]) $I(.)$ possesses a sequence of critical values tending to ∞.

3. If (p_2) is dropped, (2) possesses at least a positive and a negative solution. Likewise for (1) provided that $\lambda \in (0,\lambda_1)$ where λ_1 is the smallest eigenvalue of $-\Delta u = \lambda a(x)u$, $x \in \Omega$; $u = 0$, $x \in \partial\Omega$.

4. The role of the growth conditions (p_4) is an interesting question which is not yet completely understood. These conditions are necessary in general as can be seen using an identity of Poliozaev [4] for solutions of

$$(5) \qquad -\Delta u = f(u) \ , \ x \in \Omega \ ; \quad u = 0 \ , \ x \in \partial\Omega$$

Let $F(z) = \int_0^z f(t)dt$. Then if u satisfies (5)

$$(6) \qquad 2n \int_\Omega F(u)dt + (2-n) \int_\Omega f(u)u \, dx = \int_{\partial\Omega} x.v(x)|\nabla u|^2 \, dS$$

where $v(x)$ is the outward pointing normal to $\partial\Omega$. In particular if $0 \in \Omega$ and Ω is starshaped, then $Xx.v(x) \geqq 0$ and (6) implies

$$(7) \qquad 2n \int_\Omega F(u)dx \geqq (n-2) \int_\Omega f(u)u \, dx$$

Suppose $\qquad f(u) = |u|^{m+1}$.

Then $\qquad \frac{2n}{m+1} \int_\Omega |u|^{m+1} dx \geqq (n-2) \int_\Omega |u|^{m+1} dx$.

Hence if u is a nontrivial solution of (5) , $\frac{2n}{m+1} \geqq n-2$ or $m+1 \leqq (n+2)(n-2)^{-1}$ and in fact (6) shows equality is not possible. Thus (p_4) is required in general. For example if $n = 3$ and $0 \in \Omega$, a starshaped domain, the equation

$$(8) \qquad -\Delta u = \lambda u^2 \ , \quad x \in \Omega \ ; \ u = 0 \ , \quad x \in \partial\Omega$$

possesses no nontrivial solutions. On the other hand, if (8) is replaced by

$$(9) \qquad -\Delta u = \lambda(u+u^2)$$

degree theoretic arguments show that (9) possesses a component C of nontrivial solutions such that $(\lambda,u) \in C$ implies $\lambda \in (0,\lambda_1)$ and $u > 0$ in Ω (λ_1 being as in Remark 3). Moreover C is unbounded in $\mathbb{R} \times C^2(\Omega)$. Curiously if (7) is applied to (9), we find

$$(10) \qquad \int_\Omega u^8 dx \leqq 8\int_\Omega u^2 \, dx$$

and this easily implies a priori bounds for solutions of (9) in $L^8(\Omega) \cap E$. This suggests the projections of C on $\mathbb{R}$ may contain $(0,\lambda_1)$ (which is the case for $n = 1$). However this does not occur here. Indeed

$$(11) \qquad \lambda_1 \int_\Omega u^2 dx \leqq \int_\Omega |\nabla u|^2 dx = \lambda \int_\Omega (u^2+u^8)dx \leqq 9\lambda \int_\Omega u^2 dx$$

via (9), (10) and the Poincaré inequality. Hence $\lambda \in [\frac{\lambda_1}{9}, \lambda_1]$. More information on such phenomena for (1) and (2) can be found in [1].

Next we will show how a rather different situation from Theorem 3 occurs if (p_3) is replaced by (p_3^-). Actually we only require the weaker condition

(p_6) There exists a $\bar{z} > 0$ such that $a(x)\,\bar{z}+p(x,\bar{z}) < 0$.

This implies a priori bounds for solution of (1). To see this suppose (λ,u) is a nontrivial solution of (1). Either u has a positive maximum or a negative minimum. Both cases are handled similarly. If the first case occurred at $\bar{x}\in\Omega$, from (1) ,

$$0 \leqq -\Delta u(\bar{x}) = \lambda(a(\bar{x})u(\bar{x}) + p(\bar{x},u(\bar{x})))$$

Hence by (p_4), $u(\bar{x}) < \bar{z}$. Using (1) and e.g. the Schaunder estimates, we then get a priori bounds (depending on λ) for solutions in $R^+ \times C^2(\Omega)$. With the aid of these bounds, we can do without (p_4) - (p_5). To state our own results for this case precisely, let $0 < \lambda_1 < \lambda_2 \leqq \dots$ denote the eigenvalue of

$$-\Delta u = \lambda\, a(x)u \ , \ x\in\Omega \ ; \ u = 0 \ , \ x\in\partial\Omega \ . \tag{12}$$

As is well known, the (λn) are of finite multiplicity and form an unbounded sequence.

THEOREM 13. <u>If</u> $\lambda\in(\lambda_k,\lambda_{n+1})$ <u>and</u> p <u>satisfies</u> (p_1), (p_2), (p_6), <u>there exists at least</u> k <u>distinct pairs of solutions of</u> (1).

To treat (2), we replace (p_6) by

(p_7) $z\,p(x,z) > 0$ in a deleted neighbourhood of $z = 0$.

(p_8) There exists a $\bar{z} > 0$ such that $p(x,\bar{z}) < 0$.

Then (p_8) given a priori bounds for solution as above

THEOREM 14. <u>If</u> p <u>satisfies</u> (p_1) - (p_2),(p_7) - (p_8), <u>then for each</u> $k\in N$, <u>there exists a</u> $\underline{\lambda}_k > 0$ <u>such that for</u> $\lambda > \underline{\lambda}_k$, (2) <u>possesses at least</u> k <u>distinct pairs of solutions</u> $\bar{u}_j(\lambda)$, $\pm\,\underline{u}_j(\lambda)$, $1 \leqq j \leqq k$, <u>with</u> $I(\bar{u}_j) > 0 > I(\underline{u}_j)$.

<u>Remark</u>. The requirement that $\underline{\lambda}_k > 0$ is necessary since it is not difficult to show that (2) possesses only the trivial solution for λ small under the above hypotheses.

BIBLIOGRAPHY

[1] P.H. RABINOWITZ, Variational methods for nonlinear elliptic eigenvalue problems, to appear, Indiana U. Math. J.

[2] A. AMBROSETTI and P.H. RABINOWITZ, Dual variational methods in critical point theory and applications, to appear, J. Functional Anal.

[3] H. AMANN, Insternik-Schnirelman theory and nonlinear eigenvalue problems, Math. Ann. 199, (1972) pp. 55-72.

[4] S.I. POLIOZAEV, Eigenfunctions of the equation $\Delta u+\lambda f(u) = 0$, Sov. Math. 5 (1965) pp. 1408-1411.

[5] P.H. RABINOWITZ, Some global results for nonlinear eigenvalue problems, J. Functional Anal. 7 (1971) pp. 487-513.