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THE INITIAL VALUE PROBLEM FOR CLOSED UNIVERSES

by Dieter R. BRILL

1. Boundary conditions without boundary conditions.

In previous papers ([2], [4], [5]), everywhere regular, asymptotically Schwarzschildian solutions of the "vacuum" Einstein equations were discussed from the point of view of the initial value problem. The physical meaning of the boundary condition of asymptotically Schwarzschildian behavior is not clear from the point of view of Mach"s principle ([11], [12]), because it is not clear how the preferred "non-rotating" coordinate system is determined. The alternative boundary condition considered here, that the universe be closed in space, does not introduce any such preferred coordinate system. There is some hope that a condition of this type may be a first step toward a rigorous formulation of Mach's principle ([6], 3rd edition p. 117).

A discussion of closed universe solutions of the vacuum Einstein equations is also interesting because only one set of equations has to be solved, and no phenomenological description of matter is needed. Problems like that of stability may be most easily treated in a universe of pure gravitation, and the results for this case will throw some light on the more complicated situation when other fields are also present.

2. Universes with three-sphere topology.

In reference [4] it was shown that the time-symmetric initial value equation for an axially symmetric metric on an initial hypersurface,

(1) $$ds^2 = \Psi^4[e^{2\lambda q}(d\rho^2 + dz^2) + \rho^2 \, d\varphi^2]$$

takes on the form

(2a) $$\nabla^2 \Psi + \lambda\Phi\Psi = 0 \quad \text{(3-dimensional Schrödinger equation)}$$

(2b) $$\Phi = (\partial^2 q/\partial\rho^2) + (\partial^2 q/\partial z^2) \quad \text{(definition of "Schrödinger potential" } \Phi\text{)}.$$

As shown in references [4] and [5], by choosing the parameter λ between zero and a "critical" value λ_c for a fixed q one can obtain solutions of (2) for

Ψ which correspond to a concentration of gravitational waves of various total masses between zero and infinity.

The value λ_c is the lowest eigenvalue of (2a). The corresponding eigenfunction satisfies the boundary condition $\Psi_c \to 0(1/r)$ for $r = (\rho^2 + z^2)^{1/2} \to \infty$. By evaluating the metric (1) near the origin of the new coordinate $r' = 1/r$ one can easily see that the "point at infinity" $r' = 0$ can be added to the manifold to complete it smoothly to a closed space with the topology of a three-sphere.

Physically this result can be interpreted thus : the arbitrary function q represents a degree of freedom of the gravitational waves, and Ψ is the "static potential" corresponding to the mass-enery in the waves. (It can be shown that there is a one-to-one correspondence between q , Ψ and Arnowitt, Deser and Misner's transverse-transverse [2] and "potential" part of the gravitational field, respectively). As the "strength" λ of the waves is increased, this mass increases until it is sufficient to curl up space into a closed universe.

TAUB [13] has given a class of explicit solutions of the vacuum Einstein equations which, by proper choice of parameters and periodicity of coordinates, describe a three-sphere on each hypersurface t = Cte . These solutions are explicit examples of the kind of universes closed up by their content of gravitational radiation that we are here discussing. Taub's universes are homogeneous, so that it may seem strange from a physical point of view to say that they contain gravitational radiation. However, it should be noted that the null-congruence associated with the eingenvectors of the Riemann tensors of these metrics has non-vanishing twist [8], so that no wavefronts or phase-surfaces could occur - a type of field that is not even understood as yet in electromagnetism.

3. Universes with other topologies.

To construct solutions of the time-symmetric initial value equation ${}^3R = 0$ one can again use Lichnerowicz's method (see, for example, [5]) : Represent the solution as $ds^2 = \Psi^4 \, d\sigma^2$, choose $d\sigma^2$ arbitrarily, and solve the equation

$$\nabla_\sigma^2 \Psi + (1/8) \, R_\sigma \Psi = 0 \tag{3}$$

for Ψ . For $d\sigma^2$ one may choose, for example, a closed, multiply-connected space. (3) will then in general have solutions only for a class of measure zero of "base metrics" $d\sigma^2$, typically for one metric out of each one-parameter class. This follows from the theorem of KODAIRA and SPENCER [9], that the eigenvalues of a continuous one-parameter family of strongly elliptic operators are continuous functions of the parameter.

Explicit examples of solutions of the time-symmetric initial value equations for "wormhole universes" of topology $S_1 \times S_2$ have been given by R. LINDQUIST [10]. Some of these solutions have a high degree of symmetry, so that various identifications can be made to obtain initial solutions with other topologies. In particular, one can obtain in this way a solution with the non-orientable topology $S_1 \times P_2$.

4. Non-trivial solutions excluded for certain topologies.

In general, closed spaces will not admit the trivial flat space as a solution ; for example, there is no way of defining a flat metric on a topological three-sphere. The class of closed spaces that do admit a flat metric is well known as the Clifford-Klein spaces [7]. It is interesting that the flat space seems to be the only solution of the initial value equations for these topologies. For example, by generalizing the method of ARAKI [1] one can show that the "total mass" contained in a three-torus must vanish, at least for small departures from flatness ; therefore the metric must be exactly the flat metric.

5. Dynamics of closed universes.

The detailed dynamics of closed universes can be very complicated, particularly if they contain "wormholes" or (gravitational) geons. It has been possible to gain some insight into the large-scale time development of closed universe from the moment of time-symmetry by studying the total volume $V = \int ({}^3g)^{1/2}\, d^3x$ of the subsequent space-like hypersurfaces. If coordinates are chosen such that $g_{00} = -1$, $g_{0i} = 0$, one finds

$$\partial V/\partial t = \partial^2 V/\partial t^2 = \partial^3 V/\partial t^3 = 0$$

$$\partial^4 V/\partial t^4 = -(1/12) \int R^{ij} R_{ij} ({}^3g)^{1/2}\, d^3x < 0 \quad .$$

Therefore a time-symmetric universe will always contract, never expand, from the moment of time-symmetry. This result confirms the theorem of AVEZ [3] about non-existence of periodic solutions for compact spaces : a closed universe cannot have more than one surface of time-symmetry.

Taub's explicit solution mentioned in section 2 in fact contracts until the volume vanishes, indicating a geometrical singularity. The following table compares the dimensionless life-time $\tau/R(0)$ of the Taub universe with that of two homogeneous and isotropic universe solutions (Here $R(0)$ is the maximum radius, defined from the maximum volume) :

Taub's time-symmetric universe $\tau/R(0)$ =	$0.49\ \pi$
Dust-filled Friedmann universe	$0.50\ \pi$
Radiation-filled universe	1 .

The two latter universes are not solutions of the <u>vacuum</u> Einstein equations, but they can be considered approximations to such solutions : universes filled with randomly distributed gravitational geons, and universes filled with randomly oriented gravitational radiation. This approximation will only hold for times small compared to the total lifetime, for originally small deviations from isotropy and homogeneity may become important in the late stages of contraction. Preliminary perturbation calculations on Taub's solution to first order in an inhomogeneity parameter indicate that the inhomogeneity will increase in time as $\log(\tau - t)$. This result would confirm Wheeler's conjecture ([14] , p. 128) that closed universe solutions will generally develop a singularity after a finite proper time.

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