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SOME TOPOLOGIES CONNECTED WITH LEBESGUE MEASURE

J. B. Walsh

One of the nettles flourishing in the nether regions of the field of continuous parameter processes is the quantity $\limsup_{s\to t} X_s$. When one most wants to use it, he can't show it is measurable. A theorem of Doob asserts that one can always modify the process slightly so that it becomes separable, in which case $\limsup_{s\to t} X_s$ is the same as its less prickly relative $\limsup_{s\to t, s\in D} X_s$, where D is a countable set. If, as sometimes happens, one is not free to change the process, the usual procedure is to use the countable limit anyway and hope that the process is continuous. Chung and Walsh [3] used the idea of an essential limit—that is a limit ignoring sets of Lebesgue measure zero— and found that it enjoyed the pleasant measurability and separability properties of the countable limit in addition to being translation invariant. Doob has recently generalized and improved this, showing that there is a large class of topologies on the line enjoying similar properties, called T-topologies. (T for Lebesgue, of course!)

The first three sections of the present article are an exposition of Doob's work, including some remarks on progressive measurability, which wasn't treated in the original. The last sections give some applications, first to potential theory and then to Markov chains.

The two main properties of these topologies which are established here are first, that one can define separability relative to such a topology in a very natural way, and then every measurable process turns out to be separable, without benefit of standard modifications, and secondly, if X is a measurable process, then so is Y, where $Y_t = \limsup_{s\to t} X_s$, and in fact Y usually has better measurability properties than X.

Let T be a topology on the line R. If $A \subset R$, then A^T will denote the set of T-accumulation points of A. Lebesgue measure and Lebesgue outer measure will be denoted by m and m* respectively. We will be concerned with topologies T on R satisfying

(a) T is finer than the Euclidean topology;

(b) a set of Lebesgue measure zero has no accumulation points;

(c) for each $A \subset R$, $m(A - A^T) = 0$.

Such a topology will be called a <u>T-topology</u>. If not otherwise stated, topological notions will refer to the usual topology on the extended line. $\overline{A}$ for instance will mean the ordinary closure of A. For a real valued function f on R we will define the <u>T-cluster set</u> of f at a point t to be $\bigcap_N \overline{f(N)}$, where N ranges over all deleted neighborhoods of t; we will denote this set by $C_T(f;t)$. We define also

$$f^T(t) = \text{T-}\limsup_{s \to t} f(s) = \max C_T(f;t) \qquad \text{and}$$

$$f_T(t) = \text{T-}\liminf_{s \to t} f(s) = \min C_T(f;t) .$$

Then we have $f_T \leq f \leq f^T$ a.e. (m); for $f_T \leq f^T$ everywhere by definition, and for each $b > 0$, the set $\{t: f(t) \geq f^T(t)+b\}$ can have no accumulation points, and hence is of zero Lebesgue measure by (c). Thus $f \leq f^T$ a.e. and similarly $f \geq f_T$ a.e.

There are many T-topologies. One of the most useful is the <u>essential topology</u> L. An L-neighborhood of t is just an ordinary neighborhood minus a set of Lebesgue measure zero. A point t is an accumulation point for a set A in this topology iff $m^*(A \cap N) > 0$ for each neighborhood N of t. One can easily verify that

(i) A is L-open iff $A = O - Q$, where O is open and $m(Q) = 0$;

(ii) L is the coarsest T-topology;

(iii) the L-Borel field is exactly the Lebesgue measurable sets.

Here, (iii) is a direct consequence of (i). The essential topology has some interesting connections with the ordinary topology:

(iv) if $A \subset R$ then A^L is closed and perfect in the ordinary topology.

To see this, let t be in the closure of A^L. If N is a neighborhood of t there exists $s \in N \cap A^L$. N is also an L-neighborhood of s so $m^*(N \cap A) > 0$. Thus $t \in A^L$. Further, $m^*(N \cap A^L) > 0$ by (c), so in particular N contains points of A^L other than t.

It can be shown that this property characterizes L among T-topologies. A direct consequence of (iv) is that for a function f

(v) f^L is upper semicontinuous in the ordinary topology;

(vi) f is continuous iff f is L-continuous.

Here, (v) follows by writing $\{ f^L < a \}$ as the complement of $\bigcap_n \{ f \geq a-1/n \}^L$, which is closed by (iv). If f is continuous, it is clearly L-continuous, and if f is L-continuous, then $f = f^L = f_L$, so f is both lower and upper semicontinuous, hence continuous.

Two other examples of T-topologies are afforded by the right and left L-topologies, respectively denoted L_r and L_l. Neighborhoods of t are of the form $N \cap [t,\infty)$ and $N \cap (-\infty,t]$ respectively, where N is an L-neighborhood of t. Results quite similar to the above hold for these topologies. Quite a different example is given by the topology T_d: a point t is a T_d-limit point of a set A if $\lim_{b \downarrow 0} 1/2b \; m^* \{ A \cap (t-b,t+b) \} > 0$. It is easily verified that T_d is in fact a T-topology. T_d is called the <u>approximate topology</u>, and it is classical that a Lebesgue measurable function is approximately continuous almost everywhere.

2. MEASURABLE PROCESSES

Let $(\Omega,\underline{\underline{F}},P)$ be a probability space. Let $\underline{\underline{B}}$ denote the Borel sets and consider the product space $R \times \Omega$. We will denote the completion of $\underline{\underline{B}} \times \underline{\underline{F}}$ with respect to $m \times P$ by $\overline{\underline{\underline{B}} \times \underline{\underline{F}}}$. A function X on $R \times \Omega$ to R is <u>Borel measurable</u> if it is $\underline{\underline{B}} \times \underline{\underline{F}}$ measurable, and is <u>Lebesgue</u> measurable if it is $\overline{\underline{\underline{B}} \times \underline{\underline{F}}}$ measurable. An additional property shared by some but not all of the T-topologies is

(d) If $(t,\omega) \to X(t,\omega)$ is Borel measurable, then so is $(t,\omega) \to X^T(t,\omega)$.

Here as usual $X^T(t,\omega) = T\text{-}\limsup_{s \to t} X(s,\omega)$. Note that (d) is not satisfied by the ordinary topology! We say that two functions X and Y on $R \times \Omega$ are <u>indistinguishable</u> if for a.e. (P) ω, $X(t,\omega) = Y(t,\omega)$, all t.

<u>PROPOSITION 2.1</u> Let T be a T-topology which also satisfies (d) and let X be a Lebesgue measurable process. Then X^T is indistinguishable from a Borel measurable process.

<u>PROOF.</u> There exists a Borel measurable Y such that $Y = X$ $m \times P$ a.e. By Fubini's theorem, for a.e. ω $m\{t: X(t,\omega) \neq Y(t,\omega)\} = 0$. By (b), then, for all such ω, $X^T(t,\omega) = Y^T(t,\omega)$ for all t. By (d) Y^T is Borel measurable. qed

<u>PROPOSITION 2.2</u> The topologies L and T_d along with their right and left hand topologies satisfy (d).

<u>PROOF</u>. Suppose first that X is the indicator function of some set in $\underline{\underline{B}} \times \underline{\underline{F}}$. Notice that in that case

$$X^L(t,\omega) = \lim_{a \to 0} \left[\lim_{n \to \infty} \left[\int_{t-a}^{t+a} X(s,\omega)ds \right]^{1/n} \right]$$

and

$$X^{T_d}(t,\omega) = \lim_{n \to \infty} \left[\limsup_{a \to 0} 1/2a \int_{t-a}^{t+a} X(s,\omega)ds \right]^{1/n} .$$

These are both $\underline{\underline{B}} \times \underline{\underline{F}}$ measurable by Fubini's theorem. For general $\underline{\underline{B}} \times \underline{\underline{F}}$ measurable X, notice that for any real number b and $T = L$ or $T = T_d$:

$$I_{\{X^L \geq b\}} = \lim_{n \to \infty} \left(I_{\{X \geq b-1/n\}} \right) \text{ , which is } \underline{\underline{B}} \times \underline{\underline{F}} \text{ measurable. qed.}$$

The proof for the respective right and left hand topologies is entirely similar and is left to the reader. qed

What one often needs in applications is progressive measurability rather than just measurability. Let $(\underline{\underline{F}}_t)_{t \in R}$ be an increasing set of Borel sub fields of $\underline{\underline{F}}$ and let $\underline{\underline{B}}_a$ be the class of Borel sets of $(-\infty, a)$. Then a function X on $R \times \Omega$ is said to be <u>progressively Borel measurable</u> if for each $a \in R$ the restriction X^a of X to $(-\infty, a) \times \Omega$ (here it is important to take the open interval) is $\underline{\underline{B}}_a \times \underline{\underline{F}}_a$ measurable and is <u>progressively Lebesgue measurable</u> if X^a is $\overline{\underline{\underline{B}}_a \times \underline{\underline{F}}_a}$ measurable, the completion being again with respect to $m \times P$. (We are ignoring for the moment the question of whether or not X is adapted to $(\underline{\underline{F}}_t)$, i.e. whether or not $X(t,.)$ is $\underline{\underline{F}}_t$ measurable for each fixed t.

<u>LEMMA 2.3</u> If X is progressively Lebesgue measurable there exists a progressively Borel measurable process $\hat{X}$ such that $\hat{X} = X$ $m \times P$–a.e.

<u>PROOF</u> For each integer $n \geq 1$ and each integer k there exists a function Y_{kn} on $(-\infty, k/2^n) \times \Omega$ which is $\underline{\underline{B}}_{k/2^n} \times \underline{\underline{F}}_{k/2^n}$ measurable and is equal $m \times P$–a.e. to $X^{k/2^n}$. Define Y_n on $R \times \Omega$ by $Y_n = Y_{kn}$ on $[k-1/2^n, k/2^n) \times \Omega$, and finally define $\hat{X} = \liminf_{n \to \infty} Y_n$. Now $\hat{X}$ plainly equals X $m \times P$–a.e. To see that $\hat{X}$ is progressively Borel measurable, note that if $a = k/2^n$, then for each $m \geq n$ and $j \leq k$, Y_{jm} is $\underline{\underline{B}}_a \times \underline{\underline{F}}_a$ measurable. Thus the same is true of Y_m^a and hence of $\hat{X}^a$. For a general a, just apply this remark to a sequence of dyadic rationals which increase to a. qed

<u>THEOREM 2.4</u> Let T be a T–topology which satisfies (d). If X is progressively Borel measurable, so is X^T ; if X is progressively Lebesgue measurable, X^T is indistinguishable from a progressively Borel measurable process. In the first case X^T is adapted to $(\underline{\underline{F}}_{t+})$ and in the second it is indistinguishable from an adapted process.

<u>PROOF</u> If for all a, X^a is $\underline{\underline{B}}_a \times \underline{\underline{F}}_a$ measurable, then by (d), so is $(X^a)^T$. But $(X^a)^T = (X^T)^a$, for X^a is the restriction of X to the <u>open</u> interval $(-\infty, a)$, so X^T is also progressively Borel measurable. If X is only progressively Lebesgue measurable, it is equal $m \times P$–a.e. to a progressively Borel measurable $\hat{X}$. We have just shown that $\hat{X}^T$ is progressively Borel measurable, and the argument in the proof of Proposition 2.1 shows $\hat{X}^T$ and X^T are indistinguishable.

The statements on adaptability of X^T are consequences of the general fact that any process progressively Borel measurable with respect to $(\underline{\underline{F}}_t)$ is adapted to $(\underline{\underline{F}}_{t+})$. Indeed, for any $b > a$, $X(a,.) = X^b(a,.)$, while $X^b(a,.)$ is $\underline{\underline{F}}_b$ measurable by Fubini's theorem.
qed

The reason for insisting on the transition from progressively Lebesgue to progressively Borel is that the important stopping theorems—measurability of $X(S)$ for a stopping time S, for instance—valid for progressively Borel processes are false for progressively Lebesgue processes.

3. SEPARABILITY

Suppose now that $\{X_t, t \in R\}$ is a stochastic process. If $D \subset R$, $X|_D$ will denote $\{X_t, t \in D\}$. According to a classical theorem of Doob, there exists a countable set D and a standard modification $\hat{X}$ of X such that for a.e. ω, $\hat{X}_t$ is in the cluster set of $\hat{X}|_D$ at t for all $t \in R$. If X is progressively Lebesgue or Borel measurable, one can choose $\hat{X}$ to be progressively Lebesgue or Borel measurable respectively. Recalling our notation for cluster sets, separability for X implies

(3.1) $\quad C_0(X;t) = C_0(X|_D;t) \quad$ and $\quad C_{0_r}(X;t) = C_{0_r}(X|_D;t)$,

where 0 and 0_r are the Euclidean and Euclidean right hand topologies respectively. We will take (3.1) as a model for a definition of T-separability. Let T be a T-topology.

DEFINITION A real valued stochastic process $\{X_t, t \in R\}$ is <u>T-separable</u> (<u>right T-separable</u>) if there exists a countable set $D \subset R$, called a <u>T-separability set</u> (<u>right T-separability set</u>) and $K \subset \Omega$ with $P(K) = 0$ such that if $\omega \notin K$ then

(3.2) $\quad C_T(X;t) \subset C_0(X|_D;t) \quad$ (resp. $C_{T_r}(X;t) \subset C_{0_r}(X|_D;t)$) for all t.

X is <u>strongly T-separable</u> (strongly right T-separable) and D is a strong (right) T-separability set if there is equality in (3.2).

<u>REMARKS</u> 1. If X and $\hat{X}$ are Lebesgue measurable standard modifications of each other, then wp1 they have exactly the same T-cluster sets for any T-topology T. This follows since by Fubini's theorem, w.p.1 $m(t: X_t \neq \hat{X}_t) = 0$, hence by (b) the set $\{t: X_t \neq \hat{X}_t\}$ has no T-accumulation points.

2. A super-set of a separability set is a separability set but a super-set of a strong separability set may not be a strong separability set.

<u>THEOREM 3.1</u> Let T be a T-topology and let X be a Lebesgue measurable process. Then X is T-separable and right T-separable. X is also strongly L-separable and strongly right L-separable.

<u>PROOF</u> We will prove separability only; the same proof works for right separability. Fix for the moment a set $N \subset R$ with $m(N) = 0$. Let $\hat{X}$ be a Lebesgue measurable standard modification of X_{R-N} and let D be its separability set. Then w.p.1 we have for all t:

$$C_T(X;t) = C_T(\hat{X};t) \subset C_0(\hat{X}|_D;t) = C_0(X|_D;t) ,$$

where the first equality follows from remark 1, the second by separability and the last because, as D is countable, $X|_D$ and $\hat{X}|_D$ are indistinguishable. Thus X is T-separable.

Now we specialize to the essential topology. So far the set N has played no role, but now we choose it as follows. Let g be continuous on the compact interval $[-\infty,\infty]$. For each ω, $g(X_t(\omega)) \leq g(X_t(\omega))^L$ m-a.e. Since L satisfies (d), both sides are Lebesgue measurable so by Fubini's theorem there is an m-null set $N_g \subset R$ such that if $t \notin N_g$, then $g(X_t) \leq g(X_t)^L$ w.p.1. Choose $\{g_n\}$ dense in $C[-\infty,\infty]$ and let $N = \cup N_{g_n}$. With this N, let D be the separability set above. As $D \cap N = \emptyset$ by definition, for each $s \in D$ $g(X_s) \leq (g(X_s))^L$. This is true w.p.1 simultaneously for all s in D so, noting that $s \to g(X_s)^L$ is upper semicontinuous by (v),

$$\limsup_{\substack{s \to t \\ s \in D}} g(X_s) \le g(X_t)^L .$$

The opposite inequality follows since D is an L separability set. Thus there is equality simultaneously for all $t \in R$ and $g \in C[-\infty,\infty]$. But this implies strong separability since if a is a limit point of X^D_D at t, choose $g \in C[-\infty,\infty]$ to have a strict maximum 1 at a. Then

$$1 = \limsup_{\substack{s \to t \\ s \in D}} g(X_s) = g(X_t)^L ,$$

so a is in the L-cluster set of X at t. qed

We had no scruples about using the strongest form of Doob's separability theorem above, even though it is decidedly deeper than the theorem in question. For a demonstration more or less by hand of a similar theorem the reader may glance at §5 of [3].

We should warn the reader that we have given a slightly stronger definition of T-separability than Doob. With his definition, all Lebesgue measurable processes are strongly separable relative to any T satisfying (d). With our definition, we have the curious fact that this is true <u>only</u> for $T = L$. Indeed, if T is different from L there will be some Borel set $A \subset R$ for which $A^T \neq A^L$. As L is coarser than T, there exists t_o in $A^L - A^T$. Let f be the indicator function of A. Note that $f^T(t_o) = 0$ while $f^L(t_o) = 1$. But $f^T = f^L$ m-a.e. by (c), so t_o is an accumulation point of $\{f^T = 1\}$. By T-separability, it is also an accumulation point of $\{f = 1\} \cap D$, where D is any T-separability set. Thus the T-cluster set of f at t_o is $\{0\}$, while the cluster set of $f|_D$ at t_o also contains 1 . It follows that if Ω is a probability space, the stochastic process $X_t(\omega) = f(t)$ is T-separable but not strongly so.

4. APPLICATIONS TO POTENTIAL THEORY

One of the virtues of the limits we have been discussing is that they often exist where ordinary limits do not. For instance, nothing can be said about the existence of ordinary limits for a Lebesgue measurable submartingale $\{X_t, \underline{\underline{F}}_t, t \geq 0\}$ unless it is separable. On the other hand it has right and left L-limits (and therefore limits in any right or left T-topology) at all $t > 0$ w.p.1. This is a simple consequence of L-separability and the existence of one-sided limits along any countable parameter set.

Another use of these limits is to sidestep thorny measurability problems. Let $(\Omega, \underline{\underline{F}}, P)$ be a probability space and $(\underline{\underline{F}}_t)_{t \geq 0}$ an increasing family of Borel subfields of $\underline{\underline{F}}$. Let $\{X_t, t \geq 0\}$ be a progressively Borel measurable stochastic process taking values in a locally compact metric space E . Define a complete measure ν on E by by

$$\nu(A) = E\left\{\int_0^\infty e^{-t} I_A(X_t)\, dt\right\} \qquad \text{for Borel sets } A.$$

Then, if f is ν-measurable, the process $f(X_t)$ is progressively Lebesgue measurable: for there exist Borel measurable functions f' and f" on E for which $\nu\{f' \neq f''\} = 0$ and $f' \leq f \leq f''$; then the set $\{(s,\omega): f(X_s(\omega)) \neq f''(X_s(\omega)); s \leq t\}$ is both $\underline{\underline{B}}_t \times \underline{\underline{F}}_t$ measurable and of $m \times P$-measure null. It follows by (d) That $f(X_t)^L$ and $f(X_t)^{Lr}$ are indistinguishable from progressively Borel measurable processes.

Now let us specialize this situation. Suppose, in the notation of [1], that $X = (\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, X_t, \theta_t, P^x)$ is a Hunt process, except that we will not assume quasi left **continuity** of the paths. Measurability notions become more complex in this setting because of the presence of more than one measure on the probability space; in fact, for each probability measure μ on E there is a probability measure P^μ on $(\Omega, \underline{\underline{F}})$. Recall that $\underline{\underline{F}}_t^0$ is the Borel field generated by $\{X_s, s \leq t\}$ and $\underline{\underline{F}}^0 = \bigvee_t \underline{\underline{F}}_t^0$. Then $\underline{\underline{F}}$ and $\underline{\underline{F}}_t$ are gotten by completing $\underline{\underline{F}}^0$ and $\underline{\underline{F}}_t^0$ respectively according to the prescription in [1]. Notions of measurability in the following will be with respect to the fields $\underline{\underline{F}}$ and $\underline{\underline{F}}_t$ unless specifically stated otherwise.

We will say that a process Z is <u>(progressively) Lebesgue measurable</u> if for each probability measure μ on E, Z is (progressively) Lebesgue measurable on the space $(\Omega, \underline{\underline{F}}, P^{\mu})$. The idea of progressive Borel measurability is independent of measure, but it is convenient to introduce another notion: a process Z is <u>nearly (progressively) Borel measurable</u> if for each probability measure μ on E there exists a (progressively) Borel measurable process Z^{μ} such that Z and Z^{μ} are indistinguishable when the probability space is given the measure P^{μ}. With these notions it is clear that the analogue of Theorem 2.4 is :

<u>THEOREM 2.4'</u> Let T be a T-topology which satisfies (d). If Z is progressively Lebesgue measurable and real valued, then Z^T is nearly progressively Borel measurable and adapted.

This theorem follows upon applying the remark of the previous paragraph separately for each P^{μ} The pleasant properties of progressive Borel measurability carry over to nearly progressively Borel measurability.

<u>THEOREM 4.1</u> Suppose $Z = \{Z_t, t \geq 0\}$ is a nearly progressively Borel measurable process taking values in some separable metric space G, and S is a stopping time (always with respect to $(\underline{\underline{F}}_t)$.) Then Z_S is $\underline{\underline{F}}_S$-measurable. Furthermore, if A is an analytic set in G and $S_A = \inf\{t > 0: Z_t \in A\}$, then S_A is a stopping time.

<u>PROOF</u> Fix μ and let Z^{μ} be a progressively Borel measurable process which is P^{μ}-indistinguishable from Z. If $\underline{\underline{H}} \subset \underline{\underline{F}}$, $\underline{\underline{H}}^{\mu}$ will designate the Borel field generated by $\underline{\underline{H}}$ plus all the P^{μ}-null sets of $\underline{\underline{F}}$. By T49 of [5], Z_S^{μ} is $\underline{\underline{F}}_S$ measurable and equal to Z_S P^{μ}-a.e. Thus Z_S is measurable with respect to $\bigcap_{\mu} \underline{\underline{F}}_S^{\mu} = \underline{\underline{F}}_S$. On the other hand, according to Theorem T52 of [5], the random variable $S_A^{\mu} = \inf\{t > 0: Z_t^{\mu} \in A\}$ is a stopping time, equal to S_A P^{μ}-a.e. Thus the set $\{S_A < t\}$ is in $\underline{\underline{F}}_t^{\mu}$, hence in $\underline{\underline{F}}_t = \bigcap_{\mu} \underline{\underline{F}}_t^{\mu}$. qed

Hunt's potential theory for strong Markov processes is based on the first hitting time of a set, but one can also base a potential theory on the idea of a first penetration time, as has been done for instance by Stroock [6]. The <u>first penetration time</u> π_A of a set $A \subset E$ is the infimum of times t for which $m^*\{s \leq t : X_s \in A\} > 0$, or, in our notation

$$(4.1) \qquad \pi_A = \inf\left\{ t > 0 : (I_A \circ X_t)^{Lr} = 1 \right\} .$$

If $A \subset E$ is universally measurable, or just ${}_{\mu}R_1$-measurable for each probability measure μ on E, where ${}_{\mu}R_1(B) = E^{\mu}\left\{\int_0^{\infty} e^{-t} I_A(X_t)\,dt\right\}$, then $I_A(X_t)^{Lr}$ will be nearly progressively Borel measurable and π_A will be a stopping time. Notice that penetration times are measurable for a much larger class of sets than are first hitting times, and their measurability is a rather trivial fact. Nevertheless, we will see shortly that penetration times are in reality a subclass of hitting times.

It is natural to define a topology F, which we shall call the <u>essentially fine topology</u>, which is connected with penetration times. A set A is open in this topology if for each $x \in A$ there exists a universally measurable subset $B \subset A$ such that $x \in B$, and $P^x\{\pi_{B^c} = 0\} = 0$. The set of F-cluster points of A is denoted by A^F. A closely related notion is that of essential regularity. A point x is essentially regular for A if for each universally measurable $B \subset A$, $P^x\{\pi_B = 0\} = 0$. In fact, the set A^r of essentially regular points coincides with A^F if the resolvent charges no point, and in general we have $x \in A^F \Leftrightarrow x \in [A - \{x\}]^r$.

It is easy to see that $A^F = \bigcap B^F$ and $A^r = \bigcap B^r$, where the intersections are taken over all universally measurable sets B which contain A. The closure of a set A in the essentially fine topology is given by $A \cup A^r = A \cup A^F$.

PROPOSITION 4.2 If $A \subset E$ is Borel, so are A^F and A^r. If A is universally measurable, both A^F and A^r are nearly Borel measurable.

PROOF Let $g(t,\omega) = I_A(X_t(\omega))$. If A is Borel, this is $\underline{\underline{B}} \times \underline{\underline{F}}^0$ measurable, hence so is $g(t,\omega)^{Lr}$. Then $\{g(0;\omega)^{Lr} = 1\} \in \underline{\underline{F}}^0$ so $x \to P^x\{g(0,\omega)^{Lr} = 1\}$ is Borel measurable. But A^r is exactly the set where this probability is one. If A is universally measurable and μ is a probability measure on E, there exist Borel B and C such that $B \subset A \subset C$ and $\mu R_1(C-B) = 0$. Then $B^r \subset A^r \subset C^r$ and B^r and C^r are Borel measurable. X spends almost no time in C–B w.p.1 (P^μ); we claim $C^r - B^r$ is polar for the process with initial distribution μ. If not there is a stopping time S for which $P^\mu\{X_S \in C^r - B^r\} > 0$ [5p. 162]. But a.s. on $\{X_S \in C^r - B^r\}$ we have $0 = \pi_C \circ \theta_S < \pi_B \circ \theta_S$, implying the process spends strictly positive time in C – B, a contradiction. The proof for A^F is similar except one uses the function $h(x,t,\omega) = I_{A-\{x\}}(X_t(\omega))$ instead of g. qed.

THEOREM 4.3 Let A be universally measurable. Then $\pi_A = \pi_{A^r} = S_{A^r}$ w.p.1, and $X_{\pi_A} \in A^r$ a.s. on $\{\pi_A < \infty\}$ (where S_B is the first hitting time of B.)

PROOF If S is a stopping time, w.p.1: $X_S \in A^r$ iff $I_A(X_S)^{Lr} = 1$; for $I_A(X_S) = 1 \Leftrightarrow \pi_A \circ \theta_S = 0$ which is–by the strong Markov property–equivalent to $P^{X_S}\{\pi_A = 0\} = 1$. $I_A(X_t)$ is right upper semicontinuous in t, so $I_A(X_{\pi_A}) = 1$, implying $X_{\pi_A} \in A^r$ a.e. on $\pi_A < \infty$. Applying the above remark to fixed times S and using Fubini's theorem, we can show that the sets $\{(s,\omega): X_s(\omega) \in A^r\}$ and $\{(s,\omega): I_A(X_s(\omega)) = 1\}$ differ by a set of $m \times P$ measure zero. Thus w.p.1 $\pi_{A^r} = \pi_A$, as both equal $\inf\{s: I_A(X_s)^{Lr} = 1\}$. Now consider S_{A^r}. By the section theorem [5p. 162], for $\epsilon > 0$ we can find a stopping time S between S_{A^r} and $S_{A^r} + \epsilon$ such that $P\{X_S \in A^r\} \geq P\{S_{A^r} < \infty\} - \epsilon$. But $\pi_A \circ \theta_S = 0$ a.e. on $\{X_S \in A^r\}$ by the strong Markov property. Thus $\pi_A \leq S_{A^r}$ w.p.1. This completes the proof since S_{A^r} is always less than or equal to π_{A^r}.

qed

COROLLARY Let $A \in E$. Then A^F and A^r are closed in both the fine and essentially fine topologies.

Proof It is enough to show fine closure since the fine topology is coarser than the essentially fine topology. We need only consider universally measurable sets. Suppose x is regular for the nearly Borel set A^F. If $x \in A^F$, we are done, so suppose it is not. Then there exists a stopping time $S < \epsilon$, such that $P^x\{X_S \in A^F\} > 1 - \epsilon$. But $X_S \in A^F \Longrightarrow P\{\pi_A \circ \theta_S = 0\} = 1$; as $\pi_A \leq S + \pi_A \circ \theta_S$ we have $P^x\{\pi_A = 0\} = 1$, or $x \in A^r$. But $X_S \in A^F \Longrightarrow X_S \neq x$, so by right continuity of the paths $S_{\{x\}} \circ \theta_S > 0$; so in fact $x \in A^F$, the desired contradiction. The proof for A^r is similar. qed

This result gives some insight into why the penetration times can be defined on a larger class of sets than hitting times: penetration times reduce to hitting times of nearly Borel finely closed sets. We should note in passing that much of the above can be obtained from the general theory of additive functionals, for A^r is exactly the fine support of the continuous additive functional $B_t = \int_0^t I_A(X_s)ds$.

It is amusing, if not particularly significant, that the essentially fine topology satisfies

a') it is finer than the fine topology;

b') a set of potential zero has no accumulation points;

c') if A is universally measurable, $A - A^r$ has potential zero.

Before leaving the subject we will give one more useful corollary.

THEOREM 4.4 Let f be a real valued universally measurable function which is continuous in the essentially fine topology. Then

i) f is nearly Borel measurable;

ii) f is fine continuous;

iii) $s \to f(X_s)$ is almost surely right continuous.

PROOF Write $\{x: f(x) \geq a\} = \bigcap_n \{x: f(x) \geq a - n^{-1}\}^r$. Each set in the intersection is nearly Borel and finely closed by the corollary. This proves i) and ii). The last part is now a well-known consequence, but we will give a short proof to illustrate the methods of this paper.

We may assume f is bounded. Set $Z_s = f(X_s)^{Lr} - f(X_s)_{Lr}$. Then Z is upper semicontinuous from the right and, according to Theorem 2.4', it is nearly progressively measurable. Thus $T = \inf\{s: Z_s \geq \epsilon\}$ is a stopping time and $Z_T \geq \epsilon$. The strong Markov property and essential fine continuity of f imply $f(X_T)^{Lr} = f(X_T)_{Lr} = f(X_T)$; so $f(X_t)^{Lr}$ is right continuous. It is therefore well measurable, as is X_s and hence $f(X_s)$. If $P\{f(X_s) \neq f(X_s)^{Lr}$ some $s\} > 0$, by T21 of [5] there is a stopping time S such that $P\{f(X_S) \neq f(X_S)^{Lr}\} > 0$, which again contradicts the essential fine continuity of f, hence f(X) is identical to the right continuous process $f(X)^{Lr}$.

Once it is known that essential fine continuity (at all points!) implies fine continuity the above theorem can be derived trivially from known results. However, its novelty is precisely that only essentially fine continuity is needed. Observe, for instance, how quickly one can derive Hunt's result that an excessive function is nearly Borel and right continuous along the sample paths. We need only check that an excessive function f is essentially finely continuous. Fix $x \in E$. Then $\{f(X_s), s \geq 0\}$ is a Lebesgue measurable super-martingale with values in $[0,\infty]$ and thus has an essential limit f_0 at $s = 0$; this limit is constant with P^x probability one by the Blumenthal zero-one law. But then, if $s > 0$

$$f(x) = E^x\{f(X_0)\} \geq E^x\{f_0\} \geq E^x\{f(X_s)\} = P_s f(X_s) .$$

Now let $s \to 0$; then $P_s f(x) \to f(x)$, implying $f_0 = f(x)$. Note that the above equation is valid even if some or all of the terms are infinite.

5. APPLICATIONS TO MARKOV CHAINS

Most of the interesting aspects—and pathologies—of the sample function behavior of continuous parameter Markov processes are already present in the relatively simple case of Markov chains. The sample functions can be everywhere discontinuous, for instance. One of the bothersome features of these processes is that in general there is no separable version of the process with values in the original space. One can compactify the space in some convenient manner and then find a separable version in the enlarged space, but then the process will spend some time (of Lebesgue measure zero) outside the original state space. This suggests the suitability of the topologies we have been discussing, for they have a tolerant astigmatism which allows them to overlook sets of measure zero. We will briefly discuss some of the sample function properties of a standard Markov chain from this viewpoint. Only the viewpoint has any claim to novelty; the theorems can all be found in sections II.4 - II.9 of [2].

Let $I = \{0,1,2,\dots\}$ and let $P_t = (p_{ij}(t))$, $i,j \in I$, $t \geq 0$, be a standard transition matrix, that is, a transition matrix satisfying $p_{ij}(t) \to \delta_{ij}$ as $t \to 0$. It is known that

(5.1) $p_{ij}(t)$ is continuous in t for $t \geq 0$, and $q_i = \lim_{t \quad 0} \frac{1 - p_{ii}(t)}{t}$ exists.

The state i is said to be <u>stable</u> if $q_i < \infty$, and <u>unstable</u> if $q_i = \infty$.

Let $\{X_t,\ t \geq 0\}$ be a Markov chain having transition matrix P. The continuity of the $p_{ij}(t)$ imply that X is stochastically continuous and thus [5] can be assumed to be Borel measurable.

<u>PROPOSITION 5.1</u> Any countable dense set is a strong L, L_r, and L_l separability set.

<u>PROOF</u> If S and S' are countable dense sets, and if $s \in S \cup S'$, stochastic continuity implies that X_s is a.s. in the cluster sets of

both $X|_S$ and $X|_{S'}$ at s. Thus the cluster sets of $X|_S$ and of $X|_{S'}$ are equal at all t w.p.1. If S is a strong separability set (and one exists by Theorem 3.1) then so is S'. The same remarks apply to the left and right cluster sets. qed

NOTE For f bounded on I we write $P_t f(i) = \sum_j p_{ij}(t) f(j)$ and $R_p f(i) = \int_0^\infty e^{-pt} P_t f(i) dt$. Then for $f \geq 0$, $\{R_p f(X_t),\ t \geq 0\}$ is a supermartingale and therefore has L_r and L_l limits at each point w.p.1. This immediately implies the following basic property of the paths.

PROPOSITION 5.2 If ω is not is some set N with $P(N) = 0$, then $t \to X_t(\omega)$ has at most one finite L_r and L_l cluster point at each t.

PROOF Let $f = I_j$ and note that, as the transition matrix is standard, $pR_p f \to f$ as $p \to \infty$. Given $i \neq j$, then, we can find p large enough so that $R_p f(i) < R_p f(j)$. Let $\Omega - N$ be the set on which $R_p f(X)$ has L_r and L_l limits, and just notice that existence of an L_r limit (L_l limit) at t implies that not both i and j can be in the L_r (L_l) cluster set of X at t. qed

For $i \in I$, define $S_i(\omega) = \{t: X_t(\omega) = i\}$. This set depends on the particular version of X chosen, but the sets S_i^L, S_i^{Lr}, and S^{Ll} are invariant under standard modification, up to a set of zero probability. Proposition 5.2 doesn't exclude the possibility of having ∞ as a cluster point at some, or even all t, but it does tell us that the sets S_i and S_j aren't too badly mixed up. If i and j are distinct, then $S_i^{Lr} \cap S_j^{Lr}$ and $S_i^{Ll} \cap S_j^{Ll}$ are empty and $S_i^L \cap S_j^L$ is discrete.

PROPOSITION 5.3 W.p.1, S_i, S_i^L, and S_i^{Lr} are identical up to sets of Lebesgue measure zero.

PROOF For any set A, $m(A^L \Delta A^{Lr}) = 0$ and $m(A - A^L) = 0$. It remains to show $m(S_i^{Lr} - S_i) = 0$. But, neglecting Lebesgue null sets:

$$S_i^{L_r} - S_i = S_i^{L_r} \cap \Big[\bigcup_{j \neq i} S_j\Big] \subset S_i^{L_r} \cap \Big[\bigcup_{j \neq i} S_j^{L_r}\Big]$$

which is empty by the remarks preceeding this proposition. qed

Notice that this proposition does not quite imply Theorem 3 p. 151 of [2], which states in our notation that $P\{t \in S_i^{T_{dr}} | X_t = i\} = 1$. However, remark that $m(S_i - S_i^{T_d}) = 0$ —property (c)—so that by Fubini's theorem this probability must be one for a.e. (m) t, and hence, as it is independent of t, for every t. As in §4, π_A denotes the first penetration time of A, and we write π_i instead of $\pi_{\{i\}}$.

<u>PROPOSITION 5.4</u> For $0 \leq q_i < \infty$, and $s,t \geq 0$

(5.2) $$P\{s \in S_i^{L_r} \,|\, X_s = i\} = 1$$

(5.3) $$P\{\pi_{I-\{i\}} \geq s+t \,|\, X_s = i\} = e^{-q_i t} .$$

<u>PROOF</u> (5.2) is a consequence of the remarks preceeding the proposition and the fact that L_r is coarser than T_{d_r}. Suppose $q_i < \infty$.

(5.4) $$P\{X_{s+2^{-n}kt} = i , \ k = 0,\dots,2^n \,|\, X_s = i\} = [p_{ii}(t2^{-n})]^{2^n}$$
$$= [1 - 2^{-n}q_i t + o(2^{-n})]^{2^n}$$

which converges to $e^{-q_i t}$ as $n \to \infty$. This implies

(5.5) $$P\{X_{s+rt} = i \text{ for all dyadic } r \in [0,1] \,|\, X_s = i\} = e^{-q_i t} .$$

As any countable dense set is a strong L_r separability set this gives (5.3). Note that strong separability is needed for this implication. If q_i is infinite, (5.4) remains valid since for any $N \geq 1$ and large enough n the right hand side of (5.4) is dominated by e^{-Nt}, and hence must be zero. qed

Sooner or later one usually wishes to make the process separable; this can be done in any compact separable Hausdorff space E containing I. The main advantage gained is commonly the availability of some form of the strong Markov property; with the right choice of E, one can actually find a strongly Markov standard modification of X_t. But even the simplest choice of E gives a useful form of it. Consider the process on the Alexandroff compactification $I \cup \{\infty\}$ of I:

$$X_t^+(\omega) = X_t(\omega)_{L_r}, \quad \text{or in other language}$$

$$= \operatorname{ess} \liminf_{s \downarrow t} X_s(\omega), \text{ taken in the extended reals.}$$

By strong L_r separability (the full force of strong separability is needed) this is indistinguishable from the $\liminf_{s \downarrow t} X_s(\omega)$ over rational s, which is the version given by Chung. This process is Borel measurable, even lower semi continuous, and is a standard modification of X by (5.2) and Proposition 5.2 . Any L separability set for S—which is to say any countable dense set—will be a separability set for X^+. Set $S_i^+(\omega) = \{t: X_t(\omega) = i\}$. We already know all about this set because:

PROPOSITION 5.5 S_i and $S_i^{L_r}$ are indistinguishable.

PROOF Suppose ω is in the set of probability one in which $S_i^{L_r} \cap S_j^{L_r} = \emptyset$ whenever $i \neq j$. Then $t \in S_i^{L_r} \Rightarrow (X_t)_{L_r} \leq i$. But if $(X_t)_{L_r} = j < i$, then we would have to have $t \in S_j^{L_r}$, which can't happen. Thus $t \in S_i^{L_r} \Rightarrow X_t^+ = i$. The opposite implication is obvious from the definitions of X^+ and $S_i^{L_r}$. qed

The process X^+ is not strongly Markov for it may have ∞ for a value and there are no transition probabilities for this point. Nevertheless X^+ does satisfy a restricted version of the strong Markov property. Effectively, it is strongly Markov except when it is at ∞. Let $\underline{\underline{F}}_t$ denote the Borel field generated by $\{X_s^+, s \leq t\}$.

PROPOSITION 5.6 Let T be a stopping time with the property that $X_T^+ \in I$ a.s. on $T < \infty$. Then X^+ satisfies the strong Markov property at T.

PROOF By remarks following the definition, $T < \infty \Rightarrow X_t^+$ is a cluster point of X_s^+, $s \downarrow T$, s rational. But $T < \infty \Rightarrow X_T^+ \in I$, so in fact we see X_T^+ is a limit from the right of rationals for which $X_s^+ = X_T^+$. Thus we can find a sequence $T_n \downarrow T$ of rational valued stopping times such that $P\{X_{T_n}^+ = X_T^+\} \to P\{T < \infty\}$. Fix $j \in I$ and $t > 0$. The rationals and the rationals translated by t are both separability sets hence $\{X_{T+t}^+ = j\} \supset \limsup \{X_{T_n+t}^+ = j\}$. Thus on $\{T < \infty\}$:

$$P\{X_{T+t}^+ = j \mid \underline{\underline{F}}_T\} \geq P\{\limsup\{X_{T_n+t}^+ = j\} \mid \underline{\underline{F}}_T\} ;$$

conditioning on $\underline{\underline{F}}_{T_n}$ we have by Fatou's lemma and the Markov property

$$\geq \limsup \; E\{p_{X_{T_n}^+ j}(t) \mid \underline{\underline{F}}_T\} .$$

By the choice of T_n, X_T^+ will be a limiting value of X_{T_n} a.s. so this is

$$\geq p_{X_T^+ j}(t) .$$

To see there is actually equality, note that a.s. on $\{T < \infty,\}$

$$1 \geq \sum_j P\{X_{T+t}^+ = j \mid \underline{\underline{F}}_T\} \geq \sum_j p_{X_T^+ j}(t) = 1 ,$$

which implies equality for all j. qed

Many results follow easily from this. For instance, for any stopping time T, the post-T process $\{X_{T+t}^+ , t \geq 0\}$ is again a Markov chain with tha same transition probabilities. To see this, apply proposition 5.6 to the times $T + t_n$, where t_n is chosen such that $X_{T+t_n} \in I$ a.s. on $\{T < \infty\}$ (by Fubini's theorem, a.e. t has this property) and let $t_n \downarrow 0$. We conclude with a final proposition which concludes our description of the sets $S_i^{L_r}$.

<u>PROPOSITION 5.7</u> $S_i^{L_r}$ is perfect in the right limit topology 0_r. If $q_i = \infty$, it is a.s. nowhere dense; if $q_i < \infty$, it is a.s. the union of left semi closed intervals, finite in each finite time interval.

<u>PROOF</u> For any set $A \subset R$, A^{L_r} is perfect in the right limit topology. $S_i^{L_r}$ being closed in 0_r, to show it is nowhere dense we need only show it contains no intervals. But it is $\underline{\underline{B}} \times \underline{\underline{F}}$ measurable, so if it contained an interval with positive probability, Fubini's theorem would tell us there is a t such that $P\{t \in \text{interior of } S_i^{L_r}\} > 0$. In fact, as S_i and $S_i^{L_r}$ differ by sets of Lebesgue measure 0, we can even require $P\{t \in S_i > 0\}$. But if $q_i = \infty$, $\pi_{I-\{i\}} \circ \theta_t = 0$ by Proposition 5.4, a contradiction.

Now suppose q_i is finite. We will prove the proposition for S_i^+, which is indistinguishable from $S_i^{L_r}$. Define

$$V_0 = 0$$
$$T_n = \inf\{t > V_n : X_t^+ = i\} \qquad n = 0,1,\ldots,$$
$$V_n = \inf\{t > T_n : X_t^+ \neq i\} \qquad n = 0,1,\ldots.$$

For simplicity we assume i is recurrent and $q_i > 0$ so that both T_n and V_n will be finite valued for all n. As S_i^+ is closed in 0_r, $X_{T_n}^+ \in S_i^+$ a.s. for all n, hence $X_{T_n}^+ = i$ and Proposition 5.6 is applicable. Using this and Proposition 5.4 we see $V_n - T_n$ is a sequence of independent exponential random variables with parameter q_i. Thus S_i^+ is a union of intervals, closed on the left, and, as $\sum(V_n - T_n) = \infty$, there can be at most finitely many in any finite time. These intervals must also be open on the right, for otherwise $P\{X_{V_n}^+ = i\} > 0$ for some n, implying by the strong Markov property that X_s^+ remains in i for some time after V_n, a contradiction. qed

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