Séminaire de probabilités (Strasbourg)

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Séminaire de probabilités (Strasbourg), tome 6 (1972), p. 215-232
<http://www.numdam.org/item?id=SPS_1972__6__215_0>

TRANSITION FUNCTIONS OF MARKOV PROCESSES

J. B. Walsh

The most logical way to define "Markov process" is to say that it is a stochastic process with the Markov property. This is the point of view taken by Doob in [3] . However for some of the more interesting applications – potential theory is an example – one wants to be able to talk about the process starting from any point x . One can always do this in the classical cases, but it is by no means clear that it is possible in general. Dynkin [4] suggested that one simply define a Markov process to be a family of stochastic processes, one for each possible initial distribution. This is generally accepted as the proper definition today, yet there are cases – in time reversal, for example – where one has processes which are Markovian in Doob's sense but not in Dynkin's.

The aim of this paper is to show that the two viewpoints are compatible : if one has a right continuous strong Markov process in Doob's sense, for example, one can always construct the family required by Dynkin. A theorem like this has been proved by Meyer in [8] for the reverse of a strong Markov process, and his methods may well extend to the general case. We use a different approach, which yields as a byproduct some information on the existence of 1-supermedian functions separating points.

Some of the material in the first three sections has been part of the folklore on the subject for several years. § 3, for instance, has often appeared in the form "We can suppose that the resolvent separates points; if not we can always reduce to a quotient space ..."

Let E be a Lusin space, that is a Borel subset of a compact metric space F . $\underline{\underline{E}}$ and $\underline{\underline{E}}_b$ will denote the Borel field of E and class of bounded $\underline{\underline{E}}$ -measurable functions. The space F appears below mainly as a furnisher of a "good" class of continuous functions on E . For example, a sequence ν_n of measures on E is said to converge vaguely to a measure ν on E if $\nu_n \to \nu$ vaguely in F .

We will abbreviate "right continuous with left limits" by r.c.l.l. and "left continuous with right limits" by l.c.r.l. The pronunciations may be a bit barbaric but this will save us a lot of writing. Throughout this paper X will represent either a r.c.l.l. strong Markov process or an l.c.r.l. moderate Markov process, that is, a Markov process in Doob's sense, with a given initial distribution. We will prove all theorems for both types of processes. When a statement differs for the two cases, we will write the statement for the moderate Markov process in parentheses, e.g. " X is a strong (resp.moderate) Markov process". We recall from [1] that $\{X_t, t \geq 0\}$ is a moderate Markov process relative to an increasing family of fields $(\underline{\underline{F}}_t)$ if for each $t > 0$ there exists a kernel P_t on E such that for each predictable time T and $f \in \underline{\underline{E}}_b$

$$(1) \qquad E\{f(X_{T+t}) \mid \underline{\underline{F}}_{T-}\} = P_t f(X_T) .$$

We say a family of kernels $(P_t)_{t \geq 0}$ on E is a transition function for a moderate Markov process X if

(T 1) $\forall t \geq 0$, $\forall x \in E$: $P_t(x, \cdot)$ is a probability measure on $\underline{\underline{E}}$.

(T 2) $\forall A \in \underline{\underline{E}}$: $(t,x) \to P_t(x,A)$ is $\underline{\underline{B}} \times \underline{\underline{E}}$ -measurable.

(T 3) (P_t) satisfies (1) .

The definition of a transition function for a strong Markov process is analogous. There is no guarantee that the family of kernels in (1) satisfies (T 2) , but according to a slight modification of [1 Thm. 3.2] we can do even better :

PROPOSITION 1

Let X be a r.c.l.l. strong Markov process (resp. l.c.r.l. moderate Markov process). Then X admits a transition function (P_t) such that for all x, $t \to P_t(x,\cdot)$ is vaguely r.c.l.l. (resp. l.c.r.l.)

The following result is due to Meyer [6] in the strong Markov case and to Chung [2] for moderate Markov processes. It is usually proved for semi-groups, and full-fledged Markov processes, but the extension to our case poses no problems. We state it for further reference.

PROPOSITION 2

Suppose $\{X_t,\ t\geq 0\}$ is a r.c.l.l. (resp. l.c.r.l.) stochastic process and that for some set M of full Lebesgue measure in $\mathbb{R}^+$, $\{X_t,\ t\in M\}$ is a Markov process whose transition function (P_t) satisfies the conclusions of Proposition 1. A necessary and sufficient condition that $\{X_t,\ t\geq 0\}$ be a strong (resp. moderate) Markov process relative to (P_t) is that for $\forall f\in E_b$, $\forall p>0$

$$t \longrightarrow R_p f(X_t) \quad \text{is a.s. r.c.l.l. on } [0,\infty),$$

$$(\text{resp. l.c.r.l. on } (0,\infty) \text{ and } 0\in M)\ .$$

We say that a set $A\subset E$ is X-polar if $P\{T_A<\infty\}=0$, where $T_A=\inf\{t>0: X_t\in A\}$. We will assume from now on that our transition function satisfies the conclusions of Prop. 1.

PROPOSITION 3

Let (P_t) be a transition function for X. Then for each fixed s and t, $P_{s+t}=P_sP_t$ except possibly on an X-polar set.

Proof : If X is strongly (resp. moderately) Markov and T is a stopping (resp. predictable) time and ν_T the distribution of X_T, then

(2) $$\int_E \nu_T(dx)f(x)\ P_sP_t(x,A) = \int_E \nu_T(dx)f(x)P_{s+t}(x,A)\ ;$$

which just expresses the fact that $X_{T+s+t}\in A$ iff both $X_{T+s}\in E$ and $X_{T+s+t}\in A$. Thus ν_T does not charge $\{P_{s+t}\neq P_sP_t\}$. But X, being r.c.l.l. (resp. l.c.r.l.), is well-measurable (resp. predictable [7, p. 148]) so by the section theorem [7, p. 149], the above set is X-polar.

Dynkin's definition of a Markov process implies that its transition function (P_t) satisfies the following hypothesis, which we state separately for strong and moderate Markov processes.

<u>Hypothesis</u> (Ds).

For each $x\in E$ there exists a r.c.l.l. process $\{X_t^x, t\geq 0\}$ with initial distribution $\delta_x P_o$ which is strongly Markov relative to the transition function (P_t).

<u>Hypothesis</u> (Dm).

For each $x\in E$ there exists an l.c.r.l. process $\{X_t^x, t\geq 0\}$ with $X_o=x$ which is moderately Markov relative to the transition function (P_t).

Notice that either (Ds) or (Dm) implies (P_t) is a semi-group. A transition function satisfying (Ds) (resp. (Dm)) will be called a <u>strong</u> (resp. <u>moderate</u>) <u>Markov semi-group</u>.

§ 2. RESOLVENTS

The <u>resolvent</u> $(R_p)_{p>0}$ of (P_t) is defined by

$$R_p(x,A) = \int_0^\infty e^{-pt}\ P_t(x,A)\ dt\ .$$

We define $b(x,\cdot)=\text{vague}\lim_{p\to\infty} p\ R_p(x,\cdot)$ (which exists since $P_t(x,\cdot)$ has a vague limit as $t\downarrow 0$.)

x is a <u>branching point</u> if $b(x,\cdot)\neq\delta_x(\cdot)$. The set of branching points is Borel measurable and is denoted by E_B . If X is r.c.l.l. and strongly Markov, E_B is X-polar. If X is l.c.r.l. and moderately Markov, E_B may not be X-polar, but the set $\{t:X_t\in E_B\}$ is a.s. countable, so in either case

(3) $$E\{\int_0^\infty e^{-t} I_{E_B}(X_t)\,dt\} = 0$$

<u>PROPOSITION 4</u>

$R_p-R_q=(q-p)R_pR_q \quad \forall p,\ q>0$, except possibly on an X-polar set.

<u>Proof</u> : $R_pR_qf(x) = \int_0^\infty\int_0^\infty e^{-ps-qt}\, P_sP_tf(x)\,ds\,dt$, which is for ν_T - a.e. x (by (2))

$$= \int_0^\infty\int_0^\infty e^{-ps-qt}P_{s+t}f(x)\,ds\,dt$$. By elementary calculus this is, if $p<q$

$$= \int_0^\infty e^{(p-q)t}\,dt\int_0^\infty e^{-pv}\,P_vf(x)dv-\int_0^\infty e^{-pv}\,P_vf(x)dv\int_v^\infty e^{(p-q)t}\,dt$$

$$= \frac{1}{q-p}\,[R_pf(x) - R_qf(x)]\ .$$

Let f run thru a countable dense subset of C(F) to see that the set where the resolvent equation does not hold for some rational p and q has ν_T-measure zero for any stopping (resp. predictable) time T , and is therefore X-polar.- q.e.d.

<u>Lemma 1</u>

Let D be a Borel set such that E-D is X-polar. There exists a Borel set $C\subset D$ such that E-C is X-polar and $R_p(x,E-C)=0$, $\forall x\in C$.

Proof : We can suppose $p=1$. Let $C_o=D$ and define by induction

$$C_n = \{x \in C_{n-1} : R_1(x,E-C_{n-1}) = 0\} .$$

For any finite stopping (resp. predictable) time T

$$\nu_T R_1(E-C_n) = \int_0^\infty e^{-s} \nu_{T+s}(E-C_{n-1})\, ds = 0 ,$$

where ν_T and ν_{T+s} are the laws of X_T and X_{T+s} . The desired set is $C=\cap C_n$.- q.e.d.

THEOREM 1

There exists a transition function (P'_t) for X whose resolvent (R'_p) satisfies the resolvent equation identically, and $R'_p(x,E_B)=0$, $\forall x$.

Proof : From (3) , if T is a stopping (resp. predictable) time, $R_1(X_T,E_B)=0$, hence the set $A=\{x:R_1(x,E_B)>0\}$ is X -polar. The set A' of x for which the resolvent equation fails is also X -polar. Take $D=E-(A\cup A')$ and let $C\subset D$ be the set guaranteed by Lemma 1. Then take

$$P'_t(x,\cdot) = \begin{cases} P_t(x,\cdot) & \text{if } x\in C \\ \delta_x(\cdot) & \text{if } x\in E-C. \end{cases}$$

q.e.d.

§ 3. THE QUOTIENT SPACE AND ITS COMPLETION

Many of the obstacles in our path stem from the fact that the resolvent may not separate points. One solution is to reduce to a quotient space which is separated by the resolvent.

X is still either a r.c.l.l. strong Markov process or an l.c.r.l. moderate Markov process, its transition function satisfies the conclusions of Proposition 1 and its resolvent satisfies the resolvent equation identically and puts no mass on E_B .

Let H be the smallest inf-**stable** cone which contains $R_1(C^+(F))$ and is closed under R_p , all $p>0$. A simple but important lemma of F . Knight [5] says that H is separable in the sup-norm. Let $\{f_n\}$ be dense in H and define a semi-norm d on E by

$$(4) \qquad d(x,y) = \sum_n 2^{-n}|f_n(x)-f_n(y)|(\|f_n\|+1)^{-1} .$$

d induces an equivalence relation R on E by

$$(5) \qquad x\sim(R)\ y \iff d(x,y) = 0 \iff R_p(x,\cdot) = R_p(y,\cdot) \ \ \forall p>0 .$$

If $x\neq y$ are both in $E-E_B$, $d(x,y)>0$ since on $E-E_B$, $f=\lim_{p\to\infty} pR_pf \quad \forall f\in C(F)$.

Let E' be the quotient space E/R and h the natural homomorphism from E to E' . Since the f_n are Borel measurable, so is h . We provide E' with the metric d' :

$$d'(x,y) = d(h^{-1}(x),\ h^{-1}(y)) .$$

Let

$$X'_t = h(X_t)$$

and

$$R'_p(x,A) = R_p(h^{-1}(x),h^{-1}(A)) \ ,\ A\subset E' .$$

One readily checks that (R'_p) satisfies the resolvent equation.

Let us compactify E' : let F' be the completion of E' in the bounded metric d' . Each $f\in H$ can be extended to a function $\overline{f}\in C(F')$ by continuity and the set of differences of these functions is dense in $C(F')$ by Stone-Weierstrass. By an argument of Knight [5] it follows that there is a Ray resolvent $\overline{R}'_p$ on F' such that

(6) $$\overline{R}'_p(x,\cdot) = R'_p(x,\cdot) \quad \forall p>0 \quad \text{if} \quad x\in E'\ .$$

THEOREM 2

There exists a compact metric space $F' \supset E'$ and a Ray resolvent $\overline{R}'_p$ on F' satisfying (6). The process $X'=h(X)$ is a r.c.l.l. strong (resp. l.c.r.l. moderate) Markov process admitting $\overline{R}'_p$ as resolvent. Thus a fortiori X' admits a strong (resp. moderate) Markov semi-group on F'.

Proof : Let $M=\{t: P\{X_t\in E-E_B\}=1\}$. M has full Lebesgue measure in R^+ (E_B has potential 0 .) Then $\{X'_t, t\in M\}$ is Markovian (h is 1-1 on $E-E_B$) with resolvent R'_p. If $A\in \underline{\underline{E}}'$, $h^{-1}(A)\in\underline{\underline{E}}$ and $R_p(X_t, h^{-1}(A))=R'_p(X'_t, A)$. Thus $t\to R'_p(X'_t, A)$ is right (resp. left) continuous. By Prop. 2, X' is a r.c.l.l. strong (resp. l.c.r.l. moderate) Markov process on E' with resolvent R'_p ; and $R'_p=\overline{R}'_p$ on E'. The existence of the strong Markov semi-group is proved by Ray [9] , and that of the moderate Markov semi-group is shown in [10] .- q.e.d.

The obvious disadvantage of this theorem is that it concerns a process on F', not on E, while E and F' are by no means homeomorphic. But let us use this to prove a theorem about the original process. Let the debut and first penetration times of $A\subset E$ be defined by :

$$D_A = \inf\{t\geq 0 : X_t\in A\}$$

and

$$\pi_A = \inf\{t\geq 0 : \int_0^t I_A(X_s)ds>0\}.$$

The corresponding quantities for $X'=h(X)$ will be denoted by D'_A and π'_A respectively.

PROPOSITION 5

Let $A \subset E$ be open. Then there exists a function $\phi_A \in \underline{\underline{E}}_b$ such that for any stopping (resp. predictable) time T

$$\phi_A(X_T) = E\{e^{-D_A \circ \theta_T} | \underline{\underline{F}}_T \text{ (resp. } \underline{\underline{F}}_{T-})\} \text{ a.s.}$$

Proof : Let $\overline{X}=(\overline{\Omega},\overline{\underline{\underline{F}}},\overline{\underline{\underline{F}}}_t,\overline{X}_t,\overline{P}^x)$ be a realization of the strong (resp. moderate) Markov Ray process on F' guaranteed by Theorem 1 .

Let $A'=h(A-E_B)$. Being the 1-1 Borel image of a Borel subset of a Lusin space, $A' \in \underline{\underline{E}}'$ [12]. Define

$$\phi'_{A'} = \overline{E}^x\{e^{-\pi_{A'}}\} .$$

This is 1 -excessive and Borel measurable [11,Prop.2.2] on F' . Let

$$\phi_A(x) = \begin{cases} \phi'_{A'}(h(x)) & \text{if } x \in E-(A\cap E_B) \\ 1 & \text{if } x \in A \cap E_B \end{cases} .$$

Since $R_p(x,E_B) = 0 , \forall x ,$

$$p R_{p+1} \phi_A(x) = p \overline{R}'_{p+1} \phi'_{A'}(h(x))$$

$$\leq \phi'_{A'}(h(x)) \leq \phi_A(x) .$$

As A is open, $D_A=\pi_A=\pi_{A-E_B}$ whenever $X_o \notin A$. The same must then be true of $X'=h(X)$, i.e. $D'_{A'}=\pi'_{A'}$ if $X'_o \notin A'$. Let T be a stopping (resp. predictable) time. On $\{X_T \notin A\}$,

$$\phi_A(X_T) = \phi'_{A'}(X'_T) = E\{e^{-\pi'_{A'} \circ \theta_T} | \underline{\underline{F}}_T \text{ (resp. } \underline{\underline{F}}_{T-})\} .$$

(The restriction $X_T \notin A$ is necessary only where X is l.c. r.l.)

Applying h^{-1} :

$$= E\{e^{-\pi_A \circ \theta_T} | \underline{\underline{F}}_T \quad (\text{resp. } \underline{\underline{F}}_{T-})\} \quad .$$

On the other hand, $\phi_A = 1$ on A : this is clear on $A \cap E_B$, and on $A - E_B$, $pR_p(X_T, A) \to 1$ (A is open) $\Rightarrow p\bar{R}'_p(h(x), A') \to 1 \Rightarrow \pi'_{A'} \circ \theta_T = 0$ a.s., so $\phi'_{A'}(x) = 1$.
q.e.d.

<u>Remark</u> :

1. If $S \leq T$ are stopping (resp. predictable) times, $\{e^{-S}\phi_A(X_S),\ e^{-T}\phi_A(X_T)\}$ is supermartingale.

2. If X is r.c.l.l. and strongly Markov, $t \to \phi_A(X_T)$ is also r.c.l.l., for then E_B is X-polar, and ϕ_A is excessive on $E - E_B$.

Now let us consider the continuity properties of $\bar{X}$ with respect to the topology of E . We will look at the essential limit to avoid problems posed by the non-uniqueness of h . Let g be the 1-1 inverse of h defined on $h(E-E_B)$ by

$$g(x) = h^{-1}(x) \cap E - E_B \quad .$$

g is Borel measurable by Lusin's theorem. As in the proof of Proposition 5, $\bar{X}$ will denote the Ray process on F' .

<u>Lemma 2</u>

Let $D = \{x \in E' : \bar{P}^x\{\text{ess} \lim_{t \downarrow 0} g(\bar{X}_t) \text{ exists in } F\} = 1\}$.

Then D is Borel measurable and $E' - D$ is X'-polar, (where $X' = h(X)$). Hence the set $N = h^{-1}(E' - D)$ is X -polar.

Proof : Let $f \in C(F)$ and define

$$\Gamma_f = \{\text{ess} \limsup_{t \downarrow 0} f \circ g(\overline{X}_t) = \text{ess} \liminf_{t \downarrow 0} f \circ g(\overline{X}_t)\}$$

Then Γ_f is in the natural fields [11,Prop.2.2] hence $x \to \overline{P}^x\{\Gamma_f\}$ is Borel measurable on F'. Thus so is $D_f = \{x : \overline{P}^x\{\Gamma_f\} = 1\}$. Now X' is well-measurable (resp. predictable) because X is. Let T be a finite stopping (resp. predictable) time. Then $P\{\text{ess} \lim_{t \downarrow T} f \circ g(X'_t) \text{exists}\} = 1$. Then $E - D_f$ must be X'-polar by the section theorem [7]. The proof is completed by letting f range thru a countable dense subset of $C(F)$.- q.e.d.

Remark 3 :

It will useful to know later that if x and y are distinct points in $E-N$ such that $R_p(x,\cdot) = R_p(y,\cdot)$, there exists an open A for which

$$\phi_A(x) \neq \phi_A(y) .$$

Proof : One of the two — say x — must be in E_B since the resolvent separates $E - E_B$. Thus the branching measure $b(x,\cdot) = \text{vague} \lim_{p \to \infty} pR_p(x,\cdot)$ is not δ_x, and there exists an open neigborhood A of x such that $b(x,\overline{A}) < 1$. Thus $\overline{P}^y\{\pi_{h(A)} > 0\} > 0$ — here we use the fact that $y \notin N$ to assure ourselves that $\overline{P}^y\{\text{ess} \lim_{t \downarrow 0} g(\overline{X}_t) \text{ exists}\} = 1$ — and $\phi_A(y) < 1$. But $\phi_A(x) = 1$ by construction.- q.e.d.

§ 4. EXISTENCE OF STRONG AND MODERATE MARKOV SEMI-GROUPS

One of the main tools for constructing semi-groups is Ray's theorem on resolvents over compact spaces. A key hypothesis in this theorem is the existence of enough 1-supermedian functions to separate points. Because of this, the following proposition and its corollary have an independent interest. As in the previous sections, X is a r.c.l.l. strong (resp. l.c.r.l. moderate) Markov process whose resolvent satisfies the resolvent equation and puts no mass on E_B . The sets $N \subset E$ and $D \subset E'$ appearing below are those defined in Lemma 2.

PROPOSITION 6

There exists a countable set J of Borel measurable 1 -supermedian functions which separate points of E , such that $\forall g \in J : t \to g(X_t)$ is a.s. right (resp. left) continuous.

Proof : Suppose first that X is strongly Markov. Let $\{A_n\}$ be a countable open base for the topology of E , and let $\{f_n\}$ be dense in the positive unit ball of $C(F)$. Let $g_n = I_{A_n \cap E_B - N}$. Then J is the set of all functions

$$R_1 f_n \; , \; g_n \; , \; \phi_{A_n} \quad \text{(see Prop. 4)} \quad n=1,2,\ldots$$

Now $E_B - N$ is X -polar so $g_n(X_t) \equiv 0$ a.s. and $pR_{p+1} g_n \equiv 0 \leq g_n$. $R_1 f_n$ and ϕ_{A_n} are 1 -supermedian and right continuous along the paths (Prop. 2 and Remark 2). It remains to see that they separate points.

Since $R_p(\underline{\underline{E}}_b)$ is independent of p , $\{R_1 f_n\}$ separates any pair x and y with $h(x) \neq h(y)$. If $h(x)=h(y) \in D$, there exists an n such that $\phi_{A_n}(x) \neq \phi_{A_n}(y)$ (Remark 3). Finally, $\{g_n\}$ separates any x and y for which $h(x)=h(y) \in E' - N$.

The proof for a l.c.r.l. moderate Markov process is similar except that we must replace the functions ϕ_{A_n} by the $\psi_{\overline{A}_n}$ defined below, where $\overline{A}$ is the closure of A . The following lemma completes the proof.

Lemma 3

Let X be a l.c.r.l. moderate Markov process. Let $K \subset E$ be closed and let A_n be a decreasing sequence of open neighborhoods of K such that $K = \bigcap \bar{A}_n$. Then the function

$$\psi_K = \lim \phi_{A_n}$$

is Borel measurable, 1 -supermedian, and $t \longrightarrow \psi_K(X_t)$ is a.s. left continuous.

Proof : By decreasing convergence ψ_K is 1 -supermedian and Borel measurable. As X is predictable, so is $\psi_K(X_t)$, $e^{-t}\psi_K(X_t)$ is a supermartingale and has left and right limits along the rationals. To show it is left continuous, it is enough to show that for each sequence $T_n \uparrow\uparrow T_\infty < \infty$ of predictable times, $\psi_K(X_{T_n}) \longrightarrow \psi_K(X_{T_\infty})$ a.s. [8,p.232]. From Remark 1

$$\{e^{-T_n} \psi_K(X_{T_n}) \ , \ n=1,2,\ldots,\infty\}$$

is a bounded supermartingale. Thus $\lim\psi_K(X_{T_n}) \geq \psi_K(X_{T_\infty})$ a.s. Conversely, for all m

$$\lim_{n\to\infty} e^{-T_n} \psi_K(X_{T_n}) \leq \lim_{n\to\infty} e^{-T_n} \phi_{A_m}(X_{T_n})$$
$$= \lim_{n\to\infty} E\{e^{-T_n + D_{A_m} \circ \theta_{T_n}} | \underline{\underline{F}}_{T_n -}\} \quad .$$

If $X_{T_\infty} \notin \bar{A}_m$, then $X_{T_n} \notin \bar{A}_m$ for sufficiently large n so $T_n + D_{A_m} \circ \theta_{T_n} = T_\infty + D_{A_m} \circ \theta_{T_\infty}$, making the above limit

$$= e^{-T_\infty} \phi_{A_m}(X_{T_\infty}) \ .$$

It follows that $\lim\psi_K(X_{T_n}) \leq \psi_K(X_{T_\infty})$ a.s. on $\{X_{T_\infty} \notin K\}$. This is also true on $\{X_{T_\infty} \in K\}$ since $\psi_K = 1$ on K .- q.e.d.

If the transition function had already satisfied hypothesis (D_s) or (D_m) we would have defined ϕ_A directly : $\phi_A(x)=E^x\{e^{-D_A}\}$. The set N of Lemma 2 would be empty, and the functions $R_p f_n$ and ϕ_{A_n} , n=1,2,... would be sufficient to separate points, giving us :

COROLLARY

Suppose (P_t) satisfies hypothesis (D_s) (resp. (D_m)). Then there exists a countable family J of 1-supermedian functions which separate points and satisfy

$$\forall g\in J\ ,\ \forall x\in E : t \longrightarrow g(X_t) \text{ is } P^x \text{ - a.s.}$$

right continuous on $[0,\infty)$ (resp. left continuous on $(0,\infty)$).

This brings us to the basic theorem of this paper.

THEOREM 3

Let $\{X_t, t\geq 0\}$ be a r.c.l.l. strong (resp. l.c.r.l. moderate) Markov process. Then X admits a transition semigroup which satisfies hypothesis (D_s) (resp. (D_m)).

Proof : Let $J\subset \underline{\underline{E}}_b$ be a countable set of 1-supermedian functions which separate E and are right (resp. left) continuous along the paths of X . Let H be the smallest positive inf-stable cone which contains J and is closed under R_p for all $p>0$. H is separable in the sup-norm by Knight's lemma [5] . Let $\{f_n\}$ be dense in H , and define a distance on E by

$$d(x,y) = \sum_n 2^{-n} |f_n(x)-f_n(y)| (\|f_n\|+1)^{-1} .$$

Let E^* be the completion of E with respect to the bounded metric d . E^* is compact and contains E as a subset, tho not necessarily as a subspace. R_p can be extended to a Ray resolvent R_p^* on E^* , where

$$R_p^*(x,\cdot) = R_p(x,\cdot) \quad \text{if} \quad x \in E .$$

The original topology of E may differ from that which it inherits from E^* , so we introduce the inclusion map i from E to E^* to keep them straight. i is Borel measurable and 1-1 so that $i(E)$, being the 1-1 Borel image of a Lusin space, is Borel in E^* . The process $i(X_t)$ is strongly (resp. moderately) Markov (i is 1-1) and because $g(X_t)$ is right (resp. left) continuous, $\forall g \in J$, $i(X_t)$ is also right (resp. left) continuous. Now (R_p^*) has two semi-groups, one strongly Markov and one moderately Markov (see [10, Thm. 1]) . Let (P_t^*) be the strong (resp. moderate) Markov semi-group with resolvent R_p^* . (P_t^*) is then a transition function for $i(X)$.
To get the desired semi-group on E , it will be necessary to modify (P_t^*) slightly.

Let X^* be the canonical realization of (P_t^*) . If $f \in C(F)$, let

$$f^*(x) = \begin{cases} f(i^{-1}(x)\,) & \text{if } x \in i(E) \\ 0 & \text{otherwise.} \end{cases}$$

Consider the sets

$$\Lambda_f = \{t \to f^*(X^*_t) \text{ is not r.c.l.l. (resp. l.c.r.l.) on } (0,\infty)\} .$$

$$\Gamma_f = \{\operatorname*{ess\,lim}_{t \downarrow 0} f^*(X^*_t) \neq f^*(X_o^*)\} .$$

The measurability of Λ_f can be proved in the standard way; e.g. in the r.c.l.l. case, $\Lambda_f = \{\lim_{n\to\infty} T_n^m = \infty \ \forall m\}$ where

$$T_o^m = 0 \ , \ T_{n+1}^m = \inf\{t > T_n^m : |f^*(X^*_t) - f^*(X^*_{T_n^m})| > 1/m\} .$$

Let $k(x)=P^{*x}\{\Lambda_f\}$. This is readily seen to be P_t^* -excessive, hence nearly Borel on E^* . The function $\ell(x) = P^{*x}\{\Gamma_f\}$ is even Borel measurable since Γ_f is measurable ω.r.t the natural fields [11] . Thus let

$$A_f = \{x:k(x)>0 \quad \text{or} \quad \ell(x)>0\} .$$

Note that if $x\in E^*-A_f$, $P^{*x}\{X_t^*\in E^*-A_f \ \ \forall t\} = 1$. Furthermore, since $f^*(i(X_t))$ is r.c.l.l. (resp. l.c.r.l.) we conclude that A_f is $i(X)$ -polar. Thus the same is true for $A=\bigcup A_f$, where the union is over a countable dense subset of $C(F)$. If $x\in E^*-A$, then $i^{-1}(X_t^*)$ is P^{*x}- a.s. r.c.l.l. (resp. l.c.r.l.) in F .

Let $A'=\{x:P^{*x}\{\exists t>0:i^{-1}(X_t^*)$ or $i^{-1}(X_t^*)_-\in F-E\}>0\}$. Then A' is again a nearly Borel $i(X)$ -polar set.

We now have the exceptional set we want, except that it is only nearly Borel measurable while we would like it to be Borel measurable. We use an argument due to P.A. Meyer : there exists an $i(X)$ -polar Borel set $B_1\supset A\cup A'\cup(F'-i(E))$. The set $C_1=\{x:P^{*x}\{T_{B_1}<\infty\}>0\}$ is nearly Borel, and also $i(X)$ -polar, so there exists a second $i(X)$ -polar Borel set $B_2\supset C_1\cup B_1$. We continue, defining B_n by induction. The set $B=\bigcup B_n$ is an $i(X)$ -polar Borel set, and $E-B$ is stable, i.e. $x\in E-B \Rightarrow P^{*x}\{T_B=\infty\}=1$. Further, for $x\in E-B$, the process $i^{-1}(X_t)$ is r.c.l.l. in E (not just in F) and strongly Markov. It follows that

$$P'_t(x,K) = \begin{cases} P_t^*(i(x),\, i(K)\,) & \text{if} \quad x\in i^{-1}(E^*-B) \\ \delta_x & \text{otherwise} \end{cases}$$

defines a transition semi-group for X which satisfies hypothesis (D_s) (resp. (D_m)) on the original space E .- q.e.d.

R E F E R E N C E S

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