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Existence of Small Oscillations at Zeros of Brownian Motion

by Frank B. Knight

0. Introduction. Let $X(t)$, $X(0) = 0$, be a standard one-dimensional Brownian motion, with zero-set $Z = \{0 \leq t \leq 1 : X(t) = 0\}$. Many properties of Z are known, in the sense that they hold with probability 1. For example, Z is a closed uncountable set of Hausdorff dimension $\frac{1}{2}$ [2, 2.5]. If one asks, however, for the conditional behavior of $X(t+h)$ given that $0 < t \in Z$ one encounters the difficulty that, since $P\{t \in Z\} = 0$, the conditioning has no meaning. To be sure if $t = T(w) \in Z$ is a stopping time, then the strong Markov property implies various results, of which the most relevant here is the well-known Local Law of the Iterated Logarithm [6, VI, 51.1]: Set $\Delta_h X(t) = X(t+h) - X(t)$ and $\varphi_2(h) = (h \log\log 1/h)^{1/2}$. Then $P\{\limsup_{h \to 0+} \Delta_h X(T)(\sqrt{2}\,\varphi_2(h))^{-1} = \limsup_{h \to 0+} -\Delta_h X(T)(\sqrt{2}\,\varphi_2(h))^{-1} = 1\} = 1$. There are, however, many (random) $t \in Z$ at which this behavior does not hold, and such t we shall term "exceptional." The object of this paper is to study one type of exceptionality which occurs with probability 1.(1)

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(1) It will be noted that such a type of exceptionality will be represented not only in Z but also at all x in the range of $X(t)$ outside a random set of Lebesgue measure 0. It then follows directly from P. Lévy's modulus of continuity for $X(t)$ [6, VII, 52] that the overall exceptional set has Hausdorff dimension $\geq \frac{1}{2}$ (for this observation I am indebted to Professor N. Jain).

Our main result is to show that there exist $t \in Z$ with $\limsup_{h \to 0+} |\Delta_h X(t)|(\sqrt{2}\,\varphi_2(h))^{-1} < 1$. This gives a partial answer to a question of A. Dvoretzsky [1] which remained unanswered in [9] (without the added information that $t \in Z$). To do this we rely upon a result of B. Mandelbrot [7] and L. Shepp [10]. At the same time, our analysis seems to indicate that there do not exist $t \in Z$ for which $\limsup_{h \to 0+} |\Delta_h X(t)|(\varphi_2(h))^{-1} = 0$. Consequently, if such t exist they must be sought elsewhere than in the set where $X(t)$ has a prescribed value.

Before turning to this result, let us remark upon a type of exceptionality which is quite well understood. A time $t \in Z$ is said to be the starting time of an excursion of $X(t)$ if $X(t+h) \neq 0$ for $0 < h < \varepsilon$ sufficiently small. There are countably many such t, and for all of them the behavior of $\Delta_h X(t)$ is adequately covered by [2, 2.10]. Assuming, as we may, that $X(t+h) > 0$, we have $\limsup_{h \to 0+} \Delta_h X(t)(\sqrt{2}\,\varphi_2(h))^{-1} = 1$ as in the unexceptional case, but also $\liminf_{h \to 0+} \Delta_h X(t) h^{-\frac{1}{2}}(\log 1/h)^{(1+\varepsilon)} > 1$ for $\varepsilon > 0$, in radical contrast with the normal behavior for $-\Delta_h X(t)$. We see immediately that there cannot exist a stopping time T which equals the starting time of an excursion with positive probability.[2]

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[2] Another set of exceptional times is of course the set of local maxima and minima. Being countable, however, it does not intersect Z. The behavior of $X(t)$ following such an extremum is entirely analogous to that at the start of an excursion. This is easily seen from P. Levy's equivalence $|X(t)| = M(t) - X(t)$ where $M(t) = \max_{s \leq t} X(s)$. Moreover, by an evident reversal of time most of this exceptional behavior holds in both time directions. In short, the path exhibits a dense set of spine-like projections of sharpness exceeding $\sqrt{|h|}\,(\log \frac{1}{|h|})^{-(1+\varepsilon)}$ for every $\varepsilon > 0$.

1. Exceptional Small Oscillations at $t \in Z$.

We introduce the standard local time $f(t)$ of $X(t)$ at 0 using the indicator function $I_{(-\infty,x)}$ of $(-\infty,x)$:

$$(1.1) \qquad f(t) = \frac{1}{2}\frac{d}{dx}\int_0^t I_{(-\infty,x)}(X(s))ds]_{X=0}.$$

The existence and continuity in t of $f(t)$ is a well-known result of P. Lévy (see [2]). The exact statement of our result is as follows.

Theorem 1.1. $P\{\exists t_0 \in Z : \limsup_{h\to 0+} |X(t_0+h)|(\varphi_2(h))^{-1} < k\} = 1$, for all $k > 2^{-\frac{1}{2}}$.

Proof. The key to the proof lies in the observation that if the oscillations of $|X(t)|$ above 0 are recorded as a function of the local time $f(t)$ they generate a homogeneous Poisson point process of the type considered in [10].

Definition 1.1. Let $f^{(-1)}(\alpha) = \inf\{t : f(t) > \alpha\}$ be the right-continuous inverse local time at 0, and let $\mathcal{A}(\alpha,\alpha+\varepsilon) = \max_{f^{(-1)}(\alpha)<t<f^{(-1)}(\alpha+\varepsilon)} |X(t)|$, $0 \le \alpha < \alpha + \varepsilon$.

Lemma 1.1. The random set $\Gamma = \{(\alpha,y) : \lim_{\varepsilon\to 0+} \mathcal{A}(\alpha-\varepsilon,\alpha) = y > 0\}$ is a homogeneous Poisson point process with parameter $\alpha \ge 0$ and expectation measure $\lambda \times \mu$ where λ is Lebesgue measure and $\mu(A) = \int_A 2y^{-2}dy$ on $\{y : 0 < y < \infty\}$.

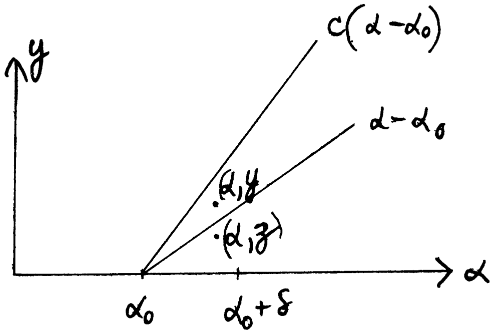
Proof. Since $f^{(-1)}(\alpha)$ is a homogeneous process with independent increments and a stopping time of $X(t)$ for each α, with $X(f^{(-1)}(\alpha)) = 0$, it is clear that Γ is a homogeneous Poisson point process. Taking into account the independence of the local times for $x > 0$ and for $x < 0$ up to time $f^{(-1)}(\alpha)$ and the fact that $P\{M(f^{(-1)}(\alpha)) < z\} = \exp - \frac{\alpha}{z}$, known from [3, Theorems 1.2 and 2.2, or 11, Proposition 2.4], we have $P\{A(0,\alpha) < z\} = \exp - \frac{2\alpha}{z}$. In view of $\frac{2\alpha}{z} = \alpha \int_z^\infty 2y^{-2}dy$ this implies the result.

The following lemma is now a direct consequence of [7].

<u>Lemma 1.2</u>. $P\{\exists\alpha_0$ with $f^{(-1)}(\alpha_0) \leq 1$ and $A(\alpha_0,\alpha_0+\varepsilon) < c\varepsilon$, $0 < \varepsilon < \delta$ for some $\delta > 0\} = 1$ for $c > 2$.

<u>Proof</u>. The property $A(\alpha_0,\alpha_0+\varepsilon) < c\varepsilon$, $0 < \varepsilon < \delta$, may be stated as saying that α_0 is not covered by the union of open intervals $(\alpha-z,\alpha)$ generated by the truncated Poisson process $\{(\alpha,z): z = \frac{y}{c} \wedge \delta, (\alpha,y) \in \Gamma\}$ where $\wedge$ denotes minimum.

Fig. 1



This process has mean density $2c^{-1}y^{-2}$; $y < c\delta$, with a point mass at $y = \delta$ of size $2(c\delta)^{-1}$. The result of [10, (40)] or [7], states that such an α_0 exists with positive probability if and only if $\int_0^\delta (\exp \int_x^\delta 2(cy)^{-1}dy)dx < \infty$, where

$2(cy)^{-1} = \int_y^\delta 2c^{-1}z^{-2}dz + 2(c\delta)^{-1}$ is the expectation measure in $[y,\infty)$. The condition is equivalent to $\int_0^1 x^{-\frac{2}{c}}\,dx < \infty$, i.e., to $c > 2$. Routine use of the scale change $X(t) \equiv k^{-\frac{1}{2}} X(kt)$ shows that $\mathcal{A}(\alpha,\alpha+\varepsilon) \equiv k^{-\frac{1}{2}}\mathcal{A}(k^{\frac{1}{2}}\alpha, k^{\frac{1}{2}}(\alpha+\varepsilon))$, and letting $k \longrightarrow 0$ we may allow $\alpha \longrightarrow \infty$ and apply the 0-1 Law to get the probability 1 as required.

The exceptional t_0 of Theorem 1.1 is essentially $t_0 = f^{(-1)}(\alpha_0)$, but to derive the result directly would involve giving a meaning to the process $X(f^{(-1)}(\alpha_0)+h)$, which is problematical. Instead, we introduce the space $\Omega' = [0,\infty) \times \Omega$, where Ω is the sample space of $X(t)$, and define the conditioning sequentially in such a way that it may be applied at a constant $\alpha = 0$. We then argue that the projection of the limit set in Ω' has positive probability in Ω, and therefore the additional condition of Theorem 1.1 is met at some $t_0 = f^{(-1)}(\alpha_0)$.

Turning to the details, let δ, c_0 and $\rho < 1$ be positive constants, $0 < r < s$ and n be integers, and consider the subset of Ω'

$$(1.2) \qquad S'(n,r,s) = S'(n) \cap M'(r,s);$$

$$S'(n) = \{(\alpha,w): f^{(-1)}(\alpha) \le 1\, \mathcal{A}(\alpha,\alpha+k2^{-n}\delta) \le ck2^{-n}\delta, 1 \le k < 2^n\}$$

$$M'(r,s) = \{(\alpha,w): \max_{f^{(-1)}(\alpha)<t<f^{(-1)}(\alpha)+\rho^m\delta} |X(t)| \le c_0\varphi_2(\rho^m\delta), r \le m \le s\}.$$

Furthermore, let $\Phi(S') = \{w:(\alpha,w) \in S'$ for some $\alpha \geq 0\}$ denote the projection onto Ω. The proof rests in showing that, for suitable c_o, c, δ, and r,

$$(1.3)\ (a) \qquad \lim_{s\to\infty} \lim_{n\to\infty} P(\Phi(S'(n,r,s))) > 0, \quad \text{and}$$

$$(b) \qquad \lim_{s\to\infty} \lim_{n\to\infty} \Phi(S'(n,r,s)) = \Phi(\lim_{s\to\infty} \lim_{n\to\infty} S'(n,r,s)) .$$

Indeed, it is clear that

$$(1.4) \qquad \lim_{s\to\infty} \lim_{n\to\infty} S'(n,r,s) = \{(\alpha,w) \in \Omega' : f^{(-1)}(\alpha) \leq 1;$$

$$\mathcal{A}(\alpha,\alpha+\varepsilon) \leq c\ \varepsilon,\ 0 < \varepsilon < \delta, \quad \text{and}$$

$$\max_{f^{(-1)}(\alpha)<t<f^{(-1)}(\alpha)+\rho^m\delta} |X(s)| \leq c_o\varphi_2(\rho^m\delta),\ r \leq m < \infty\} .$$

Since φ_2 is increasing this will imply the result when $c_o \sim 2^{-\frac{1}{2}}$ and $(1-\rho) \sim 0$, for in view of (b) the set of w for which there exists an exceptional $t_o = f^{(-1)}(\alpha_o)$ will have positive probability (the scale change used in Lemma 1.2 again shows easily that the probability must be 0 or 1).

The first step in proving (a) is

Lemma 1.3. $P\{\Phi(S'(n,r,s))\} \geq P\{\Phi(S'(n))\} \times$ $P\{(0,w) \in M'(r,s) | (0,w) \in S'(n)\}$.

Proof. We set $\alpha_n = \inf\{\alpha:(\alpha,w) \in S'(n)\}$ if this is non-null and $\alpha_n = f(1) + 1$ otherwise. Although α_n is not a stopping time, we can reduce it to stopping time on the set $\{\alpha_n \leq f(1)\} = \{(\alpha_n,w) \in S'(n)\}$. On this set, either $\alpha_n = 0$ or else α_n is the local time of an excursion of $X(t)$ such that $\mathcal{A}(\alpha_n-,\alpha_n) > c2^{-n}\delta$. To see this, note that if $0 < \alpha_n \leq f(1)$ then for $\alpha < \alpha_n$ we have $\mathcal{A}(\alpha,(\alpha+k2^{-n}\delta) \wedge \alpha_n) > ck2^{-n}\delta$ for some $k < 2^n$, and the assertion follows as α increases to α_n. The set of $f^{(-1)}(\alpha)$ with $\mathcal{A}(\alpha-,\alpha) > c2^{-n}\delta$ is contained in the sequence $T_1,\ldots,T_n,\ldots$ of stopping times $T_1 = \inf\{t:X(t) = 0$ and $\max_{0<s<t} |X(s)| > c2^{-n}\delta\}$, $T_{n+1} = T_n + T_1 \circ \theta_{T_n}$, where θ_t is the usual translation operator. Setting $T_o = 0$ and using the strong Markov property, we have

$$P\{\Phi(S'(n,r,s))\} \geq \sum_{k=0}^{\infty} P\{(\alpha_n,w) \in S'(n,r,s),\alpha_n = f(T_k)\}$$

$$= P\{\Phi(S'(n))\}P\{(0,w) \in M'(r,s)|(0,w) \in S'(n)\},$$

as required.

The next step is to obtain an estimate of the above conditional probability. The analytical content is contained in

Lemma 1.4. For $\beta > 0$, $x > 0$, $K > 0$ and large r,

$$\lim_{n\to\infty} P\{f^{(-1)}(\beta\varphi_2(\rho^m\delta)) < x\rho^m\delta|(0,w) \in S'(n)\}$$
$$\leq |\log \rho^m\delta|^{Kx-2\beta\sqrt{2K}} (c\beta\sqrt{2K} \log|\log \rho^m\delta|)^{\frac{c}{2}}.$$

Proof. Given $(0,w) \in S'(n)$ the increments $f^{(-1)}(j2^{-n}\delta) - f^{(-1)}((j-1)2^{-n}\delta)$ remain independent, $1 \leq j < n$, and their conditional distribution is the same as that of $f^{(-1)}(2^{-n}\delta)$ given that $\max\limits_{0<t<f^{(-1)}(2^{-n}\delta)} |X(t)| < cj2^{-n}\delta$. The Laplace transform of this conditional distribution is readily obtained from [4, Theorem 2.1], in which we set $\alpha = 2^{1-n}\delta$, $a = cj2^{-n}$, and square the result since the f of [4] is twice the present f and the sojourns in $(0,a)$ and $(-a,0)$ are independent.(3) Multiplying from $j = 1$ to k we obtain

$$(1.5) \qquad E(\exp - \lambda f^{(-1)}(k2^{-n}\delta) \mid (0,w) \in S'(n))$$
$$= \exp \sum_{j=1}^{k} \left(\frac{2}{cj} - 2^{1-n}\delta\sqrt{2\lambda}\,\operatorname{cotanh}(cj2^{-n}\delta\sqrt{2\lambda})\right).$$

Letting $k2^{-n}\delta = \varepsilon$ remain fixed as $n \longrightarrow \infty$ the exponent becomes

$$\lim_{n\to\infty} 2^{-n}\delta \sum_{j=1}^{k} \left(\frac{2^{n+1}}{cj\delta} - 2\sqrt{2\lambda}\,\operatorname{cotanh} cj2^{-n}\delta\sqrt{2\lambda}\right)$$
$$= \lim_{\varepsilon'\to 0} \int_{\varepsilon'}^{\varepsilon} \left(\frac{2}{cx} - 2\sqrt{2\lambda}\,\operatorname{cotanh} c\sqrt{2\lambda}\,x\right)dx$$
$$= -\frac{2}{c}\left(\log(\varepsilon^{-1}\sinh \varepsilon c\sqrt{2\lambda}) - \lim_{\varepsilon'\to 0}(\log c\sqrt{2\lambda} + o(\varepsilon'))\right)$$
$$= -\frac{2}{c}\log((\varepsilon c\sqrt{2\lambda})^{-1}\sinh \varepsilon c\sqrt{2\lambda}),$$

and so the Laplace transform converges to

(3) The "check" on p. 179 of [4] has a mistaken integrand. It should be $\exp - (\alpha\sqrt{2\lambda}\,\operatorname{cotanh} a\sqrt{2\lambda})$.

$$(1.6) \qquad ((\varepsilon c\sqrt{2\lambda})(\sinh \varepsilon c\sqrt{2\lambda})^{-1})^{+\frac{2}{c}} .$$

Accordingly, the conditional distributions converge weakly, and the limits may be bounded by using $P\{R < k\} < e^{\lambda k} Ee^{-\lambda R}$, valid for any positive random variable R and $\lambda > 0$. Setting $\varepsilon = \beta\varphi_2(\rho^m\delta)$, $k = x\rho^m\delta$, and $\lambda = K(\rho^m\delta)^{-1}\log|\log \rho^m\delta|$, we have $\lambda k = Kx \log|\log \rho^m\delta|, \varepsilon c\sqrt{2\lambda} = c\beta\sqrt{2K} \log|\log \rho^m\delta|$, and using the fact that for large values of the argument we may replace $\sinh(\cdot)$ by $\frac{1}{2}\exp(\cdot)$ in (1.6), for large r and $m > r$ we obtain the required upper bound of Lemma 1.4.

To continue, let $\varepsilon' > 0$ be fixed and note that $\lim_{n\to\infty} P\{|X(t)| \leq cf(t) + \varepsilon', 0 < f(t) < \delta|(0,w) \in S'(n)\} = 1$, as follows from $|X(t)| \leq \mathcal{A}(0,f(t))$ in view of (1.4). On the other hand, since the limit distribution with transform (1.6) is concentrated near 0 for small ε, we have for any $\delta' > 0$ and $m > r$ sufficiently large, $\lim_{n\to\infty} P\{\rho^m\delta < f(\delta)|(0,w) \in S'(n)\} > 1-\delta'$. Therefore

$$(1.7) \qquad \lim_{n\to\infty} P\{\bigcup_{m=r}^{s}(\max_{0<t<\rho^m\delta}|X(t)| > c_0\varphi_2(\rho^m\delta))|(0,w) \in S'(n)\}$$

$$\leq \lim_{n\to\infty} P\{\bigcup_{m=r}^{s}(cf(t) + \varepsilon' \geq |X(t)|, 0 < t < \rho^m\delta,$$

$$\text{and} \max_{0<t<\rho^m\delta}|X(t)| > c_0\varphi_2(\rho^m\delta))|(0,w) \in S'(n)\} + \delta'.$$

Thus if we set $T(m) = \inf\{t : cf(t) > c_o'\varphi_2(\rho^m\delta)\}$ for $c_o' < c_o$, and let $\varepsilon' \longrightarrow 0$, (1.7) will be bounded by

$$\lim_{n \to \infty} P\{\bigcup_{m=r}^{s} (T(m) < \rho^m\delta \text{ and } \max_{T(m)<t<\rho^m\delta} |X(t)| > c_o\varphi_2(\rho^m\delta))$$

$$|(0,w) \in S'(n)\} + \delta'.$$

Next, since $T(m)$ is a stopping time and $X(T(m)) = 0$, this limit is seen to be bounded by

$$(1.8) \qquad \lim_{n \to \infty} \sum_{m=r}^{s} \int_0^1 P\{\max_{0<t<\rho^m\delta(1-x)} |X(t)| > c_o\varphi_2(\rho^m\delta)\} dF_{m,n}(x) + \delta',$$

where $F_{m,n}(x)$ is the conditional distribution function of $T(m)(\rho^m\delta)^{-1}$ given $\{(0,w) \in S'(n)\}$. Here we have $F_{m,n}(x) \leq P\{cf(x\rho^m\delta) > c_o'\varphi_2(\rho^m\delta) | (0,w) \in S'(n)\} \leq P\{f^{(-1)}(\frac{c_o'}{c}\varphi_2(\rho^m\delta)) < x\rho^m\delta | (0,w) \in S'(n)\}$. In applying Lemma 1.4 we may simply set $c_o' = c_o$ since the bound is continuous. Moreover, for large m the last factor may be absorbed by an arbitrarily small increase in the exponent $Kx - 2\beta\sqrt{2K}$, where $\beta = \frac{c_o}{c}$.

As for the integrand in (1.8), we use the standard inequality

$$(1.9) \qquad P\{\max_{0<s<t} |X(s)| > k\} \leq 4P\{X(t) > k\} \leq \left(\frac{4}{k}\sqrt{\frac{t}{2\pi}}\right)\exp - \frac{k^2}{2t},$$

where the first factor on the right will be small for large m and may be replaced by unity. It follows from this and the weak convergence of the distributions in Lemma 1.4 that (1.8) is bounded by

$$(1.10)\quad \sum_{m=r}^{s} \int_0^1 \exp - \frac{c_o^2\varphi_2^2(\rho^m\delta)}{2\rho^m\delta(1-x)}\, d_x(|\log \rho^m\delta|^{Kx - \frac{2c_o}{c}\sqrt{2K}})$$

$$= \sum_{m=r}^{s} K \log|\log \rho^m\delta| \int_0^1 |\log \rho^m\delta|^{(-\frac{c_o^2}{2(1-x)} + Kx - \frac{2c_o}{c}\sqrt{2K})} dx + \delta'.$$

Now for given K the exponent is maximized at $x = 1 - c_o(\frac{1}{2K})^{+\frac{1}{2}}$, where it becomes $K - K^{\frac{1}{2}}c_o(\sqrt{2} + \frac{2^{3/2}}{c})$. We can easily minimize this over $K > 0$ to obtain the value $E(c_o) = -\frac{c_o^2}{4}(2 + \frac{8}{c} + \frac{8}{c^2})$. If we choose c_o to make this less than -1, then the integrals in (1.8) are of the order $m^{E(c_o)}$, which is the general term of a convergent series.

By choosing $c - 2$ small, this may be accomplished for any $c_o > \frac{1}{\sqrt{2}}$. Recalling that δ' in (1.7) does not depend on s we can then let $s \longrightarrow \infty$ and (1.7) will be strictly less than 1 if r is large. In view of Lemmas 1.3 and 1.2 this proves property (1.3), (a): $\lim_{s\to\infty} \lim_{n\to\infty} P(\Phi(S'(n,r,s))) > 0$, for any $c_o > \frac{1}{\sqrt{2}}$ when c and r are suitably chosen.

It remains only to prove (1.3), (b). The inclusion from right to left is obvious. Conversely, let $w \in \lim_{s\to\infty} \lim_{n\to\infty} \Phi(S'(n,r,s))$, and let $(w,\alpha_{n,s}) \in S'(n,r,s)$ for each (n,s). Keeping s fixed and choosing a subsequence we may assume that $\lim_{n\to\infty} \alpha_{n,s} = \alpha_s$ exists. We will show that $\lim_{n\to\infty} f^{(-1)}(\alpha_{n,s}) = f^{(-1)}(\alpha_s)$. In the contrary case, α_s would be the local time of an excursion of $X(t)$, and $\alpha_{n,s} < \alpha_s$ would hold for infinitely many n. This would contradict the definition of $S'(n,r,s)$ since $\mathcal{A}(\alpha_s-,\alpha_s) > 0$ is impossible when $\mathcal{A}(\alpha_{n,s},\alpha_{n,s} + k2^{-n}\delta) \leq ck2^{-n}\delta$, $1 \leq k < 2^n$, for $0 < \alpha_s - \alpha_{n,s}$ sufficiently small. It thus follows from the definitions that $(w,\alpha_s) \in \lim_{n\to\infty} S'(n,r,s)$. Similarly, let $\lim_{s\to\infty} \alpha_s = \alpha$ exist along a subsequence. Then $\lim_{s\to\infty} f^{(-1)}(\alpha_s) = f^{(-1)}(\alpha)$ in view of (1.4), and so $(w,\alpha) \in \lim_{s\to\infty} \lim_{n\to\infty} S'(n,r,s)$. This implies the result.

A very slight change in this proof also shows the existence of two-sided exceptional times.

__Corollary 1.2__. $P\{\exists t_o \in Z: \limsup_{h\to 0+} |X(t_o+\varepsilon_1)-X(t_o-\varepsilon_2)|(\varphi_2(h))^{-1} < k$, $0 < \varepsilon_1,\varepsilon_2;\ \varepsilon_1 + \varepsilon_2 = h\} = 1$ for all $k > \frac{4}{3}$.

__Remark__. It is shown in [12] that for $t > 0$ $P\{\limsup_{h\to 0+} |X(t+\varepsilon_1)-X(t-\varepsilon_2)|(\varphi_2(h))^{-1} = \sqrt{2}\} = 1$ for t fixed. Since $\frac{4}{3} < \sqrt{2}$ the t_o obtained above is exceptional.

__Proof__. The argument of Lemma 1.2 also shows that $P\{\exists\alpha_o$ with $f^{(-1)}(\alpha_o) \leq 1$, and both $\mathcal{A}(\alpha_o,\alpha_o+\varepsilon) < c\varepsilon$ and $\mathcal{A}(\alpha_o-\varepsilon,\alpha_o) < c\varepsilon$, $0 < \varepsilon < \delta\} = 1$ for $c > 4$. Indeed, this is equivalent to α_o not being covered by the intervals $(\alpha-z,\alpha+z)$, and by the homogeniety of the Poisson process this is equivalent to replacing z

by $2z$. The mean density is then $4c^{-1}y^{-2}$ and the integral converges for $c > 4$. Since the problem only involves the increments of $X(t)$ we can assign to $X(0)$ a uniform initial measure on $(-\infty,\infty)$ and obtain a stationary process, $-\infty < t < \infty$. Then the same proof given above, but with $c > 4$, applies both to $X(t_o+\varepsilon_1) - X(t_o)$ and to $X(t_o-\varepsilon_2) - X(t_o)$. The condition that $E(c_o) < -1$ becomes $c_o > \frac{2}{3}\sqrt{2}$, and since $\varphi_2(\varepsilon_1) + \varphi_2(\varepsilon_2) < \sqrt{2}\,\varphi_2(h)$ when $h = \varepsilon_1 + \varepsilon_2$ is small (as is not difficult to show) we obtain the constant $\frac{4}{3}$. The Corollary is proved.

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