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MULTIPLICATIVE EXCESSIVE MEASURES AND DUALITY BETWEEN
EQUATIONS OF BOLTZMANN AND OF BRANCHING PROCESSES

by Masao NAGASAWA ¹

Duality between Boltzmann's equations (corresponding Markov processes) and so-called S-equations of branching Markov processes (branching Markov processes) was discovered by H.Tanaka[11,12], Y.Takahashi[10] and the author(unpublished). This will be explained in the following sections. To make the duality to be realistic, existence of a special class of excessive measures of branching Markov processes will be discussed in the later sections.

I. Boltzmann's equation

Suppose we are observing the distribution of speed of a gas molecule. The speed is assumed to stay constant until collision occurs. Suppose after some time interval of the exponential distribution, two particles with speeds a_1 and a_2 collide and the speed of the first particle will be distributed in db with a probability distribution $\bar{\pi}_2(a_1, a_2; db)$ depending on a_1 and a_2 . Then, if the initial distribution of speed of each particle is $f(da)$, the speed distribution $u_t(da)$ at t satisfies

$$u_t = e^{-t}f + \int_0^t ds e^{-(t-s)} \iint u_s(da_1)u_s(da_2)\bar{\pi}_2(a_1, a_2; \cdot).$$

This is the so-called Boltzmann's equation of gas in a simple form. McKean[3,4,5] discussed probabilistic aspects of the equation and gave a model as a temporally inhomogeneous Markov processes(with non-constant transition mechanism in his terminology).

Let's generalize the equation allowing; (a) the speed before collision not to be constant but varying as a right continuous

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Markov process \bar{x}_t on a state space S with a collision time of the distribution $\exp(-c_t)$ where c_t is the Kac's additive functional of a non-negative function c , and (b) n -particle collision ($n = 2, 3, \dots$) with a distribution $\bar{\pi}_n(a_1, a_2, \dots, a_n; db)$. Let $\bar{q}_n(a_1, \dots, a_n)$ be a weight of n -collision and \bar{P}_t^0 be the transition probability of the Markov process, then a generalized Boltzmann's equation is

$$(1) u_t = f \bar{P}_t^0 + \int_0^t ds \sum_{n=2}^{\infty} \int_{S^n} \prod_{j=1}^n u_s(da_j) \bar{q}_n(\underline{a}) \int_S \bar{\pi}_n(\underline{a}, dc) \bar{P}_{t-s}^0(c, \cdot),$$

where $\underline{a} = (a_1, \dots, a_n)$.

This equation was treated by H. Tanaka [11, 12] and T. Ueno [13, 14]. Tanaka constructed a Markov process on a large state space $\underline{S} = \bigcup_{n=1}^{\infty} S^n$ such that: Let H_t be the transition probability of the Markov process and \hat{f} be the measure on \underline{S} whose restriction on S^n is the n -fold product of f . Then the solution of (1) is given by

$$u_t(B) = \int_{\underline{S}} \hat{f}(d\underline{a}) H_t(\underline{a}, B), \quad B \subset S.$$

Moreover he characterized the Markov process in terms of the following convolution property: Let $\underline{\phi} = (\phi_1, \phi_2, \dots)$ and $\underline{\psi} = (\psi_1, \psi_2, \dots)$ be defined on \underline{S} where ϕ_n is the restriction of $\underline{\phi}$ on S^n . Define a convolution $\underline{\phi} * \underline{\psi}$ by

$$(\underline{\phi} * \underline{\psi})_n = \text{sym.} \sum_{\substack{i+j=n \\ i, j \geq 1}} \phi_i(a_1, \dots, a_i) \psi_j(a_{i+1}, \dots, a_n).$$

Then the transition probability satisfies

$$H_t(\underline{\phi} * \underline{\psi}) = (H_t \underline{\phi}) * (H_t \underline{\psi}).$$

Let's call it Tanaka's collision property. He also proved that if we put

$$P^f(t, a, B) = \int_{\underline{S}} \hat{f}(d\underline{b}) H_t(a \cdot \underline{b}, B), \quad a \in S, B \subset S, \uparrow$$

then it is a temporally inhomogeneous transition probability of McKean's non-constant transition mechanism, i.e.

$$P^f(t+s, a, B) = \int P^f(t, a, db) P^u_t(s, b, B),$$

$$u_t(B) = \int f(da) P^f(t, a, B).$$

2. Duality between Boltzmann's and S-equations

To have the duality, we need assumptions on existence of density functions with respect to a finite (or σ -finite) measure $\mu(da)$ on S .

Assumption.

(A.1) There exists a density function $\bar{\pi}_n(a_1, \dots, a_n; b)$ with respect to μ such that

$$\bar{\pi}_n(a_1, \dots, a_n; db) = \bar{\pi}_n(a_1, \dots, a_n; b) \mu(db).$$

Set

$$q'_n(b) = \int_{\underline{S}^n} \hat{\mu}(d\underline{a}) \bar{q}_n(\underline{a}) \bar{\pi}_n(\underline{a}, b)$$

and

$$c(b) = \sum_{n=2}^{\infty} q'_n(b) < \infty.$$

(A.2) There exists a transition density function $P_t^0(a, b)$ with respect to μ , and

$$P_t^0(a, db) = P_t^0(a, b) \mu(db), \text{ and}$$

$$\bar{P}_t^0(a, db) = \mu(db) P_t^0(b, a)$$

are transition probabilities of $\exp(-c_t)$ -subprocesses of right continuous Markov processes in duality with respect to μ .

Put

$$\begin{aligned} q_n(b) &= q'_n(b)/c(b), & \text{if } c(b) > 0, \\ &= 0, & \text{if } c(b) = 0. \end{aligned}$$

¹ $\underline{a} \cdot \underline{b} = (a_1, \dots, a_n, b_1, \dots, b_m)$ when $\underline{a} = (a_1, \dots, a_n)$ and $\underline{b} = (b_1, \dots, b_m)$.

$$\begin{aligned}\pi_n(b, d\underline{a}) &= \bar{q}_n(\underline{a}) \bar{\pi}_n(\underline{a}, b) \hat{u}(d\underline{a}) / q_n'(b), & \text{if } q_n'(b) > 0, \\ &= \text{any probability measure on } S^n, & \text{if } q_n'(b) = 0,\end{aligned}$$

where $\underline{a} = (a_1, \dots, a_n) \in S^n$.

Then we can write the Boltzmann's equation (1) in terms of density functions, i.e. if $f(da) = f(a)\mu(da)$ ($\|f\| \leq 1$), then $u_t(da)$ has a density function $u_t(a)$ with respect to μ , and it satisfies

$$u_t(a) = P_t^0 f(a) + \int_0^t ds P_s^0 \left\{ c \sum_{n=2}^{\infty} q_n \int_{S^n} \pi_n(\cdot, d\underline{b}) \hat{u}(\underline{b}) \right\} (a),^1$$

Clearly this is the so-called S-equation for a branching Markov process determined by $\{P_t^0, q_n, \pi_n\}$, $q_0 = q_1 = 0$ in the present case (cf. [2,8]), and the solution $u_t(a)$ of the S-equation is given in terms of the branching Markov process $(X_t, P_{\underline{a}}, \underline{a} \in \underline{S}^\partial)$ on $\underline{S}^\partial = \bigcup_{n=0} S^n$, $S^0 = \{\partial\}$, an extra point;

$$u_t(a) = E_a[\hat{f}(X_t)], \quad a \in S,$$

(cf. [2,8]).

PROPOSITION 1. There exists a transition density $T_t(\underline{a}, \underline{b})$ of the branching Markov process with respect to $\hat{\mu}$, and it satisfies the branching property in density form:

$$T_t(v_1 * v_2, \cdot) = T_t(v_1, \cdot) * T_t(v_2, \cdot).^2$$

PROOF. Put

$$T_t^0(\underline{a}, \underline{b}) = \prod_{i=1}^m P_t^0(a_i, b_i), \quad \text{when } \underline{a} \text{ and } \underline{b} \text{ are in } S^m,$$

and define

$$T_t^n(\underline{a}, \underline{b}) = E_{\underline{a}}[T_{t-\tau}^{n-1}(X_\tau, \underline{b})], \quad n \geq 1,$$

where τ is the first branching time of the process X_t .

¹ $\hat{u}(\underline{b}) = \prod_{j=1}^n u(b_j)$, when $\underline{b} = (b_1, \dots, b_n)$

² $\int_{\underline{S}^\partial} v_1 * v_2(d\underline{a}) F(\underline{a}) = \int_{\underline{S}^\partial} \int_{\underline{S}^\partial} v_1(d\underline{a}_1) v_2(d\underline{a}_2) F(\underline{a}_1 \cdot \underline{a}_2)$.

Then

$$T_t(\underline{a}, \underline{b}) = \sum_{n=0}^{\infty} T_t^n(\underline{a}, \underline{b})$$

provides a desired transition density. For the branching property refer to ([8], where the property is proved not in the density form).

Put

$$H_t(\underline{a}, d\underline{b}) = \hat{\mu}(d\underline{b})T_t(\underline{b}, \underline{a}).$$

Then if $\hat{\mu}$ is an excessive measure for T_t , H_t is a transition probability.

PROPOSITION 2. H_t satisfies Tanaka's collision property.

PROOF. Because of the branching property of $T_t(\underline{a}, \underline{b})$

$$\begin{aligned} H_t(\phi * \psi)(\underline{a}) &= \int \phi * \psi \hat{\mu}(d\underline{b})T_t(\underline{b}, \underline{a}) = \int (\phi \hat{\mu}) * (\psi \hat{\mu})(d\underline{b})T_t(\underline{b}, \underline{a}) \\ &= (\phi \hat{\mu} T_t) * (\psi \hat{\mu} T_t)(\underline{a}) = H_t \phi * H_t \psi(\underline{a}). \end{aligned} \quad 1$$

3. Excessive measures

This is a preparatory remark on excessive measures of Markov processes. Let T_t be the transition probability of a right continuous Markov process on a locally compact Hausdorff space with a countable open base.

PROPOSITION 3. Let μ be a measure which is finite on every compact sets. Then the following statements are equivalent

(i) μ is an excessive measure for the Markov process,

(ii) $\mu T_t f \leq \mu f$ for every $f \in C_k^+$,

(iii) $\mu(\alpha G_\alpha f) \leq \mu f$ for every $\alpha > 0$ and $f \in C_k^+$,

(iv) $\mu A u \leq 0$ for every u in $\{G_\alpha f; \alpha > 0, f \in C_k^+\}$,

where C_k^+ is the space of non-negative continuous functions with compact supports, G_α is the resolvent of T_t , and A is the generator of T_t .

PROOF. Clearly (i) \leftrightarrow (ii) \rightarrow (iii). (iii) \rightarrow (ii) is proved in Nagasawa-Sato ([6] lemma 3.3). Since $AG_\alpha f = \alpha G_\alpha f - f$, (iii) \leftrightarrow (iv) is clear.

¹ When $q_0 \neq 0$, this convolution property must be modified as we see in the later section.

4. Multiplicative excessive measures of CGW

In the theory of branching Markov processes the state space is $\underline{S}^\partial = \bigcup_{n=0}^{\infty} S^n$. However, since ∂ is a trap, there is no excessive measure on \underline{S}^∂ in general. Therefore we exclude ∂ and consider the process on $\underline{S} = \bigcup_{n=1}^{\infty} S^n$.

Let us call an excessive measure of the form $\hat{\mu}$ to be multiplicative. We will first prove the existence of m-excessive measures for continuous parameter Galton-Watson processes (abbreviated as CGW). We don't assume $q_0 = 0$ in this section ($q_0 = 0$ for the dual of collision processes). The existence of the unique invariant measure for CGW is proved in Harris([1], p.111) under the assumption $q_0 > 0$, but the invariant measure is not multiplicative except when $q_0 = q_2 = 1/2$.

The CGW process is a Markov chain on $Z^+ = \{0, 1, 2, \dots\}$ satisfying

- (i) $P_n[t < \tau] = \exp(-nct)$,
(ii) $P_n[X_\tau = m] = q_{m-n+1}$, $P_0[X_t = 0] = 1$,

where c is a non-negative constant, τ is the first jumping time, and $q_n \geq 0$, $\sum_{n=0}^{\infty} q_n = 1$ ($q_1 = 0$).

As mentioned above we exclude $\{0\}$ from the state space in the following. Since the generator of the transition semi-group of the CGW process is

$$Af(n) = cn \left\{ \sum_{m=n-1}^{\infty} q_{m-n+1} f_m - f_n \right\}, \quad n = 1, 2, \dots,$$

and since $\hat{\mu}$ is excessive if and only if $\hat{\mu}Af \leq 0$, we have

LEMMA 1. Take $\mu > 0$. $\hat{\mu} = \{\mu^n; n = 1, 2, \dots\}$ is m-excessive measure for CGW process if and only if

$$(2) \quad \sum_{k=0}^m (m+1-k) q_k \mu^{-k} - m\mu^{-1} \leq 0, \quad \text{for } m = 1, 2, 3, \dots$$

PROOF. For $g = G_\alpha f$, $f \in C_k^+$, we have

$$0 \geq \sum_{n=1}^{\infty} \mu^n Ag(n) = \sum_{n=1}^{\infty} \mu^n cn \left\{ \sum_{m=n-1}^{\infty} q_{m-n+1} g_m - g_n \right\}$$

$$= c \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{m+1} n q_{m-n+1} \mu^n - m \mu^n \right\} g_m.$$

Since we can find $f \in C_k^+$ such that $g_m > 0$ for $m = 1, 2, 3, \dots$, we have

$$\sum_{n=1}^{m+1} n q_{m-n+1} \mu^m - m \mu^n \leq 0,$$

multiplying μ^{-m-1} and putting $m-n+1 = k$,

$$\sum_{k=0}^m (m+1-k) q_k \mu^{-k} - m \mu^{-1} \leq 0.$$

LEMMA 2. $\hat{\mu}$ is m-excessive measure if and only if

$$(a) \quad 2q_0 \leq \mu^{-1}$$

$$(b) \quad h(\mu^{-1}) \leq \mu^{-1},$$

where $h(u) = \sum_{n=0}^{\infty} q_n u^n$ is the probability generating function of q_n .

REMARK. The lemma implies $q \leq \mu^{-1} \leq r$, where q and r are non-negative roots of $h(u) - u = 0$.

PROOF OF LEMMA 2. If $\hat{\mu}$ is excessive, Lemma 1 implies, putting $m=1$,

$$2q_0 \leq \mu^{-1},$$

and

$$\sum_{k=0}^{\infty} a_k^m q_k \mu^{-k} - m \mu^{-1} / (m+1) \leq 0,$$

where

$$\begin{aligned} a_k^m &= 1 - k/(m+1), & k \leq m, \\ &= 0, & k > m. \end{aligned}$$

Since a_k^m increases to 1 when m tends to infinity, we have

$$h(\mu^{-1}) - \mu^{-1} \leq 0.$$

Conversely, suppose (a) and (b) are satisfied. Then (2) follows by induction; assuming (2) for $m \geq 1$,

$$\begin{aligned}
& \sum_{k=0}^{m+1} (m+2-k)q_k \mu^{-k} - (m+1)\mu^{-1} \\
&= \sum_{k=0}^m (m+1-k)q_k \mu^{-k} + \sum_{k=0}^{m+1} q_k \mu^{-k} - m\mu^{-1} - \mu^{-1} \\
&\leq \sum_{k=0}^{m+1} q_k \mu^{-k} - \mu^{-1} \leq h(\mu^{-1}) - \mu^{-1} \leq 0.
\end{aligned}$$

THEOREM 1. M-excessive measures for a CGW process exist

- (i) if critical ($h'(1) = 1$) when and only when $q_0 = q_2 = 1/2$
and $\mu = 1$,
- (ii) if supercritical ($h'(1) > 1$) when and only when $q_0 \leq 1/2$
and $1 \leq \mu \leq 1/2q_0$,
- (iii) if subcritical ($h'(1) < 1$) when and only when $1/2 \leq q_0 \leq r/2$
and $1/r \leq \mu \leq 1/2q_0$,

where $0 \leq q \leq r$ are two roots of $h(u) - u = 0$.

PROOF. (i) When the process is critical, $q = r = 1$. Therefore we have $\mu = 1$ by (b) of lemma 2. Suppose $2q_0 < 1$, then

$$\sum_{n=2}^{\infty} q_n > 1/2, \text{ and}$$

$$\sum_{n=2}^{\infty} nq_n \geq 2\sum_{n=2}^{\infty} q_n > 2 \cdot (1/2) = 1,$$

which contradicts to $h'(1) = \sum_{n=2}^{\infty} nq_n = 1$. Thus we have $2q_0 = 1$ by (a) of lemma 2. Because $2q_0 = 1$, we have

$$\sum_{n=2}^{\infty} nq_n \geq 2\sum_{n=2}^{\infty} q_n = 1.$$

However, since the equality must be held, we have $q_3 = q_4 = \dots = 0$. Hence, $q_0 = q_2 = 1/2$.

(ii) When $0 \leq u \leq 1$, we have

$$(3) \quad h(u) - u \leq q_0 + (1-q_0)u^2 - u.$$

If the process is supercritical, $0 \leq q < r = 1$. By lemma 2 and the remark, we have $2q_0 \leq 1$. Then by (3)

$$h(2q_0) - 2q_0 \leq -q_0(1-2q_0)^2 \leq 0.$$

This implies $q \leq 2q_0$ and hence $2q_0 \leq \mu^{-1} \leq 1$.

(iii) When subcritical, $q = 1 < r$. Let us prove $1 \leq 2q_0$. Suppose $2q_0 < 1$, then by (3) $h(2q_0) - 2q_0 < 0$. This means the existence of a root q such that $q < 2q_0 < 1$, contradicting to subcriticality. If $1 \leq 2q_0 \leq r$, then $2q_0 \leq \mu^{-1} \leq r$ provides $\hat{\mu}$, completing the proof.

From the theorem we find an interesting fact. For example, let us suppose $q_0 + q_3 = 1$ ($q_3 \neq 0$). Then there is no m -excessive measure if $1/2 < q_0 < (1+\sqrt{5})/4$. When $q_0 + q_2 = 1$ ($q_0 \neq 0$), there is at least one m -excessive measure. When $q_0 = 0$ (this is the case if the CGW process is the dual of collision processes) the CGW process is supercritical and $\mu \geq 1$ provides m -excessive measure (cf. [9]).

It is quite natural to have the following question: If $q_0 \neq 0$ and if m -excessive measure $\hat{\mu}$ exists, what is the $\hat{\mu}$ -dual markov process? Let's prove that even in this case, the dual process has a property similar to Tanaka's collision property. Let H_t be the dual transition probability with respect to $\hat{\mu}$ defined by

$$H_t(n, m) = \mu^m T_t(m, n) \mu^{-n}, \quad n, m \geq 1$$

$$H_t(0, 0) = 1, \quad H_t(0, m) = 0, \quad m \geq 1,$$

where $T_t(n, m)$ is the transition probability of the CGW process. Defining a convolution $f * g$ of $f = (f_1, f_2, \dots)$ and $g = (g_1, g_2, \dots)$ by

$$(f * g)(n) = f_n + f_{n-1}g_1 + \dots + f_1g_{n-1} + g_n, \quad n \geq 1,$$

we have

PROPOSITION 4. For $n = 1, 2, 3, \dots$

$$(4) \quad H_t(f * g)(n) = (H_t f * H_t g)(n) + H_t f(n) H_t^0 g(0) + H_t^0 f(0) H_t g(n),$$

where $H_t^0(0, m) = \mu^m T_t(m, 0)$.

REMARK. When $q_0 = 0$, $T_t(m, 0) = 0$. Therefore (4) reduces to Tanaka's collision property

PROOF OF PROPOSITION. Let $n \geq 1$. By the definition of H_t

$$H_t(f * g)(n) = \sum_{m=0}^{\infty} (f\hat{\mu} * g\hat{\mu})(m) T_t(m, n) \mu^{-n},$$

where a term for $m=0$ is formally added because $T_t(0, n) = 0$, then by the branching property of T_t ,

$$= \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 0}} (f\hat{\mu} T_t)(n_1) (g\hat{\mu} T_t)(n_2) \mu^{-n}.$$

Because

$$\begin{aligned} f\hat{\mu} T_t(n) &= H_t f(n) \mu^n, \quad n \geq 1, \\ &= H_t^0 f(0), \quad n = 0, \end{aligned}$$

we get (4).

REMARK. Since the generator B of the $\hat{\mu}$ -dual Markov process of the CGW process is given by

$$Bf(m) = c \{ \sum_{n=1}^{m+1} n \mu^{n-m} q_{m+1-n} f_n - m f_m \}, \quad m \geq 1,$$

we have

PROPOSITION 5. The necessary and sufficient condition for $\hat{\lambda}$ to be m -excessive measure for the $\hat{\mu}$ -dual of the CGW process is

$$\mu q \leq \lambda \leq \mu r.$$

PROOF. Because

$$\sum_{m=1}^{\infty} \lambda^m Bf(m) = c \sum_{n=1}^{\infty} n f_n \lambda^n \{ \sum_{k=0}^{\infty} (\lambda/\mu)^{k-1} q_k - 1 \},$$

$\hat{\lambda}$ is m -excessive measure if and only if

$$\sum_{k=0}^{\infty} q_k (\lambda/\mu)^k - \lambda/\mu \leq 0,$$

thus we have

$$q \leq \lambda/\mu \leq r.$$

5. General cases

Let us extend the result in the previous section to wider class of branching Markov processes. Let S be a state space of one particle as usual. Given a right continuous Markov process on S with a transition probability P_t , bounded non-negative measurable function c , $q_n \geq 0$ with $\sum q_n(x) = 1$, and probability kernels $\pi_n(x, d\underline{y})$ defined on $S \times S^n$, $n = 0, 1, 2, \dots$, we can construct a branching Markov process $(X_t, P_{\underline{a}})$ on \underline{S}^{∂} (cf. [2, 8]).

Let \underline{P}_t and \underline{P}_t^0 be direct products of P_t and the transition probability P_t^0 of $\exp(-c_t)$ -subprocess, respectively. Then the transition probability \underline{T}_t of the branching Markov process satisfies for $\underline{x} = (x_1, x_2, \dots, x_n)$ and for a symmetric bounded measurable function $F = (F_n)$ on \underline{S}^{∂} ,

$$(5) \quad \underline{T}_t F(\underline{x}) = \underline{P}_t^0 F(\underline{x}) + \int_0^t dr \int_{S^n} \underline{P}_r^0(\underline{x}, d\underline{y}) \phi(\underline{y}, \underline{T}_{t-r} F),$$

where

$$\phi(\underline{y}, F) = \sum_{k=1}^n c(y_k) \sum_{m=1}^{\infty} q_m(y_k) \int_{S^m} \pi_m(y_k, d\underline{z}) F(y_1, \dots, y_{k-1}, \underline{z}, y_{k+1}, \dots, y_n).$$

This is verified through construction of the process (cf. [2, 8]). Therefore we have

$$(\underline{T}_t F(\underline{x}) - F(\underline{x}))/t = (\underline{P}_t^0 F(\underline{x}) - F(\underline{x}))/t + 1/t \cdot \int_0^t dr \int_{S^n} \underline{P}_r^0(\underline{x}, d\underline{y}) \phi(\underline{y}, \underline{T}_{t-r} F).$$

The second term is bounded by the uniform norm of F , because $\underline{P}_r^0(\underline{x}, d\underline{y}) \sum_{k=1}^n c(y_k)$ is a probability measure on S^n , and the rest part of the integrand on S^n is bounded by

$$\|F\| \sum_{m=0}^{\infty} q_m(y_k) \int_{S^m} \pi_m(y_k, d\underline{z}) \leq \|F\| \sum_{m=0}^{\infty} q_m(y_k) = \|F\|,$$

and it converges to $\phi(\underline{x}, F)$, ($t \downarrow 0$). Therefore if F belongs to the domain of the weak generator \underline{G}^0 of \underline{P}_t^0 , then so does to the domain of the weak generator \underline{A} of \underline{T}_t , and vice versa. Therefore we have $D(\underline{A}) = D(\underline{G}^0)$ and $\underline{A}f = \underline{G}^0 f + \phi(\cdot, F)$. Thus we have

PROPOSITION 6. Let \underline{G} be the weak generator of \underline{P}_t , then $D(\underline{A}) = D(\underline{G}^0) \subset D(\underline{G})$ and for symmetric $F \in D(\underline{A})$ and \underline{x} in S^n

$$\underline{A}F(\underline{x}) = \underline{G}F(\underline{x}) + \Psi(\underline{x}, F)$$

where $\Psi(\underline{x}, F) = \Phi(\underline{x}, F) - \sum_{k=1}^n c(x_k) F(\underline{x})$.

We assume in the following that there is a P_t -invariant measure dx on S .

Assumption B. (i) There exists a density function $\pi_n(x, \underline{z})$ with respect to $\widehat{d\underline{z}}$;

$$\pi_n(x, d\underline{z}) = \pi_n(x, \underline{z}) \widehat{d\underline{z}} :$$

$$(ii) \quad \underline{q}_0 = \int_S q_0(x) dx < \infty,$$

$$\underline{q}_k = \sup_{\underline{z} \in S^k} \int_S dx q_k(x) \pi_k(x, \underline{z}) < \infty, \quad k = 1, 2, 3, \dots$$

$$(iii) \quad c(x) \equiv 1.$$

Put

$$\underline{h}(u) = \sum_{k=0}^{\infty} \underline{q}_k u^k.$$

THEOREM 2. If there are non-negative solutions $0 \leq \eta_1 \leq \eta_2$ (at most two) of $\underline{h}(u) - u = 0$, then $\widehat{\mu d\underline{x}}$ (μ is a positive constant) is m -excessive measure for the branching Markov process determined by $\{P_t, c=1, q_n, \pi_n\}$, when

$$\eta_1 \leq 2\underline{q}_0 \leq \eta_2$$

and

$$1/\eta_2 \leq \mu \leq 1/2\underline{q}_0.$$

PROOF. Because μdx is an invariant measure of P_t ,

$$\int \widehat{\mu d\underline{x}} \underline{G}F = 0$$

for non-negative $F \in D(\underline{G})$. Therefore $\widehat{\mu d\underline{x}}$ is excessive for \underline{T}_t if and only if

$$\int \widehat{\mu d\underline{x}} \Psi(\underline{x}, F) \leq 0.$$

This is equivalent to

$$\sum_{k=0}^m (m+1-k) \sup_{\underline{z} \in S^k} \int \mu dx q_k(x) \pi_k(x, \underline{z}) / \mu^k - m \leq 0, \quad m = 1, 2, 3, \dots.$$

Therefore we get the theorem by the same arguments as in the case of CGW processes.

EXAMPLE 1. When $q_0 = 0$, $h(u) = u$ has two solutions $u=0$ and $\eta > 0$, where η is the solution of

$$\sum_{n=2}^{\infty} q_n u^{n-1} = 1.$$

If q_n is constant, then $q_n = q_n$ and $\eta = 1$. Thus $\mu \geq 1$ gives an m -excessive measure $\widehat{\mu dx}$. Therefore the duality described in §2 and §3 is justified in this case.

EXAMPLE 2. When dx is P_t -invariant probability measure, $q_n =$ constant, and $\pi_n(x, \underline{z}) \equiv 1, n=1, 2, \dots$, then $h(u) = h(u) = \sum q_n u^n$. Therefore we can state the same conclusion as for CGW processes.

EXAMPLE 3. (due to K.Uchiyama) Take n -dimensional Brownian motion. The Lebesgue measure dx is the invariant measure for the motion. Assume $\{q_n, \pi_n\}$ satisfy that q_0 is bounded by 1 and belongs to L^1 , and

$$q_k(x) = a_k(1 - q_0(x)), \quad \sum_{k=2}^{\infty} a_k = 1, \quad a_k \geq 0;$$

$$\pi_k(x, \underline{z}) = p_k(x - z_1) \times \dots \times p_k(x - z_k), \quad \text{for } \underline{z} = (z_1, \dots, z_k),$$

where p_k is a probability density function which belongs to L^k . Then a sufficient condition for existence of m -excessive measure $\widehat{\mu dx}$ is

$$2q_0 \leq 1/\mu, \quad a_0 + \sum_{k=2}^{\infty} a_k \|p_k\|^k (1/\mu)^k \leq 1/\mu.$$

where $\|p_k\|$ is L^k norm. This follows from

$$\sup_{\underline{z}} \int q_k(x) dx \pi_k(x, \underline{z}) \leq a_k \|p_k\|^k.$$

For example, take 1-dimensional Brownian motion and

$$q_0(x) = \exp(-ax^2), \quad q_2(x) = 1 - q_0(x),$$

$$\pi_2(x, \underline{z}) = (2\pi b)^{-1/2} \exp\{-1/2b \cdot ((x - z_1)^2 + (x - z_2)^2)\}.$$

Then $q_0 = \sqrt{\pi/a}$, $q_2 \leq 1/2\sqrt{\pi b}$, and a sufficient condition for $\hat{\mu}dx$ to be excessive is given by

$$2\sqrt{\pi/a} \leq 1/\mu \leq \sqrt{\pi b} + \sqrt{\pi b - 2\pi\sqrt{b/a}}$$

i.e., $ab \geq 4$.

APPENDIX. When $q_2 = 1$, the resolvent of the CGW process is

$$\begin{aligned} G(n,m) &= 1/m, & 1 \leq n \leq m, \\ &= 0, & \text{otherwise.} \end{aligned}$$

In the case, $\mu \geq 1$ gives an m -excessive measure $\hat{\mu}$. Let's find which initial distribution ν gives $\hat{\mu}$ as a potential such that $\nu G = \hat{\mu}$. If $\nu G = \hat{\mu}$, we have

$$\sum_{n=1}^m \nu_n = m\mu^m,$$

and

$$\nu_n = n\mu^n - (n-1)\mu^{n-1}, \quad n = 1, 2, 3, \dots,$$

is the one we need. Let's take this initial distribution. Then the reversed process of the CGW process from an L -time (cf. [7]) is a collision process with the $\hat{\mu}$ -dual transition probability.

Since the resolvent of the $\hat{\mu}$ -dual process is given by

$$\begin{aligned} \bar{G}(n,m) &= \mu^{m-n}/n, & 1 \leq m \leq n, \\ &= 0, & \text{otherwise,} \end{aligned}$$

and since $0 \leq \lambda \leq \mu$ gives m -excessive measure $\hat{\lambda}$ for the $\hat{\mu}$ -dual, the initial distribution ν which gives $\hat{\lambda}$ as a potential ($\nu \bar{G} = \hat{\lambda}$) is given by

$$\nu_m = m\lambda^m(1 - \lambda/\mu).$$

As a special case if we take $\mu = 1$ then $\nu_n = 1$ ($n \geq 1$) gives $\nu \bar{G}(m) = 1$ ($m \geq 1$), but $\hat{\lambda}$ is not a potential for the $\hat{\lambda}$ -dual process. The author does not know when m -excessive measure is given as a potential in general.

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