DAVID C. HEATH

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SKOROKHOD STOPPING IN DISCRETE TIME

by David HEATH[1]

I. Introduction. This note presents a more general version, for discrete-time processes, of a construction presented in [1] which generalized the construction presented by Skorokhod in [3] for stopping Brownian motion to achieve a given distribution. Many of the ideas presented here are due to Mokobodzki -- in particular the integral representation of one excessive measure in terms of another (3) is simply the version for measures of the theorem of Mokobodzki presented in [1].

II. Statement Of The Theorem. We use basically the same notation as Watanabe [4]. Let N be a sub-Markov kernel on the measurable space $(E,\mathcal{E})$ and let $(\Omega,\mathcal{F},X_k,\mathcal{F}_k,P_x,x\in E)$ be a realization of the Markov chain on E with transition kernel N. We shall suppose that on this space there is also a random variable S with distribution uniform on $[0,1]$ independent of $(X_k,k\geq 0)$ and measurable with respect to each $\mathcal{F}_k$. Let $G=\sum_{n\geq 0} N^n$ be the potential associated with N; we suppose that G1 is bounded. We then have the following:

THEOREM. Suppose μ_0 and μ_1 are (sub-) probability measures on $(E,\mathcal{E})$ with $\mu_0 G \geq \mu_1 G$. There is then an increasing collection $(A(s),s\in[0,1])$ of sets in $\mathcal{E}$ such that if T is defined by $T=\inf\{k\geq 0 : X_k\in A(S)\}$, then for every $B\in\mathcal{E}$, $P^{\mu_0}(X_T\in B)=\mu_1(B)$.

REMARK. It is easy to show that if there is any stopping time T satisfying the condition stated for the (sub-) probability measures μ_0 and μ_1, then $\mu_0 G \geq \mu_1 G$.

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III. <u>Proof Of The Theorem</u>. For $t \in [0,1]$ let $\bar{t}=1-t$ and define: $\nu_t = (\mu_1 - \bar{t}\mu_0)G$, $\sigma_t = \nu_t L_E$, and $\beta_t = \sigma_t - \nu_t$. Clearly β_t is a (positive) measure; since $\nu_t \leqslant \bar{t}\mu_0 G$ and $\sigma_t \leqslant \mu_1 G \leqslant \mu_0 G$, β_t is absolutely continuous with respect to $\mu_0 G$.

Let $A^o(t) = \left\{x \in E : \dfrac{d\beta_t}{d\mu_0 G} = 0\right\}$ where any version of the Radon-Nikodym derivative is used; $A^o(t)$ is then unique up to $\mu_0 G$-equivalence. Moreover we have $\sigma_t L_{A^o(t)} = \sigma_t$; this follows from the easy-to-prove result for measures corresponding to Corollary 6 of Mokobodzki [2].

We wish to show now that $(A^o(s), s \in [0,1])$ is "almost increasing": since for $s<t$, $\sigma_s + (t-s)\mu_0 G$ is excessive and dominates $\nu_s + (t-s)\mu_0 G = \nu_t$ we clearly have $\sigma_s + (t-s)\mu_0 G \geqslant \sigma_t$ which implies $\sigma_s - \nu_s \geqslant \sigma_t - \nu_t$, so (β_s) is a decreasing family. Thus if $s<t$, $\mu_0 G(A^o(s) \setminus A^o(t))=0$. We thus obtain that for $s<t$, $\sigma_s L_{A^o(t)} = \sigma_s$.

Since $\nu_t = \nu_s + (t-s)\mu_0 G$, we obtain $\sigma_t \leqslant \sigma_s + (t-s)\mu_0 G$ and applying $L_{A^o(t)}$ gives $\sigma_t \leqslant \sigma_s + (t-s)\mu_0 G L_{A^o(t)}$ which implies:

$$(1) \qquad \frac{\sigma_t - \sigma_s}{t-s} \leqslant \mu_0 G L_{A^o(t)} .$$

In the other direction, $\sigma_t \geqslant \nu_t = \nu_s + (t-s)\mu_0 G$ which, on $A^o(s)$, is equal to $\sigma_s + (t-s)\mu_0 G$, so, by the additivity of $L_{A^o(s)}$ on excessive measures (see Watanabe [4]) we obtain $\sigma_t \geqslant \sigma_t L_{A^o(s)} \geqslant \sigma_s L_{A^o(s)} + (t-s)\mu_0 G L_{A^o(s)} = \sigma_s + (t-s)\mu_0 G L_{A^o(s)}$, which implies

$$(2) \qquad \mu_0 G L_{A^o(s)} \leqslant \frac{\sigma_t - \sigma_s}{t-s} .$$

Combining (1) and (2) we conclude:

$$\sigma_1 - \sigma_0 = \int_0^1 \mu_0 G L_{A^o(s)}\, ds.$$

We now modify the collection $(A^o(s), s \in [0,1])$ to make it monotone: Let $\mathbb{Q}$ be the set of rationals in $[0,1]$; for $s \in [0,1]$ define

$$A(s) = \bigcap_{\substack{r \geq s \\ r \in \mathbb{Q}}} A^{o}(r).$$

Clearly $(A(s), s \in [0,1])$ is increasing, and $\mu_0 G(A^{o}(r) \setminus A(r)) = 0$ for every rational r, so $\mu_0 G L_{A(r)} = \mu_0 G L_{A^{o}(r)}$ for each $r \in \mathbb{Q}$. Since any two positive monotone functions on [0,1] which agree on $\mathbb{Q}$ have the same integral on [0,1], we obtain:

$$(3) \quad \mu_1 G = \int_0^1 \mu_0 G L_{A(s)} \, ds.$$

Now let T be defined as in the statement of the theorem; clearly the distribution of X_T (when the process is started according to μ_0) is given by $\int_0^1 \mu_0 H_{A(s)} \, ds$; we wish to show that this measure is μ_1.

Clearly the potential of this measure is $(\int_0^1 \mu_0 H_{A(s)} ds) G = \int_0^1 \mu_0 H_{A(s)} G \, ds$ = (see (2.12) of [4]) $\int_0^1 \mu_0 G K_{A(s)} \, ds$ = (by Theorem 1 of [4]) $\int_0^1 \mu_0 G L_{A(s)} \, ds$, which, according to (3) is the potential of μ_1. Applying (I-N) we obtain the desired conclusion.

REFERENCES

[1] D. HEATH, Skorokhod stopping via potential theory, Séminaire VIII*, vol. 381 (1974).

[2] G. MOKOBODZKI, Densité relative de deux potentiels comparables, Séminaire IV*, vol. 124 (1970).

[3] A. V. SKOROKHOD, Studies in the theory of random processes, Addison-Wesley (1965).

[4] T. WATANABE, On balayées of excessive measures and functions with respect to resolvents, Séminaire V*, vol. 191 (1971).

* refers to Séminaire de Probabilités, Université de Strasbourg, Lecture notes in Math., Springer.