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A PROBABILISTIC APPROACH TO NON-LINEAR DIRICHLET PROBLEM

By Masao Nagasawa

(1) Given a continuous strong Feller process (x_t, P_x) on a nice topological state space S (which will be called the base process), an open set D of S, a bounded continuous non-negative function c(x) in D (put c = 0 on the complement D^c of D), and bounded continuous functions $q_n(x)$ in D ($q_n = 0$ on D^c) satisfying

$$\sum_{n=0}^{\infty} |q_n(x)| = 1, \quad \text{for } x \in D.$$

Let us consider a non-linear Dirichlet problem, given a bounded measurable function ϕ on the boundary ∂D,

$$(1) \quad \begin{cases} \mathbf{A}u(x) + c(x)\left(\sum_{n=0}^{\infty} q_n(x) u(x)^n - u(x)\right) = 0, \text{ in } D, \\ u(b) = \phi(b), \text{ on } \partial D, \end{cases}$$

where $\mathbf{A}$ is Dynkin's characteristic operator for the base process (x_t, P_x).

We will show that solutions (not necessarily unique) of the non-linear Dirichlet problem can be obtained in terms of a branching Markov process under the condition $\|\phi\| \leq 1$.

(2) As is well known in the theory of Markov processes, the unique solution of linear Dirichlet problem

$$(2) \quad \begin{cases} \mathbf{A}u(x) = 0 \text{ in } D, \\ u(b) = \phi(b) \text{ on } \partial D, \\ \lim_{\substack{x \in D \\ x \to b \in \partial D}} u(x) = \phi(b), \text{ if } b \text{ is regular and } \phi \text{ is continuous at } b, \end{cases}$$

is obtained in terms of the base process under the assumption

$$P_x[T < \infty] = 1, \text{ for } x \in \bar{D},$$

where $T = \inf\{t \geqq 0;\ x_t \in \partial D\}$ is the first hitting time to the boundary ∂D. One expression is

$$u(x) = E_x[\phi(x_T)],$$

(cf. e.g. [1] p.32, Theorem 13.1). We have another expression in terms of the stopped process at the boundary

$$\bar{x}_t = x_{t \wedge T}.$$

Let $\bar{P}_t$ be the transition probability of $\bar{x}_t$, and f be a bounded measurable function on $\bar{D}$ which coincides with ϕ on the boundary. Then

$$(3) \qquad u(x) = \lim_{t \to \infty} \bar{P}_t f(x)$$

gives the same solution. The solution does not depend on the value of f in D. For, since $P_x[T < \infty] = 1$,

$$u(x) = \lim_{t \to \infty} E_x[f(\bar{x}_t)] = E_x[\lim_{t \to \infty} f(\bar{x}_t)]$$
$$= E_x[\phi(x_T)].$$

We will express solutions of (1) in the form of (3) taking the transition probability of a branching Markov process and $\hat{f}$ instead of $\bar{P}_t$ and f ($\hat{f}$ will be defined by (4)).

($\underline{\underline{3}}$) For simplicity, we assume $q_n(x) \geqq 0$, but the same arguments can be carried over for general case.

Let $(\mathbf{X}_t, \mathbf{P}_{\mathbf{x}})$ be $(\bar{x}_t, c, q_n)$-branching Markov process(*) on $\mathbf{S}$, where $\bar{x}_t$ is the stoped process of x_t at ∂D and

$$\mathbf{S} = \bigcup_{n=0}^{\infty} \bar{D}^n \cup \{\Delta\}.^{(**)}$$

For a bounded measurable function f on $\bar{D}$, we define $\hat{f}$ on $\mathbf{S}$ by

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(*) Cf. [2],[3]. Here, we take $\pi_n(x,d\mathbf{y}) = \delta_{(x,\cdots,x)}(d\mathbf{y})$, i.e. n-particles created at x start continuously.

(**) $\bar{D}^n$ is the n-fold Cartesian product of $\bar{D}$, and $\bar{D}^0 = \{\delta\}$ an extra point.

$$(4)\quad \begin{cases} \hat{f}(\mathbf{x}) = f(x_1)\times\cdots\times f(x_n), \text{ when } \mathbf{x} = (x_1,\cdots,x_n), \\ \hat{f}(\delta) = 1, \\ \hat{f}(\Delta) = 0. \end{cases}$$

If $\|f\| \leqq 1$, $\hat{f}$ is bounded on $\mathbf{S}$.

Let $\mathbf{P}_t$ be the transition probability of the branching Markov process. Taking a bounded measurable function f on $\bar{D}$ with the uniform norm $\|f\| \leqq 1$, we assume the existence of the limit

$$(5)\qquad u(\mathbf{x}) = \lim_{t\to\infty} \mathbf{P}_t\hat{f}(\mathbf{x}).$$

(We will discuss the existence of the limit in the next section.)

(I) $u(\mathbf{x})$ <u>is</u> $\mathbf{P}_t$-<u>invariant</u>.

For, $\mathbf{P}_s u(\mathbf{x}) = \lim_{t\to\infty} \mathbf{P}_{t+s}\hat{f}(\mathbf{x}) = u(\mathbf{x})$.

(II) $u(\mathbf{x})$ <u>is multiplicative</u>, <u>i.e.</u>, $u(\mathbf{x}) = \hat{u}(\mathbf{x})$.

For, since $\mathbf{P}_t$ satisfies the branching property

$$\mathbf{P}_t\hat{f}(\mathbf{x}) = \widehat{\mathbf{P}_t\hat{f}}\,(\mathbf{x}),$$

we have

$$u(\mathbf{x}) = \lim_{t\to\infty} \widehat{\mathbf{P}_t\hat{f}}\,(\mathbf{x}) = \hat{u}(\mathbf{x}).$$

(III) <u>If</u> $\hat{f}$ <u>belongs to the domain of the weak generator</u> $\mathbf{G}$ <u>of</u> $\mathbf{P}_t$, <u>then</u> f <u>belongs to the domain of the weak generator of</u> $\bar{P}^o_t$, <u>the transition probability of the killed process of</u> $\bar{x}_t$ <u>by</u> $\exp(-\int_0^t c(\bar{x}_s)ds)$.

Proof. $\mathbf{P}_t\hat{f}$ satisfies S-equation; for $x \in \bar{D}$,

$$\mathbf{P}_t\hat{f}(x) = \bar{P}^o_t f(x) + \int_0^t ds \int \bar{P}^o_s(x,dy)c(y)F(y,\mathbf{P}_{t-s}\hat{f}).$$

where $F(x,u) = \sum_{n=0}^{\infty} q_n(x)u(x)^n$. Therefore

$$\frac{\mathbf{P}_t\hat{f}(x) - f(x)}{t} = \frac{\bar{P}_t^o f(x) - f(x)}{t} + \frac{1}{t}\int_0^t ds\bar{P}_s^o(cF(\cdot,\mathbf{P}_{t-s}\hat{f}))(x).$$

The second term of the right hand side converges to cF(x,f) when t tends to zero. Therefore, if the left hand side converges, then so does the first term of the right hand side.

(IV) $u(\mathbf{x})$ defined by (5) belongs to the domain of the weak generator $\mathbf{G}$ of $\mathbf{P}_t$ and $\mathbf{G}u(\mathbf{x}) = 0$.

Since u is $\mathbf{P}_t$-invariant, u belongs to the domain of $\mathbf{G}$, and $\mathbf{G}u(\mathbf{x}) = 0$.

Therefore, $u(x)$, $x \in \bar{D}$ belongs to the domain of the weak generator of $\bar{P}_t^o$, and by Kac's theorem it belongs to the domain of the weak generator of $\bar{P}_t$. Thus we have, by (II), (III) and (IV),

PROPOSITION 1. If $u(x)$, $x \in \bar{D}$, defined by (5) exists, then it satisfies

$$\mathbf{A}u(x) + c(x)\{\sum_{n=0}^{\infty} q_n(x)u(x)^n - u(x)\} = 0 \text{ in } D,$$

and $u(b) = f(b)$, $b \in \partial D$, where $\mathbf{A}$ is Dynkin's characteristic operator of the base process.

Remark. Even when $\|f\| \not\leq 1$, if the limit in (5) exists and if $F(\cdot,f)$ is bounded, then (I)$\sim$(IV) and Proposition 1 hold.

(4) Let τ be the killing time of the base process by $\exp(-\int_0^t c(x_s)ds)$ and T be the first hitting time to the boundary ∂D, and we assume

(6) $$P_x[T < \tau] \geq \varepsilon > 0, \quad \text{for all } x \in \bar{D}.$$

Remark. (6) is satisfied if $E_x[\exp(-\|c\|T)] \geq \varepsilon$. For, $P_x[T < \tau] = E_x[\exp(-\int_0^T c(x_s)ds)] \geq E_x[\exp(-\|c\|T)]$.

LEMMA 1. Under the assumption (6)

$\mathbf{P}_x[X_t^i \in \partial D$ for all i or $\mathbf{X}_t = \delta$ at some $t < \infty$, or the number of particles in D tends to ∞ when $t \to \infty] = 1$,

where $\mathbf{X}_t = (X_t^1, \cdots, X_t^{n(t)})$.

Proof. Let σ be the first hitting time to $\bar{D}^m$, and define sequences of Markov times $\{\sigma_n\}$ and $\{\eta_n\}$ by

$$\sigma_1 = \sigma, \qquad \eta_1 = \sigma_1 + \tau \circ \theta_{\sigma_1},$$
$$\sigma_2 = \eta_1 + \sigma_1 \circ \theta_{\eta_1} \qquad \eta_2 = \sigma_2 + \tau \circ \theta_{\sigma_2},$$

and so on. Then

$$\begin{aligned} &\mathbf{P}_x[\mathbf{X}_t \text{ visits } \bar{D}^m \text{ infinitely often}] \\ &= \mathbf{P}_x[\bigcap_n \{\sigma_n < +\infty\}] \\ &= \lim_{n\to\infty} \mathbf{P}_x[\sigma_n < +\infty] \\ &\leqq \lim_{n\to\infty} (1-\varepsilon)(1-\varepsilon^m)^n = 0, \end{aligned}$$

because

$$\begin{aligned} \mathbf{P}_x[\sigma_1 < +\infty] &\leqq 1 - P_x[T < \tau] \leqq 1 - \varepsilon, \\ \mathbf{P}_x[\sigma_2 < +\infty] &= \mathbf{E}_x[\mathbf{P}_{\mathbf{X}_{\sigma_1}}[\tau + \sigma_1 \circ \theta_\tau < +\infty]; \sigma_1 < +\infty] \\ &\leqq \mathbf{E}_x[(1 - \prod_{i=1}^m P_{X_{\sigma_1}^i}[T < \tau]); \sigma_1 < +\infty] \\ &\leqq (1-\varepsilon)(1-\varepsilon^m), \end{aligned}$$

and so on. Thus we have the lemma.

As a corollary of Lemma 1, we have

PROPOSITION 2. Given a measurable function ϕ on the boundary ∂D with $\|\phi\| \leqq 1$, set

$$f = \begin{cases} \phi & \text{on } \partial D \\ 0 & \text{in } D. \end{cases} \tag{7}$$

Then,

(8) $$u(x) = \lim_{t\to\infty} \mathbf{E}_x[\hat{f}(\mathbf{X}_t)], \quad x \in \bar{D}, \text{ exists.}$$

PROPOSITION 3. Given ϕ and define f as in Proposition 2, then u defined by (8) satisfies

(9) $$\lim_{\substack{x\in D\\ x\to b\in\partial D}} u(x) = \phi(b),$$

if b is a regular point of the boundary ∂D and if ϕ is continuous at b.(*)

Proof.

$$|u(x) - \phi(b)| \leqq \lim_{t\to\infty} \mathbf{E}_x[|\hat{f}(\mathbf{X}_t) - \phi(b)|].$$

Put $B = \{X_t$ hits first to the boundary before branching$\}$.

$$I = \lim_{t\to\infty} \mathbf{E}_x[|\hat{f}(\mathbf{X}_t) - \phi(b)|;B]$$

$$= \lim_{t\to\infty} \mathbf{E}_x[|f(\bar{x}_t) - \phi(b)|;B],$$

because $\mathbf{X}_t = \bar{x}_t = x_{t\wedge T}$ on B;

$$\leqq E_x[|\phi(x_T) - \phi(b)|;\ T < \tau]$$

$$\leqq E_x[|\phi(x_T) - \phi(b)|],$$

where E_x is the expectation with respect to the base process. If b is regular and if ϕ is continuous at b, then there exists a neighbourhood U_b of b and

(10) $$E_x[|\phi(x_T) - \phi(b)|] < \varepsilon \quad \text{for all } x \in U_b,$$

(cf.e.g. [1] p.32, Theorem 13.1). Thus we have $I < \varepsilon$.

$$II = \lim_{t\to\infty} \mathbf{E}_x[|\hat{f}(\mathbf{X}_t) - \phi(b)|;B^c]$$

$$\leqq 2\|\phi\|\,\mathbf{P}_x[B^c] \leqq 2P_x[T \geqq \tau] \leqq 2(1-P_x[T < \tau])$$

$$\leqq 2(1 - P_x[T < s < \tau]), \quad \text{for any } s > 0.$$

(*) The regularity is for the base process.

$$P_x[T < s < \tau] = P_x[\exp(-\int_0^s c(x_s)ds),\ T < s]$$

$$\geqq \exp(-\|c\|s)P_x[T < s].$$

Take s sufficiently small so that $\exp(-\|c\|s) \geqq 1 - \varepsilon$. Since $P_x[T < s]$ is lower semicontinuous in x (cf.e.g.[1] p.28 Lemma 13.2) and $P_b[T < s] = 1$ because b is regular, there exists a neighbourhood U_b' of b such that

$$P_x[T < s] \geqq 1 - \varepsilon, \quad \text{for all } x \in U_b'.$$

Therefore

$$P_x[T < s < \tau] \geqq (1-\varepsilon)^2 > 1 - 2\varepsilon.$$

Thus we have $II < 4\varepsilon$, and

$$|u(x) - \phi(b)| < 5\varepsilon, \text{ for all } x \in U_b \cap U_b'.$$

Since ε is arbitrary, (9) is proved.

Remark. We assumed $\|\phi\| \leqq 1$ in Proposition 3. However, if ϕ is bounded and if the limit exists in (8), then (9) is valid.

Thus we have

THEOREM. *Under the assumption* (6), *there exists*

$$u(x) = \lim_{t \to \infty} \mathbf{E}_x[\hat{f}(\mathbf{X}_t)], \quad x \in \bar{D},$$

where f *is defined by* (7) *for a given* ϕ *on* ∂D ($\|\phi\| \leqq 1$), *and* u *is a solution of non-linear Dirichlet problem* (1) *satisfying the boundary limit property* (9).

We proved Theorem in the case of $q_n \geqq 0$. When q_n is not non-negative, we can prove the theorem using the branching Markov process with sign (cf.[3],[4]) instead of usual branching Markov process.

Moreover, there is no difficulty to generalize the result to the system

$$\begin{cases} \mathbf{A}_i u_i + c_i\{\sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} q^i_{n_1\cdots n_k}(u_1)^{n_1}\cdots(u_k)^{n_k} - u_i\} = 0 \\ \qquad\qquad\qquad \text{in } D, \text{ for } i = 1,2,\ldots,k, \\ u_i(b) = \phi_i(b) \text{ on } \partial D, \end{cases}$$

where $\Sigma\cdots\Sigma|q^i_{n_1\cdots n_k}(x)| = 1$, $x \in D$ (= 0 outside D).
To do this, what we need is just to introduce an appropriate branching Markov processes(cf. [3] pp.505-507).

(5) Instead of (7), let us take

$$f = \begin{cases} \phi & \text{on } \partial D, \\ g & \text{in } D, \end{cases} \tag{11}$$

as an initial value, where g is a measurable function in D with $\|g\| \leqq 1$. When $\|g\| < 1$, the limit

$$u(x) = \lim_{t\to\infty} \mathbf{E}_x[\hat{f}(\mathbf{X}_t)] \tag{12}$$

exists and it does not depend on the choice of the initial value g in D. Let n_t^D be the number of particles in D at t. By lemma 1,

$$\begin{aligned} u(x) &= \lim_{t\to\infty} \mathbf{E}_x[\hat{f}(\mathbf{X}_t);\ X_s^i \in \partial D \text{ for all } i \text{ or } \mathbf{X}_s = \delta \\ &\qquad\qquad \text{at some } s < \infty] \\ &+ \lim_{t\to\infty} \mathbf{E}_x[\hat{f}(\mathbf{X}_t);\ n_s^D \uparrow \infty \text{ when } s \uparrow \infty], \end{aligned} \tag{13}$$

where the second term is equal to zero when $\|g\| < 1$ and the first term does not depend on g.

In general, the limit in (12) depends on the choice of the initial value g in D if

$$\mathbf{P}_x[n_t^D \uparrow \infty \text{ when } t \uparrow \infty] > 0 \tag{14}$$

at some point x_0 in D. For example, taking $\phi \equiv 1$ on the boundary for simplicity, if we take $f_1 \equiv 1$ on $\bar{D}$, then

$$u_1(x) = \lim_{t\to\infty} \mathbf{E}_x[\hat{f}_1(\mathbf{X}_t)] = 1, \text{ for all } x \in \bar{D},^{(*)}$$

while if we take $f_0 = 1$ on ∂D (= 0 in D), then

$$u_0(x) = \lim_{t\to\infty} \mathbf{E}_x[\hat{f}_0(\mathbf{X}_t)]$$

takes value less than one at $x_0 \in D$, because of (13) and (14). Actually, $u_0(x)$ is the extinction probability of particles from D (cf.[5],[6]).

Remark. In order to express the stochastic solution, defined in (5) or (8), in terms of "the first hitting time to the boundary", we must introduce a vector of hitting times of every branches of the branching Markov process. When $X_t^i \in \partial D$ for all i or $\mathbf{X}_t = \delta$ at some $t < \infty$, let T_i be the first hitting time of X_t^i to the boundary, where $\mathbf{X}_t = (X_t^1, \cdots, X_t^{n(t)})$. Then put

$$\mathbf{T} = (T_1, T_2, \cdots, T_n),$$

(under the assumption, the total number of paticles is finite, say, n). When the number of particles in D tends to infinity, let's put $\mathbf{T} = \infty$. Let us call $\mathbf{T}$ the first hitting time of the branching Markov process to the boundary. Then we have

$$(15) \qquad u(x) = \lim_{t\to\infty} \mathbf{E}_x[\hat{f}(\mathbf{X}_t)] = \mathbf{E}_x[\hat{f}(\mathbf{X}_\mathbf{T}); \mathbf{T} < \infty] = \mathbf{E}_x[\hat{\phi}(\mathbf{X}_\mathbf{T}); \mathbf{T} < \infty],$$

where

$$\mathbf{X}_\mathbf{T} = (X_{T_1}^1, X_{T_2}^2, \cdots, X_{T_n}^n).$$

(*) We assume here that the branching Markov process does not explode in finite time. When explosion occurs, $u_1(x) = 1$ - explosion probability.

References

[1] E.B.Dynkin, Markov Processes, Vol.II.(1965) Springer.

[2] N.Ikeda-M.Nagasawa-S.Watanabe, Branching Markov processes I,II,III, Journal of Math.Kyoto Univ. Vol.8 (1968)233-278,365-410, Vol.9(1969)95-160.

[3] M.Nagasawa, Construction of branching Markov processes with age and sign, Kodai Math.Sem.Rep. Vol.20(1968) 469-508.

[4] M.Nagasawa, Branching property of Markov processes, Lecture note in Math. Vol.258, Séminaire de Probabilités de Strasbourg VI,(1973)177-198, Springer.

[5] B.A.Sevast'yanov, Branching stochastic processes for particles diffusing in a bounded domain with absorbing boundaries, Theory of Probability and its application Vol.3(1958)111-126(English translation).

[6] S.Watanabe, On the branching process for Brownian particles with an absorbing boundary, Journal of Math.Kyoto Univ. Vol.4(1965)385-398.

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