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# One-dimensional Potential Embedding

by R.V. Chacon and J.B. Walsh

Let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion from zero. Skorokhod's embedding theorem tells us that if $\mu$ is a probability measure of mean zero and finite second moment there exists a stopping time $T$ such that $B_T$ has distribution $\mu$, and such that $E\{T\} < \infty$. (We say $\mu$ is embedded in B.) This theorem has inspired a large number of extensions and ramifications. Notably, H. Rost has shown how to decide if a given measure can be embedded into a given Markov process. In general, one must use randomized stopping times for this embedding, but non-randomized stopping times suffice in many interesting special cases. For n-dimensional Brownian motion, for instance, one can restrict oneself to natural stopping times as long as the target measure has a continuous potential [1]. The construction of the stopping time in that paper is somewhat complicated to describe in general, but it is quite transparent in the case $n = 1$, where it serves to prove Skorokhod's theorem. We thought it would be amusing to give an account of this construction: not only is it one of the few places we know of where one can use classical one-dimensional potential theory with a straight face, but the heart of the proof can be explained with four pictures.

Let's recall a few facts about potential theory on the line. The potential kernel is $k(x) = -|x|$. If $\mu$ is a measure on R, its potential $U\mu$ is given by

$$U\mu(x) = -\int_{-\infty}^{\infty} |x - y| \, \mu(dy) .$$

Then:

1° $U\mu(x)$ is a concave function, finite iff $\int |y| \, \mu(dy) < \infty$.

2° If $\mu$ is a probability measure with mean zero and if $\delta_0$ is the unit mass at zero, $U\mu \leq U\delta_0$. Furthermore, $U\delta_0(x) - U\mu(x) \to 0$ as $|x| \to \infty$.

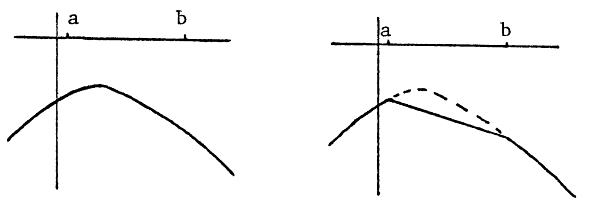
3° If $\mu, \mu_1, \mu_2, \ldots$ are measures such that $U\mu_n(x) \to U\mu(x)$ for all $x$, then $\mu_n \to \mu$ weakly.

We need one further fact concerning the balayage of potentials.

$4^{\circ}$ Let $\nu$ be a probability measure with finite mean and let $[a,b]$ be a finite interval. Let $B$ be Brownian motion with initial distribution $\nu$ and define

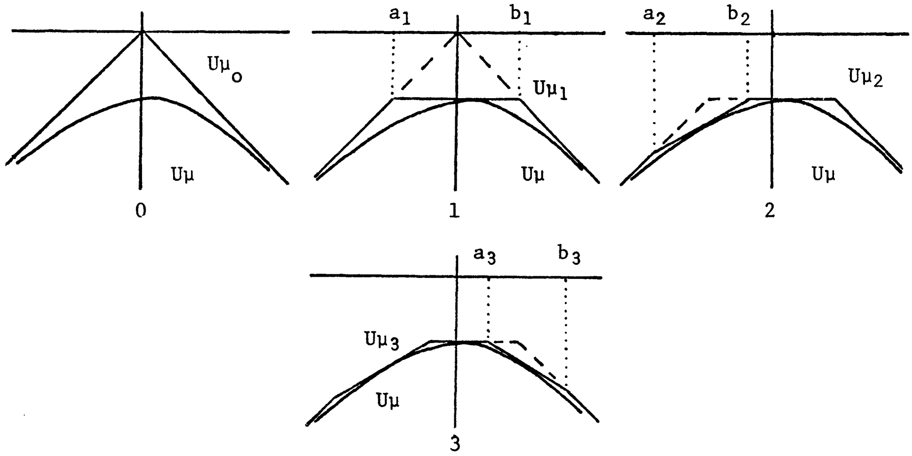
$$T_{ab} = \inf \{t : B_t \leq a \quad \text{or} \quad B_t \geq b\} .$$

Then if $\nu'$ is the distribution of $B_{T_{ab}}$, $U\nu'$ is linear in $[a,b]$ and $U\nu' = U\nu$ outside $[a,b]$:




$U\nu$ $\qquad$ $U\nu'$

Now let $\mu$ be a probability measure with mean zero and let $B$ be a Brownian motion from zero. We will construct an increasing sequence $T_0 \leq T_1 \leq T_2 \leq \ldots$ of (non-randomized) stopping times increasing to a limit $T$, such that $B_T$ has distribution $\mu$. Let $\mu_n$, $n = 0, 1, \ldots$ be the distribution of $B_{T_n}$. The following pictures will explain our construction:




... and the following words will explain our pictures.

Choose $T_0 = 0$ , so $\mu_0 = \delta_0$ . Then the potentials $U\mu_0$ $(= -|x|)$ and $U\mu$ are as shown in 0 . (see $2^o$). Next, choose an $x \ni U\mu(x) < U\mu_0(x)$ , draw a tangent to the graph of $U\mu$ at $x$ , and let $a_1$ and $b_1$ be as in 1 . (The fact that $a_1$ and $b_1$ are finite follows from $2^o$.) Now let $T_1 = T_{a_1 b_1}$ (= first exit from $(a_1, b_1)$.) Then by $4^o$ , $B_{T_1}$ has the distribution $\mu_1$ whose potential is graphed in 1 . Continuing in the same vein, choose another $x \ni U\mu(x) < U\mu_1(x)$ , and draw a tangent to the graph of $U\mu$ at $x$ . If $a_2$ and $b_2$ are as in 2 , let $T_2 = T_1 + T_{a_2 b_2} \circ \theta_{T_1}$ (i.e. the first exit from $(a_2, b_2)$ after $T_1$ . Here, $\theta_t$ is the usual translation operator). Then $U\mu$ and $U\mu_2$ are as shown in 2 . At the next step, we set $T_3 = T_2 + T_{a_3 b_3} \circ \theta_{T_2}$ , etc. At each stage, $U\mu_n$ is piecewise linear and $U\mu_n \geq U\mu$ . We haven't been too specific as to exactly how we choose the functions $U\mu_n$ , and in fact it doesn't much matter. What is important is that we can choose them so that they decrease to the function $U\mu$ - indeed, any concave function can be written as the infimum of a countable number of affine functions, and each $U\mu_n$ is just the infimum of finitely many.

But now $U\mu_n \downarrow U\mu$ , so that by $3^o$ , $\mu_n \to \mu$ weakly. At the same time, $B_{T_n} \to B_T$ by continuity, hence the distribution of $B_T$ is $\mu$ . It remains to show that $E\{T\} = \int x^2 d\mu$. The main step in this is the observation that $B_t^2 - t$ is a martingale. However, to conclude from this that $E\{T\} = E\{B_T^2\}$ we need an additional argument to show that $E\{T\} < \infty$ .

There are a number of ways to see this. Here is one in the spirit of this paper; based on the fact that if $\nu$ is a measure on the line having mean zero and potential $U\nu$, then $\int x^2 d\nu$ is equal to the area between the curves $y = U\nu(x)$ and $y = U\delta_0(x)$ $(= -|x|)$. (This follows from a direct calculation, for, since $\int x d\nu = 0$, we can write

$$-|y| - U\nu(y) = \begin{cases} 2\int_{-\infty}^{y} (y-x)\nu(dx) & \text{if } y \leq 0 \\ 2\int_{y}^{\infty} (x-y)\nu(dx) & \text{if } y > 0, \end{cases}$$

so that the area between the two curves is

$$\int_{-\infty}^{\infty} (-|y| - U\nu(y))dy = 2\int_{-\infty}^{0} dy \int_{-\infty}^{y} (y-x)\nu(dx) + 2\int_{0}^{\infty} dy \int_{y}^{\infty} (x-y)\nu(dx)$$

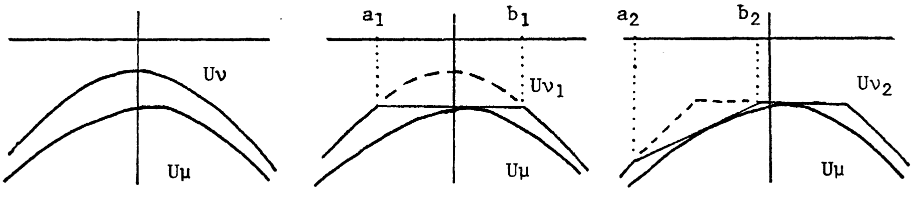
$$= 2\int_{-\infty}^{0} \nu(dx)\int_{x}^{0} (y-x)dy + 2\int_{0}^{\infty} \nu(dx)\int_{0}^{y} (x-y)dy$$

$$= \int_{\infty}^{\infty} x^2 d\nu \text{ .)}$$

But now, since $B_{t\wedge T_n}^2 - t\wedge T_n$ is a martingale and $B_{t\wedge T_n}$ is bounded, we can let $t \to \infty$ to see that $E\{T_n\} = E\{B_{T_n}^2\}$. This last equals the area between $-|x|$ and the potential $U\mu_n$ of the distribution of $B_{T_n}$, which was itself constructed to be between $-|x|$ and $U\mu(x)$. Thus this area is bounded by the area between $-|x|$ and $U\mu(x)$, i.e. by $\int x^2 d\mu$. Since there is clearly equality in the limit,

$$E\{T\} = \lim_n E\{T_n\} = \int x^2 d\mu \ .$$

Three remarks are worth adding here. First, Dubins' scheme for constructing the "Skorokhod time" [2] is actually a special case of the above. Indeed, his method gives what is essentially a canonical method for choosing the intervals $[a_n, b_n]$. Secondly, we need not necessarily start with the distribution $\delta_0$ . Indeed, if $\mu$ and $\nu$ are probability distributions with finite potentials, and if $U\mu \leq U\nu$ , let $B_t$ have initial distribution $\nu$ . Then there exists a non-randomized stopping time T for which the distribution of $B_T$ is $\mu$ . The proof is by picture:




Finally, if $\mu$ does not have a finite second moment but, say, $\int x^p \, d\mu < \infty$ for some $p > 1$ , this method yields a stopping time T for which $E\{T^{p/2}\} < \infty$ , though the proof of this last is more complicated.

References

[1] Baxter, J. R. and Chacon, R. V., Potentials of stopped distributions Illinois J. of Math., 18(1974), 649-656.

[2] Chacon, R. V., Potential Processes, to appear, Trans. Amer. Math. Soc.

[3] Dubins, L. E., On a theorem of Skorokhod, Ann. Math. Stat., 39(1968), 2094-2097.