RICHARD M. DUDLEY
SAM GUTMANN

Stopping times with given laws

Séminaire de probabilités (Strasbourg), tome 11 (1977), p. 51-58
<http://www.numdam.org/item?id=SPS_1977__11__51_0>

Stopping times with given laws

by R. M. Dudley[1] and Sam Gutmann

Abstract. Given a stochastic process X_t, $t \in T \subset R$, and $s \in R$, then a) iff b): a) For every probability measure μ on $]s,\infty]$, there is a stopping time τ for X_t with law $L(\tau) = \mu$; b) If A_t is the smallest σ-algebra for which X_u are measurable for all $u \le t$, then P restricted to A_t is nonatomic for all $t > s$.

This note began with a question of G. Shiryaev, connected with the following example. Let W_t be a standard Wiener process, $t \in T = [0,\infty]$. Any exponential distribution on $]0,\infty]$ will be shown to be the law of a stopping time. Using this, one can obtain a standard Poisson process P_t from W_t by a non-anticipating transformation, $P_t = g(\{X_s: s \le t\})$.

Definitions. A probability space (Ω, A, P), or A (for P), is nonatomic iff for every $A \in A$ and $0 < p < P(A)$ there is a $B \subset A$, $B \in A$, with $P(B) = p$.

A stochastic process (here) is a map $X: (t,\omega) \longrightarrow X_t(\omega)$, $t \in T \subset R$, $\omega \in \Omega$, where (Ω, A, P) is a complete probability space. Each X_t has values in some measurable space $(S_t, \mathcal{F}_t)$ where S_t is a set, $\mathcal{F}_t$ is a σ-algebra of subsets of S_t, and X_t is measurable from $\mathcal{A}$ to $\mathcal{F}_t$. Let $\mathcal{A}_t$ be the smallest sub-σ-algebra of $\mathcal{A}$ for which X_s is measurable for all $s \leq t$ and for which $A \in \mathcal{A}_t$ whenever $A \subset B$ and $P(B) = 0$. Let $NA(X) := \inf\{t: \mathcal{A}_t$ is nonatomic$\}$.

Note. X_t is said to be nonatomic if $\mathcal{F}_t$ is nonatomic for $P \circ X_t^{-1}$. Then if X_t (or any other $\mathcal{A}_t$-measurable random variable) is nonatomic, $\mathcal{A}_t$ is nonatomic. After R. Dudley proved Theorem 2 below, and a weaker form of Theorem 1 considering only nonatomicity of individual X_t, S. Gutmann found the present Theorem 1.

A stopping time for the process X_t is a random variable τ on Ω with values in $]-\infty,\infty]$ such that for any $t \in T$, $\{\omega: \tau(\omega) < t\} \in \mathcal{A}_t$.

Theorem 1. For any stochastic process X_t and $s \in R$, $s \geq NA(X)$ iff for every Borel probability measure (law) μ on $]s,\infty]$, there is a stopping time τ for X_t with $\mathcal{L}(\tau) = \mu$. If $s \in T$ and $\mathcal{A}_s$ is nonatomic, the same holds for any μ on $[s,\infty]$.

Proof. If $\mathcal{A}_s$ is nonatomic, and μ is any law on $[s,\infty]$, then there is an $\mathcal{A}_s$-measurable random variable g with $\mathcal{L}(g) = \mu$, as follows. We take a nonatomic countably generated sub-σ-algebra

$\mathcal{B}$ of $\mathcal{A}_s$. Then there is a measure-preserving map ϕ of $(\Omega, \mathcal{B}, P)$ into $[0,1]$ with Lebesgue measure (Halmos, 1950, p. 173). Its range has outer measure 1. Let $F_\mu(t) := \mu(]-\infty, t])$, $F_\mu^{-1}(x) := \inf\{t: F_\mu(t) \geq x\}$. Then $g = F_\mu^{-1} \circ \phi$ is as desired.

Now $\{\omega: g(\omega) < t\}$ is empty for $t \leq s$, and belongs to $\mathcal{A}_s \subset \mathcal{A}_t$ for $t > s$. Thus, g is a stopping time, as desired. If for all $\epsilon > 0$ there is a stopping time τ with uniform distribution on $(s, s+\epsilon)$ then τ is $\mathcal{A}_{s+\epsilon}$-measurable, hence $\mathcal{A}_{s+\epsilon}$ is nonatomic and $s \geq NA(X)$.

Now suppose $\mathcal{A}_s$ has an atom, $t(n) \downarrow s$ with $\mathcal{A}_{t(n)}$ nonatomic, and μ is any law on $]s, \infty]$. Let $t(0) = +\infty$, $p_n := \mu(]t(n), t(n-1)])$, $n = 1, 2, \ldots$. By assumption, $\Sigma_{n \geq 1} p_n = 1$. Suppose there is a stopping time $\mathcal{T}$ with $P(\mathcal{T} = t(n)) = p_n$ for all n, and $\{\mathcal{T} = t(n)\} \in \mathcal{A}_{t(n)}$.

Whenever $p_n > 0$, the conditional law of P restricted to $\mathcal{A}_{t(n)}$, given $\mathcal{T} = t(n)$, is nonatomic. Thus for each n there is a real $\mathcal{A}_{t(n)}$-measurable random variable g_n such that

$$P(g_n \in A | \mathcal{T} = t(n)) = \mu(A \cap]t(n), t(n-1)])/p_n.$$

Let $\tau := g_n$ iff $\mathcal{T} = t(n)$. Then τ is measurable and $\mathcal{L}(\tau) = \mu$. If $t \in T$ and $t \leq s$, $\{\tau < t\}$ is empty. If $t > s$,

$$\{\tau < t\} = \left(\bigcup_n \{\mathcal{T} = t(n) < t(n-1) < t\}\right) \cup \{\mathcal{T} = t(n) < t \leq t(n-1)$$
$$\text{and } g_n < t\} \in \bigcup_{t(n)<t} \mathcal{A}_{t(n)} \subset \mathcal{A}_t.$$

Then τ is a stopping time with law μ. The problem is now reduced to the case $T = \{t(n)\}$ or equivalently where T is the set of negative integers and all A_t are nonatomic. This will be treated in the following Lemma and Theorem 2.

Lemma. Given a nonatomic probability space (Ω, A, P) and events A, B, D with $A \subset B$, $P(B) > 0$ and $P(D) > 0$, there is an event $C \subset D$ such that $P(C|D) = P(A|B)$ and $P(C \Delta A) \leq 2P(B \Delta D)$, where $C \Delta A := (C \setminus A) \cup (A \setminus C)$.

Proof. Let $p := P(D)P(A)/P(B)$, $E := A \cap D$. If $p \leq P(E)$, choose $C \subset E$ with $P(C) = p$. Then $P(C \Delta A) = P(A \setminus C)$ $= P(A) - p \leq P(B \setminus D)$ since $P(A)P(B) \leq P(A)P(D) + P(A)P(B \setminus D)$.

$$\leq P(A)P(D) + P(B)P(B \setminus D).$$

If $p > P(E)$, choose C with $E \subset C \subset D$ and $P(C) = p$. Then $P(A \Delta C) = P(A \setminus D) + p - P(E)$.

We need to prove

$P(A \setminus D)P(B) + P(A)P(D) \leq P(B)P(E) + 2P(B)P(B \Delta D)$. Now $P(A \setminus D) \leq P(B \setminus D)$, and $P(A)P(D) \leq P(A)P(B) + P(A)P(D \setminus B)$

$\leq P(B)P(E) + P(B)P(A \setminus D) + P(B)P(D \setminus B)$

$\leq P(B)P(E) + P(B)P(B \Delta D)$, as desired. In either case $C \subset D$ and $P(C|D) = P(A|B)$, Q.E.D.

Note. If $B = \Omega$ and $A = B \setminus D$, then $P(C \Delta A) = P(A) + P(D)P(A)$ $= 2P(A) - P(A)^2 \sim 2P(B \Delta D)$ as $P(A) \longrightarrow 0$. In this case, the constant 2 is best possible.

Theorem 2. Given a probability space (Ω, A, P) and non-increasing sub-σ-algebras A_n, $n = 1,2,\ldots$, $A \supset A_1 \supset A_2 \supset \cdots$, such that P is nonatomic on each A_n, and given any $p_n \geq 0$ with $\Sigma_{n\geq 1} p_n = 1$, there exist disjoint $A_n \in A_n$ with $P(A_n) = p_n$.

Proof. Let $n(0) := 1$, choose $n(1)$ large enough so that $r_1 := \Sigma_{j<n(1)} p_j > 0$, and let $n(k) \uparrow +\infty$ fast enough so that $\Sigma_{n\geq n(k)} p_n \leq 4^{-k}$ for all $k \geq 2$. Let $r_k := \Sigma_{n(k-1)\leq n<n(k)} p_n$. If we can find disjoint $B_k \in A_{n(k)}$ with $P(B_k) = r_k$ for all k, then we can choose A_n for $n(k-1) \leq n < n(k)$ as disjoint subsets of B_k with $P(A_n) = p_n$, $A_n \in A_{n(k)} \subset A_n$. Thus, we may assume $p_1 > 0$ and $\Sigma_{n\geq 1} 3^n p_n < \infty$.

Let $\pi_n := p_n / \Sigma_{1\leq j\leq n} p_j$. Take $A_{n1} \in A_n$ with $P(A_{n1}) = \pi_n$ for each n. Given A_{nj} for all n and for $j < k$, let $B_{n1} := \Omega$ and for $k \geq 2$ let $B_{nk} := \Omega \setminus \bigcup_{1\leq j<k} A_{n+j,k-j}$. We choose A_{nk} for each n by the Lemma so that $A_{nk} \in A_n$, $A_{nk} \subset B_{nk}$, $P(A_{nk} | B_{nk}) = \pi_n$ (or if $P(B_{nk}) = 0$, $A_{nk} = \phi$), and

$$P(A_{nk} \Delta A_{n,k-1}) \leq 2p_{nk} := 2P(B_{nk} \Delta B_{n,k-1}). \quad \text{Then}$$

$$(*) \qquad p_{nk} \leq \pi_{n+k-1} + \Sigma_{1\leq j<k-1} 2p_{n+j,k-j}.$$

Claim: $p_{nk} \leq 3^{k-2} \pi_{n+k-1}$ for all $k \geq 2$.

This will be proved by induction on k. For $k = 2$, $(*)$ gives $p_{n2} \leq \pi_{n+1}$ as desired. For the induction step, $(*)$ gives

$$p_{n,k+1} \leq \pi_{n+k} + 2\Sigma_{1 \leq j<k} 3^{k-j-1}\pi_{n+k}$$

$$= \pi_{n+k}[1 + 2(1 + 3 + \cdots + 3^{k-2})]$$

$$= \pi_{n+k}[1 + 2(3^{k-1} - 1)/(3-1)] = 3^{k-1}\pi_{n+k},$$

proving the Claim.

Now $\Sigma 3^n \pi_n \leq \Sigma 3^n p_n/p_1 < \infty$. So A_{nk} converges to some event A_n as $k \longrightarrow \infty$, specifically

$$P(A_n \Delta A_{nk}) \leq \Sigma_{j>k} P(A_{nj} \Delta A_{n,j-1})$$

$$\leq 2\Sigma_{j>k} 3^{j-2}\pi_{n+j-1} = 2\Sigma_{i \geq k} 3^{i-1}\pi_{n+i}.$$

Since A_{nk} is disjoint from $A_{n+j,k-j}$ for all $j < k$, we can let $k \longrightarrow \infty$ for fixed j to obtain $P(A_n \cap A_{n+j}) = 0$ for all $j \geq 1$. Thus, we may take all the A_n to be disjoint. Let $B_n := \Omega \setminus \bigcup_{m>n} A_m$. Then

$$P(B_n \Delta B_{nk}) \leq (\Sigma_{1 \leq j<k} P(A_{n+j} \Delta A_{n+j,k-j})) + \Sigma_{j \geq k} P(A_{n+j})$$

$$\leq 2\Sigma_{1 \leq j<k} \Sigma_{i \geq k-j} 3^{i-1}\pi_{n+j+i} + \Sigma_{j \geq k}\pi_{n+j}$$

$$\leq \Sigma_{j \geq k}\pi_{n+j} + 2\Sigma_{r \geq k}\pi_{n+r}\Sigma_{1 \leq j<k} 3^{r-j-1}$$

$$\leq \Sigma_{j \geq k}\pi_{n+j} + \Sigma_{r \geq k} 3^{r-1}\pi_{n+r} \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Thus, $B_{nk} \longrightarrow B_n$. For each n, $P(A_n) \leq \pi_n$. So, at least for n large enough, $P(B_n) > 0$ and

$$P(A_n | B_n) = \lim_{k \longrightarrow \infty} P(A_{nk} | B_{nk}) = \pi_n.$$

For such n, $P(A_n) = \pi_n(1 - \Sigma_{k>n}P(A_k))$. Then for $m \geq n$, $P(B_m|B_{m+1}) = 1 - \pi_{m+1}$ and

$$P(A_n|B_m) = \pi_n \Pi_{n<j\leq m}(1 - \pi_j) = p_n/(p_1+\cdots+p_m).$$

Thus

$$P(A_n) = p_n(1 - \Sigma_{k>m}P(A_k))/(p_1+\cdots+p_m).$$

Letting $m \longrightarrow \infty$ gives $P(A_n) = p_n$ for n large. Then, since $p_1 > 0$, $P(B_n) > 0$ for all n and the above holds for all n (by induction downward). Thus, Theorem 2 is proved.

Letting $A_n = A_{t(n)}$ and $A_n = \{\mathcal{T} = t(n)\}$ Theorem 1 is also proved.

<u>Example</u>. It may happen that for every law μ on the closed interval $[0,\infty]$, there is a stopping time with law μ, even though A_0 is trivial. Let $T = [0,1]$ and $X_t(\omega) := \omega t$ where ω is uniformly distributed on $[0,1]$. Let $\omega \longrightarrow g(\omega)$ have law μ. The identity $\omega \longrightarrow \omega$ is measurable from $(\Omega, \bigcap_{t>0}A_t)$ into R, so g is a stopping time.

<u>Proposition</u>. There is a stopping time τ with any law μ on $[s,\infty]$ iff both a) $s \geq NA(X)$ and b) for any $p \in (0,1)$ there is an event $A \in \bigcap_{t>s}A_t$ with $P(A) = p$.

<u>Proof</u>. By Theorem 1, a) is necessary. To show b) necessary, pick a law μ with $p = \mu\{s\}$ and let $A = \{\tau = s\}$. Conversely,

given a law μ with $\mu\{s\} = p < 1$, choose A as in b) and apply Theorem 1 to $\mu'(\cdot) = \mu(\cdot|(s,\infty])$ and $P'(\cdot) = P(\cdot|A^c)$. This proves the proposition.

If $\mathcal{C}$ is a σ-algebra generated by atoms of size 2^{-n}, $n = 1,2,\ldots,$ then $\mathcal{C}$ contains A with $P(A) = p$ for each $p \in (0,1)$, although $\mathcal{C}$ is purely atomic.

REFERENCE

Halmos, P. (1950), Measure Theory (Princeton, Van Nostrand).

Footnote

1. This research was partially supported by the Danish Natural Science Council and by the U.S. National Science Foundation, Grant no. MCS76-07211.