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ON THE INTEGRABILITY OF BANACH SPACE VALUED WALSH POLYNOMIALS

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1. Introduction

In [2] the author claims that the integrability of Banach space valued Wiener polynomials follows from the Nelson hypercontractivity theorem [5]. Here, using a similar idea, we will study the integrability of Banach space valued Walsh polynomials. Our conclusion extends the familiar result of Khintchine-Kahane-Kwapień for the linear case [4].

To start with we introduce several definitions.

We let δ_a denote the Dirac measure at the point $a \in \mathbb{R}$ and set

$$\mu = (\delta_{-1} + \delta_{+1})/2 .$$

The functions $e_0(x) = 1$, $e_1(x) = x$, $x \in \mathbb{R}$, form an orthonormal basis for $L_2(\mu;\mathbb{R})$. We introduce the infinite product measure

$$\mu_\infty = \prod_{i\in\mathbb{N}} \mu_i \qquad (\mu_i = \mu)$$

on $\mathbb{R}^{\mathbb{N}}$ and define

$$e_\alpha(x) = \prod_{i\in\mathbb{N}} e_{\alpha_i}(x_i) , \quad x = (x_i) \in \mathbb{R}^{\mathbb{N}}$$

for every $\alpha = (\alpha_i) \in M$, where

$$M = \{\alpha \in \{0,1\}^{\mathbb{N}} ; \ |\alpha| = \sum_{i\in\mathbb{N}} \alpha_i < +\infty\} .$$

Clearly, the e_α constitute an orthonormal basis for $L_2(\mu_\infty;\mathbb{R})$.

Suppose now that $E = (E,\| \ \|)$ is a fixed Banach space. The vector space of all functions

$$c : M \to E$$

such that

$$\# \{\alpha ; c_\alpha \neq 0\} < +\infty$$

is denoted by $\mathcal{F}(E)$. For every fixed positive integer d, we define

$$W_d(E) = \{\Sigma c_\alpha e_\alpha ; c \in \mathcal{F}(E),\ c_\alpha = 0,\ |\alpha| \neq d\}$$

and

$$\overline{W}_d(E) = \text{closure of } W_d(E) \text{ in } L_0(\mu_\infty, E),$$

respectively. The elements of $\overline{W}_d(E)$ are called E-valued d-homogeneous Walsh polynomials.

<u>Theorem 1.1</u>. <u>The vector space</u> $\overline{W}_d(E)$ <u>is a closed subspace of</u> $L_p(\mu_\infty, E)$ <u>for every</u> $p \in [0, +\infty[$. <u>Moreover, for every fixed</u> $1 < p < q < +\infty$, <u>the norm of the canonical injection of</u> $(\overline{W}_d(E), \|\cdot\|_{p,\mu_\infty})$ <u>into</u> $(\overline{W}_d(E), \|\cdot\|_{q,\mu_\infty})$ <u>does not exceed</u>

$$(1.1)_d \qquad [(q-1)/(p-1)]^{d/2} .$$

<u>In particular,</u> $\exp(\|f\|^{2/d}) \in L_1(\mu_\infty; \mathbb{R})$ <u>for all</u> $f \in \overline{W}_d(E)$.

In the special case $d = 1$, Theorem 1.1 essentially reduces to the Khintchine-Kahane-Kwapień result [4]. However, in the Banach space valued case the constant in $(1.1)_1$ is slightly better than in [4].

2. <u>Proof of Theorem 1.1</u>.

Let $1 < p < q < +\infty$ be fixed and choose the real number λ so that

$$|\lambda| \leq [(p-1)/(q-1)]^{1/2} .$$

Theorem 1.1 turns out to be a simple consequence of the elementary inequality

$$\left[\tfrac{1}{2}\left(|c_0 - \lambda c_1|^q + |c_0 + \lambda c_1|^q\right)\right]^{1/q} \leq \left[\tfrac{1}{2}\left(|c_0 - c_1|^p + |c_0 + c_1|^p\right)\right]^{1/p}, \quad c_0, c_1 \in \mathbb{R},$$

which is well-known ([3, Th. 3], [1, pp. 180]). To see this, we define

$$K(x,y) = e_0(x)e_0(y) + \lambda e_1(x)e_1(y), \quad x, y \in \mathbb{R},$$

and

$$\overline{K} f = \int K(\cdot, y) f(y) dy, \quad f \in \mathbb{R}^{\mathbb{R}},$$

respectively. Then

$$\|\overline{K} f\|_{q,\mu} \leq \|f\|_{p,\mu}, \qquad f \in \mathbb{R}^{\mathbb{R}},$$

and by applying the Segal lemma [1, Lemma 2] we also have

$$\|(\overset{n}{\underset{1}{\otimes}} \overline{K}) f\|_{q, \overset{n}{\underset{1}{\otimes}} \mu} \leq \|f\|_{p, \overset{n}{\underset{1}{\otimes}} \mu}, \quad f \in \mathbb{R}^{\mathbb{R}^n}, \quad n \in \mathbb{N}_+,$$

that is

$$\| \Sigma \lambda^{|\alpha|} c_\alpha e_\alpha \|_{q,\mu_\infty} \le \| \Sigma c_\alpha e_\alpha \|_{p,\mu_\infty} \tag{2.1}$$

for every $c \in \mathcal{F}(\mathbb{R})$. Since $K \geq 0$ a.s. $[\mu]$ the inequality (2.1) remains true for every $c \in \mathcal{F}(E)$. In particular,

$$\|f\|_{q,\mu_\infty} \le [(q-1)/(p-1)]^{d/2} \|f\|_{p,\mu_\infty} , \quad f \in W_d(E) .$$

Letting $\mathcal{T}_p$ denote the topology of the metric space $(W_d(E), \| \ \|_{p,\mu_\infty})$ we now have that $\mathcal{T}_p = \mathcal{T}_q$ for all $p, q \in [0, +\infty[$ and Theorem 1.1 follows at once.

3. An unsolved problem

Assume $\varphi: E \longrightarrow [0, +\infty]$ is a Borel measurable seminorm, which may take on the value $+\infty$. Let $f \in \overline{W}_d(E)$ and suppose

$$\varphi(f) < +\infty \quad \text{a.s.} \quad [\mu_\infty] .$$

Does it follow that

$$\exp[(\varphi(f))^{2/d}] \in L_1(\mu_\infty ; \mathbb{R}) \ ?$$

At present, we do not know the answer to this question for any $d \in \mathbb{Z}_+$. Note, however, that the corresponding question has an affirmative answer for Banach space valued Wiener polynomials if f is replaced by εf and $\varepsilon > 0$ is sufficiently small [2].

REFERENCES

1. Beckner, W., Inequalities in Fourier analysis. Annals of Math. 102, 159-182 (1975)

2. Borell, C., Tail probabilities in Gauss space. Lecture Notes in Math. 644, 71-82, Springer-Verlag, Berlin-Heidelberg-New York 1978.

3. Gross, L., Logarithmic Sobolev inequalities. Amer. J. Math. 97, 1061-1083 (1975).

4. Kwapień, S., A theorem on the Rademacher series with vector valued coefficients. Lecture Notes in Math. 526, 157-158, Springer-Verlag, Berlin-Heidelberg-New York 1976.

5. Nelson, E., The free Markoff field. J. Functional Anal. 12, 211-227 (1973).