MASATAKA IZUMISAWA
T. SEKIGUCHI

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WEIGHTED NORM INEQUALITIES FOR MARTINGALES

M. Izumisawa and T. Sekiguchi

In this note we extend THEOREM 4 in M. Izumisawa and N. Kazamaki [1], to the case when the weight is not continuous as a martingale.

1. Theorem.

Let $(\Omega, \underline{\underline{F}}, P)$ be a probability space with an inceasing right continuous family $(\underline{\underline{F}}_t)_{t \geqq 0}$ of sub-σ-fields of $\underline{\underline{F}}$ such that $\underline{\underline{F}} = \bigvee_{t \geqq 0} \underline{\underline{F}}_t$. We use the same notations $[X, Y]$, X^* and so on as in P. A. Meyer [2]. Let Z be a uniformly integrable martingale with $E[Z_\infty] = 1$ and $Z_\infty > 0$ a. s. . We put $\hat{P} = Z_\infty \cdot P$ and

$$(1) \qquad \hat{M} = - \int_{]0,\cdot]} Z_{s-} \, d(Z_s^{-1}) \ .$$

Then $\hat{M}$ is a local martingale with respect to $\hat{P}$ as is shown later.

THEOREM. Let X be any local martingale with respect to P. Then we have the inequalities

$$(2) \quad (1/2)\{1 - (2\sqrt{2}+1)\|\hat{M}\|_{B(\hat{P})}\}\hat{E}[[X, X]_\infty^{1/2}] \leqq \hat{E}[X^*]$$

$$\leqq \sqrt{2}(4+5\|\hat{M}\|_{B(\hat{P})})\hat{E}[[X, X]_\infty^{1/2}],$$

where $\hat{E}[\ ]$ and $\|\ \|_{B(\hat{P})}$ denote the expectation and the BMO-norm with respect to the probability measure $\hat{P}$ respectively.

By applying Garsia's lemma (see [2] V.24, p. 347) to the above theorem, we obtain the following corollary.

COROLLARY. Let Φ be a continuous increasing convex function on $[0, \infty[$ with $\Phi(0) = 0$ and satisfy the growth condition, that is, there exists a constant A such that $\Phi(2t) \leqq A\Phi(t)$ for all t. Assume that $\|\hat{M}\|_{B(\hat{P})} < (2\sqrt{2} + 1)^{-1}$. Then for any local martingale X with respect to P.

$$c\hat{E}[\Phi(X^*)] \leqq \hat{E}[\Phi[X, X]_\infty^{1/2})] \leqq C\hat{E}[\Phi(X^*)]. \tag{3}$$

Here, the choice of c and C depends only on the growth parameter A of Φ.

2. Proof of the Theorem.

For a local martingale X with respect to P we define

$$\hat{X} = X - \int_{]0,\cdot]} Z_s^{-1}\, d[X, Z]_s. \tag{4}$$

Then $\hat{X}$ is a local martingale with respect to $\hat{P}$ (see [2] VI. 22-26, p. 376). We put

$$(5) \qquad M = \int_{]0,\cdot]} Z_{s-}^{-1}\, dZ_s.$$

We apply Ito's formula to $1 = Z_t Z_t^{-1}$ obtaining

$$M_t + \int_{]0,t]} Z_{s-}\, d(Z_s^{-1})$$

$$= - \int_{]0,t]} d[Z, Z^{-1}]_s$$

$$= \int_{]0,t]} d\langle Z^c, \frac{1}{Z_{\cdot-}^2} \cdot Z^c\rangle_s - \sum_{0<s\leqq t} \Delta Z_s \Delta(Z^{-1})s$$

$$= \int_{]0,t]} Z_{s-}^{-1}\, d\langle M^c, Z^c\rangle_s + \sum_{0<s\leqq t} Z_s^{-1} \Delta M_s \Delta Z_s$$

$$= \int_{]0,t]} Z_s^{-1}\, d[M, Z]_s \;.$$

Therefore $\hat{M}$ defined in (1) is a local martingale with respect to $\hat{P}$.

We proceed to prove the following relations:

$$(6) \quad \hat{X} = X - [X, \hat{M}],$$

$$(7) \quad (1-\|\hat{M}^d\|_{B(\hat{P})})[X, X]^{1/2} \leqq [\hat{X}, \hat{X}]^{1/2} \leqq (1+\|\hat{M}^d\|_{B(\hat{P})})[X, X]^{1/2}$$

$$(8) \quad (1/2)\hat{E}[[\hat{X}, \hat{X}]^{1/2}] \leqq \hat{E}[\hat{X}^*] \leqq 4\sqrt{2}\hat{E}\,[[\hat{X}, \hat{X}]^{1/2}],$$

and

$$(9)\quad \hat{E}[\int_{]0,\infty]}|d[X, M]_s|] \leqq \sqrt{2}\hat{E}[[X, X]_\infty^{1/2}]\|\hat{M}\|_{B(\hat{P})}.$$

From (4)

$$\hat{M}_t = \int_{]0,t]} Z_{s-}^{-1}\, dZ_s - \int_{]0,t]} Z_s^{-1}\, d[Z_{\cdot-}^{-1}\cdot Z, Z]_s,$$

we have

$$[X, \hat{M}]_t = \int_{]0,t]} Z_{s-}^{-1}\, d\langle X^c, Z^c\rangle_s - \sum_{0<s\leqq t} \Delta X_s\{\Delta Z_s/Z_{s-} - Z_s^{-1}(\Delta Z_s/Z_{s-})\Delta Z_s\}$$

$$= \int_{]0,t]} Z_s^{-1}\, d\langle X^c, Z^c\rangle_s + \int_{]0,t]} Z_s^{-1}\, d[X^d, Z^d]_s$$

$$= \int_{]0,t]} Z_s^{-1}\, d[X, Z]_s$$

and so we obtain the equality (6). According to the equalities $\langle\hat{X}^c\rangle = \langle X^c\rangle$ (see [2] VI. 25, p. 378) and $\Delta\hat{X}_t = \Delta X_t - \Delta X_t\Delta\hat{M}_t = \Delta X_t(1-\Delta\hat{M}_t)$, we get easily the inequality (7). The inequality (8) is nothing but Davis' inequality (see [2] V. 29, P. 349). The last inequality (9) is of Feffermann's type.
The proof of [2] V. 9, p. 337 is still valid in our case where X is a semi-martingale with respect to $\hat{P}$.

Now it follows from the above equation and inequalities (6) - (9) that

$$\hat{E}[X^*] = \hat{E}[(\hat{X} + [X, \hat{M}])^*]$$

$$\geqq \hat{E}[\hat{X}^*] - \hat{E}[\int_0^\infty |d[X, \hat{M}]_s|\ ]$$

$$\geqq (1/2)\hat{E}[[\hat{X}, \hat{X}]_\infty^{1/2}] - \sqrt{2}\|\hat{M}\|_{B(\hat{P})}\hat{E}[[X, X]_\infty^{1/2}]$$

$$\geqq (1/2)(1-\|\hat{M}^d\|_{B(\hat{P})})\hat{E}[[X, X]_\infty^{1/2}] - \sqrt{2}\|\hat{M}\|_{B(\hat{P})}\hat{E}[[X, X]_\infty^{1/2}]$$

$$= (1/2)\{1 - (2\sqrt{2}+1)\|\hat{M}\|_{B(\hat{P})}\}\hat{E}[[X, X]_\infty^{1/2}]$$

and

$$\hat{E}[X^*] = \hat{E}[(\hat{X} + [X, \hat{M}])^*]$$

$$\leqq \hat{E}[\hat{X}^*] + \hat{E}[\int_0^\infty | d[X, \hat{M}]_s|]$$

$$\leqq 4\sqrt{2}\hat{E}[[\hat{X}, \hat{X}]_\infty^{1/2}] + \sqrt{2}\|\hat{M}\|_{B(\hat{P})}\hat{E}[[X, X]_\infty^{1/2}]$$

$$\leqq 4\sqrt{2}(1 + \|\hat{M}^d\|_{B(\hat{P})})\hat{E}[[X, X]_\infty^{1/2}] + \sqrt{2}\|\hat{M}\|_{B(\hat{P})}\hat{E}[[X, X]_\infty^{1/2}]$$

$$= \sqrt{2}(4 + 5\|\hat{M}\|_{B(\hat{P})})\hat{E}[[X, X]_\infty^{1/2}]$$

Finally we remark that, even though M is not continuous, for each continuous local martingale X with respect to P the constants of the inequalities (2) $(1/2)\{1 - (2\sqrt{2}+1)\|\hat{M}\|_{B(\hat{P})}\}$ and $\sqrt{2}(4 + 5\|\hat{M}\|_{B(\hat{P})})$ can be replaced by $(1/2)(1 - 2\sqrt{2}\|\hat{M}\|_{B(\hat{P})})$ and $\sqrt{2}(4 + \|\hat{M}\|_{B(\hat{P})})$ respectively, which are the same constants as in M. Izumisawa and N. Kazamaki [1].

[1] M. Izumisawa and N. Kazamaki, Weighted norm inequalities for martingales, Tôhoku Math. Journ. 29(1977), 115-124.

[2] P. A. Meyer, Un cours sur les intégrales stochastiques, Séminaire de Probabilités X, Univ. de Strasbourg, Springer-Verlag, Berlin, (1976), 246-400.

Mathematical Institute
Tôhoku University
Sendai, Japan