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Random Fourier Series on Locally Compact Abelian Groups

M. B. Marcus and G. Pisier

In 1930 Paley and Zygmund [9] introduced the problem of whether the random series

$$(1) \quad \sum_{k=0}^{\infty} a_k \epsilon_k \cos(kt + \alpha_k), \quad t \in [0, 2\pi]$$

converges uniformly a.s., where $\{a_k\}$ and $\{\alpha_k\}$ are sequences of real numbers and $\{\epsilon_k\}$ is a Rademacher sequence, that is, a sequence of independent, symmetric random variables taking on the values ± 1 . This problem was subsequently studied by Salem and Zygmund [11], Kahane [6] and others (see [8], [10]). In [8] we give a necessary and sufficient condition for the uniform convergence of the series in (1). An interesting aspect of this result is that the condition remains valid when the sequence $\{\epsilon_k\}$ is replaced by other sequences of random variables, for example, independent gaussian random variables with mean zero and variance 1 ($N(0,1)$). Our results in [8] are a consequence of the Dudley-Fernique [2], [3] necessary and sufficient condition for the continuity of stationary Gaussian processes and a line of approach initiated in [4] (see also [7]). In this paper, by adding some technical modifications we show that the results in [8] extend directly to the more general class of random series mentioned in the title. The case of compact abelian groups is included in [10].

Let G be a locally compact abelian group with identity

element 0 . Let $K \subset G$ be a compact symmetric neighborhood of 0 . Let Γ denote the characters of G and let $A \subset \Gamma$ be countable. Therefore, $\{\gamma \mid \gamma \in A\}$ is a countable collection of characters of G . (We only consider Fourier series with spectrum in A . Therefore, in all that follows, we may as well assume that Γ is separable, so that the compact subsets of G are metrizable.) We also define the following sequences of random variables indexed by $\gamma \in A$: $\{\epsilon_\gamma\}$ a Rademacher sequence, $\{g_\gamma\}$ independent $N(0,1)$ random variables and $\{\xi_\gamma\}$ complex valued random variables satisfying

$$(2) \quad \sup_{\gamma \in A} E|\xi_\gamma|^2 < \infty \text{ and } \liminf_{\gamma \in A} E|\xi_\gamma| > 0.$$

Let $\{a_\gamma\}$ be complex numbers satisfying $\sum_{\gamma \in A} |a_\gamma|^2 = 1$ and consider the random Fourier series

$$(3) \quad Z(x) = \sum_{\gamma \in A} a_\gamma \epsilon_\gamma \xi_\gamma \gamma(x), \quad x \in K.$$

For each fixed $x \in K$ the series converges a.s. so the sum is well defined. We will give a necessary and sufficient condition for the series (3) to converge uniformly a.s. on K .

Define $K \oplus K = \{x + y \mid x \in K, y \in K\}$ and in a similar fashion define $\bigoplus_{i=1}^n K_i$. Let $\tau(x)$ be a non-negative function on $K \oplus K$ and let

$$(4) \quad m_\tau(\epsilon) = \mu(x \in K \oplus K \mid \tau(x) < \epsilon)$$

where μ is the Haar measure on G . Define

$$(5) \quad \overline{\tau(u)} = \sup\{y \mid m_\tau(y) < u\}$$

and let $\mu_n = \mu(\bigoplus_{i=1}^n K_i)$. Therefore $0 \leq m_\tau(\epsilon) \leq \mu_2$ so that the domain

of $\bar{\tau}$ is the interval $[0, \mu_2]$. Note that $\bar{\tau}$ viewed as a random variable on $[0, \mu_2]$ has the same probability distribution with respect to normalized Lebesgue measure on $[0, \mu_2]$ that $\tau(x)$ has with respect to normalized Haar measure on $K \otimes K$. In keeping with classical terminology we call $\bar{\tau}$ the non-decreasing rearrangement of τ (with respect to $K \otimes K$). In terms of μ , τ and K we define the integral

$$(6) \quad \begin{aligned} I(K, \mu, \tau(s)) &= I(\tau(s)) = I(\bar{\tau}) \\ &= \int_0^{\mu_2} \frac{\bar{\tau}(s)}{s(\log \frac{\mu_2}{s})^{1/2}} ds. \end{aligned}$$

Finally, we define a translation invariant pseudo-metric σ on G by

$$(7) \quad \begin{aligned} \sigma(x-y) &= \left(\sum_{\gamma \in A} |a_\gamma|^2 |\gamma(x) - \gamma(y)|^2 \right)^{1/2} \\ &= \left(\sum_{\gamma \in A} |a_\gamma|^2 |\gamma(x-y) - 1|^2 \right)^{1/2}. \end{aligned}$$

To see the motivation for this note that when $E|\xi_\gamma|^2 = 1$ for all $\gamma \in A$ then $\sigma(x-y) = (E|Z(x) - Z(y)|^2)^{1/2}$. We can now state our result.

Theorem 1: Employing the notation and definitions given above let

$\|Z\| = \sup_{x \in K} |Z(x)|$. If $I(\sigma) < \infty$ the series (3) converges uniformly

a.s. and

$$(8) \quad (E\|Z\|^2)^{1/2} \leq C \left(\sup_{\gamma} E|\xi_\gamma|^2 \right)^{1/2} \left[\left(\sum_{\gamma \in A} |a_\gamma|^2 \right)^{1/2} + I(\sigma) \right]$$

where C is a constant independent of $\{a_\gamma\}$ and σ . Let $\{\gamma_k, k = 1, 2, \dots\}$ be an ordering of $\gamma \in A$ and let $\{a_k\}$, $\{\epsilon_k\}$ and $\{\xi_k\}$ be the corresponding orderings of $\{a_\gamma\}$, $\{\epsilon_\gamma\}$ and $\{\xi_\gamma\}$. If $I(\sigma) = \infty$ then for all open sets $U \subset K$

$$(9) \quad \sup_n \sup_{x \in U} \left| \sum_{k=1}^n a_k \epsilon_k \xi_k \gamma_k(x) \right| = \infty$$

on a set of measure greater than zero. (Note that neither (2) nor (7) depend on the order of $\{\gamma_k\}$ so that the implications of $I(\sigma) < \infty$ are also valid for all orderings $\{\gamma_k\}$ of $\gamma \in A$.)

Proof: The first step is a adaptation of Dudley's theorem on a sufficient condition for continuity of the sample paths of a Gaussian process. It is well known that this theorem is also valid for processes with sub-gaussian increments. Let $\{Y(t), t \in T\}$, T an arbitrary index set, be a real valued stochastic process. The process is said to have subgaussian increments if there exists a $\delta > 0$ such that for all $s, t \in T$ and $\lambda > 0$

$$E\{\exp(\lambda(X(s)-X(t)))\} \leq \exp\{\lambda^2 \delta^2 E(X(t)-X(s))^2/2\}.$$

Let (S, ρ) be a metric (or pseudo-metric) space. We denote by $N_\rho(S, \epsilon)$ the minimum number of balls in the metric (or pseudo-metric) ρ that is necessary to cover S . The following theorem is an immediate consequence of Theorem 4.1 [7]; it is similar to a theorem of Fernique, [13].

Theorem 2: Let $\tilde{S} = \{\tilde{X}(t), t \in T\}$, T a compact topological space, be a stochastic process with subgaussian increments and let

$\rho(t,s) = (E(\tilde{X}(t) - \tilde{X}(s))^2)^{1/2}$ be continuous on $T \times T$. Define $\hat{\rho} = \sup_{s,t \in T} \rho(s,t)$ and assume that

$$(10) \quad J(\tilde{S}, \rho) = J(\rho) = \int_0^{\hat{\rho}} (\log N_{\rho}(\tilde{S}, u))^{1/2} du < \infty.$$

Then there exists a version $S = \{X(t), t \in T\}$ of the process, with continuous sample paths, satisfying the inequality

$$(11) \quad E[\sup_{t \in T} |X(t)|] \leq C'[E|X(t_0)| + \hat{\rho} + J(S, \rho)]$$

where $t_0 \in T$ and $C' = C'(\delta)$ is a constant independent of ρ . (Note that $N_{\rho}(S, u) = N_{\rho}(\tilde{S}, u)$ so, in particular, $J(\tilde{S}, \rho) = J(S, \rho)$.)

We will use this theorem in the special case in which ρ is translation invariant. In this case we can relate the integrals defined in (6) and (10). In order to do this we need the following lemma which is a generalization of Lemma 2.1 [4].

Lemma 3. Let τ be a translation invariant pseudo-metric on G then

$$(12) \quad \frac{\mu_1}{m_{\tau}(\epsilon)} \leq N_{\tau}(K \oplus K, \epsilon) \leq \frac{\mu_4}{m_{\tau}(\epsilon/2)}.$$

Proof: Since this lemma is the only ingredient in the proof of Theorem 1 that is not supplied in [8] or [10] we will sketch the proof. Note that when G is compact we can take $K = G$. In this case the proof is elementary and (12) reduces to

$$\frac{\mu(G)}{m_{\tau}(\epsilon)} \leq N_{\tau}(G, \epsilon) \leq \frac{\mu(G)}{m_{\tau}(\epsilon/2)}.$$

Let $B(t, \epsilon) = \{x \in G \mid \tau(x-t) < \epsilon\}$ and let $M_\tau(K \oplus K, \epsilon)$ denote the maximal number of balls of radius ϵ in the τ pseudo-metric centered in $K \oplus K$ and disjoint in $\bigoplus_{i=1}^4 K_i$. Then for all $t \in K \oplus K$ we have

$$\mu\{B(t, \epsilon) \cap \bigoplus_{i=1}^4 K_i\} \geq \mu\{B(0, \epsilon) \cap K \oplus K\}$$

and

$$M_\tau(K \oplus K, \epsilon/2) \geq N_\tau(K \oplus K, \epsilon)$$

Denote the centers of the $M_\tau(K \oplus K, \epsilon/2)$ balls of radius $\epsilon/2$ centered in $K \oplus K$ and disjoint in $\bigoplus_{i=1}^4 K_i$ by $\{t_j, j = 1, \dots, M_\tau(K \oplus K, \epsilon)\}$ then

$$\begin{aligned} \mu_4 &\geq \mu\left(\bigcup_{j=1}^{M_\tau(K \oplus K, \epsilon/2)} \{B(t_j, \epsilon/2) \cap \bigoplus_{i=1}^4 K_i\}\right) \\ &\geq M_\tau(K \oplus K, \epsilon/2) \mu\{B(0, \epsilon/2) \cap K \oplus K\} \\ &\geq N_\tau(K \oplus K, \epsilon) m_\tau(\epsilon/2) \end{aligned}$$

This proves the right side of (12); the proof of the left side is similar.

We note two other standard results

$$(13) \quad N_\tau(K, \epsilon) \leq N_\tau(K \oplus K, \epsilon)$$

$$(14) \quad N_\tau(K \oplus K, 2\epsilon) \leq N_\tau^2(K, \epsilon)$$

and define the integral expression

$$(15) \quad \tilde{I}(K, \mu, \tau(u)) = \tilde{I}(\tau(u)) = \tilde{I}(\tau)$$

$$= \int_0^{\mu_2} \frac{\int_0^s \overline{\tau(u)} du}{s^2 (\log \frac{4\mu_4}{s})^{1/2}} ds$$

for $\overline{\tau}$ as defined in (5). The next lemma follows from (12), (13), (14) and integration by parts.

Lemma 5: Let $\hat{\tau} = \sup_{x \in K \oplus K} \tau(x)$ and assume that $J(K, \tau) = J(\tau) < \infty$, then

the following inequalities hold:

$$(16) \quad -C_1 \hat{\tau} + I(\tau) \leq \tilde{I}(\tau) \leq 2I(\tau)$$

$$(17) \quad -C_2 \hat{\tau} + \frac{1}{2\sqrt{2}} I(\tau) \leq J(\tau) \leq C_2 \hat{\tau} + 2I(\tau)$$

$$(18) \quad -C_3 \hat{\tau} + \frac{1}{4\sqrt{2}} \tilde{I}(\tau) \leq J(\tau) \leq C_3 \hat{\tau} + 2\tilde{I}(\tau)$$

where $C_1, C_2, C_2', C_3, C_3'$ are all positive and finite.

The next step in the proof is a Jensen type inequality for the non-decreasing rearrangements of a family of random functions.

Let (Ω, \mathcal{F}, P) be some probability space with expectation operator E and let $\tau(x, \omega)$, $x \in K \oplus K$, $\omega \in \Omega$ be a family of random non-negative functions such that $E|\tau(x, \omega)|^2 < \infty$ for $x \in K \oplus K$. Following (4) and (5) we define the random families $m_{\tau(\cdot, \omega)}(\epsilon)$ and $\overline{\tau(\cdot, \omega)}$. We have

Lemma 6: For $0 \leq h \leq \mu_2$

$$(19) \quad (E|\int_0^h \overline{\tau(u, \omega)} du|^2)^{1/2} \leq \int_0^h (E|\tau(u, \omega)|^2)^{1/2} du.$$

This lemma is a generalization of Lemma 1.1 [7]. The proof is essentially the same as the one given in [8].

We can now obtain the implications of $I(\sigma) < \infty$ in Theorem 1. Let $(\Omega_1, \mathcal{F}_1, P_1)$ denote the probability space of $\{\xi_\gamma\}$ and $(\Omega_2, \mathcal{F}_2, P_2)$ denote the probability space of $\{\epsilon_\gamma\}$ and denote the corresponding expectation operators by E_1 and E_2 . The series (3) is defined on the probability space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2)$. We shall refer to this space as (Ω, \mathcal{F}, P) and denote the corresponding expectation operator by E (not to be confused with the space used to explain Lemma 6).

Without loss of generality we can assume $\sup_{\gamma \in A} E|\xi_\gamma|^2 \leq 1$; the second assumption of (2) is not used in this part of the proof.

Fix $\omega_1 \in \Omega_1$ and consider

$$(20) \quad Z(x, \omega_1) = \sum_{\gamma \in A} a_\gamma \epsilon_\gamma \xi_\gamma(\omega_1) \gamma(x), \quad x \in K$$

as a random series on $(\Omega_2, \mathcal{F}_2, P_2)$. Note that

$$Z_1(x, \omega_1) = \sum_{\gamma \in A} \epsilon_\gamma \operatorname{Re}[a_\gamma \xi_\gamma(\omega_1) \gamma(x)] \text{ and}$$

$$Z_2(x, \omega_1) = \sum_{\gamma \in A} \epsilon_\gamma \operatorname{Im}[a_\gamma \xi_\gamma(\omega_1) \gamma(x)] \text{ are both processes with sub-}$$

gaussian increments (see e.g. Chapter 2, Section 2 [5]) and both

$(E_2|Z_1(x, \omega_1) - Z_1(y, \omega_1)|^2)^{1/2}$ and $(E_2|Z_2(x, \omega_1) - Z_2(y, \omega_1)|^2)^{1/2}$ are less than or equal to

$$(21) \quad \sigma(x-y, \omega_1) = \left(\sum_{\gamma \in A} |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2 |\gamma(x-y) - 1|^2 \right)^{1/2}.$$

By Theorem 2 with $t_0 = 0$ and (18) we have

$$(22) \quad E_2[\sup_{x \in K} |Z(x, \omega_1)|] \leq D[(\sum_{\gamma \in A} |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2)^{1/2} + \tilde{I}(\sigma(u, \omega_1))],$$

for some constant D , where we use the facts that

$$\hat{\sigma} = \sup_{x \in K \oplus K} \sigma(x) \leq 2(\sum_{\gamma \in A} |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2)^{1/2}$$

and

$$\begin{aligned} E_2 |Z(0, \omega_1)| &\leq (E_2 |Z(0, \omega_1)|^2)^{1/2} \\ &= \left(\sum_{\gamma \in A} |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2 \right)^{1/2}. \end{aligned}$$

The series (20) is a Rademacher series therefore by Kahane's inequality we have

$$(23) \quad E_2 \left[\sup_{x \in K} |Z(x, \omega_1)|^2 \right]^{1/2} \leq C E_2 \left[\sup_{x \in K} |Z(x, \omega_1)| \right]$$

where C is a constant independent of the values of $\{a_\gamma \xi_\gamma(\omega_1) | \gamma \in A\}$. By Lemma 6 we have

$$(24) \quad \begin{aligned} &(E_1 |\tilde{I}(\sigma(u, \omega_1))|^2)^{1/2} \\ &\leq \int_0^{\mu_2} \frac{(E_1 \left| \int_0^s \sigma(u, \omega_1) du \right|^2)^{1/2}}{s^2 \left(\log \frac{4\mu_2}{s} \right)^{1/2}} ds \leq \tilde{I}(\sigma) \end{aligned}$$

where σ is given in (7). Also

$$(25) \quad (E_1 \sum_{\gamma \in A} |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2)^{1/2} \leq \left(\sum_{\gamma \in A} |a_\gamma|^2 \right)^{1/2}.$$

Using (23), (24), (25) and (16) in (22) we obtain (8).

We now show that the series (3) converges uniformly a.s. It follows from (24) and Lemma 5 that $I(\sigma) < \infty$ implies $J(K, \sigma(\cdot, \omega_1)) < \infty$ a.s. (P_1). Therefore by Theorem 2 there exists a set $\bar{\Omega}_1 \subset \Omega_1, P(\bar{\Omega}_1) = 1$, such that for $\omega_1 \in \bar{\Omega}_1$, $Z(x, \omega_1)$ has a version which is continuous a.s. (P_2). Therefore by the Ito-Nisio theorem (Theorem 2.3.4 [5]) the series (20) converges uniformly a.s. (P_2)

for each $\omega_1 \in \bar{\Omega}_1$. This implies, by Fubini's theorem, that the series (3) converges uniformly a.s. (P).

We now obtain the implications of $I(\sigma) = \infty$. The major result in this direction is Fernique's necessary condition for the continuity of stationary Gaussian processes. Consider

$$(26) \quad G(x) = \sum_{\gamma \in A} a_{\gamma} g_{\gamma} \gamma(x), \quad x \in K.$$

We use the following version of Fernique's theorem.

Theorem 7. A necessary condition for the series (26) to converge uniformly a.s. is that $J(K, \sigma) < \infty$.

Proof: Fernique's theorem (Theorem 8.1.1 [3]) is proved for real valued processes on R^n but only minor modifications are necessary to adapt the proof to the case considered here. Instead of $G(x)$ it is sufficient to prove Theorem 7 for the real valued process

$$(27) \quad Y(x) = \sum_{\gamma \in A} g_{\gamma} \operatorname{Re}(a_{\gamma} \gamma(x)) + \sum_{\gamma \in A} g'_{\gamma} \operatorname{Im}(a_{\gamma} \gamma(x)), x \in K,$$

where $\{g'_{\gamma} | \gamma \in A\}$ is an independent copy of $\{g_{\gamma} | \gamma \in A\}$, since $E(G(x) - G(y))^2 = E(Y(x) - Y(y))^2 = \sigma^2(x - y)$ and the series (26) and (27) either both converge uniformly a.s. or neither does.

The only point in the proof of Theorem 8.1.1 [3] that needs to be extended is Lemma 8.1.2. Let $H = \{x \in G | \sigma(x) = 0\}$ and form the quotient group $G' = G/H$. There exists a canonical mapping of G onto G' ; let K' be the image of K under this mapping. Denote by σ' the metric on K' that corresponds to the pseudo-metric σ on K .

Lemma 8: There exists a $\delta_0 > 0$ and a compact symmetric neighborhood of 0 $S \subset K$ such that if $s, t \in \bigoplus_{i=1}^4 S_i$ then $\sigma'(s-t) \leq \delta_0$ implies $s-t \in S$.

Proof: Let S be a compact symmetric neighborhood of $0 \in K'$ such that $\bigoplus_{i=1}^8 S \subset K'$. Let $\beta = \min\{\sigma'(x), x \in \bigoplus_{i=1}^8 S_i/S\}$. Since 0 is the unique zero of σ' on K' we have $\beta > 0$. Let $s, t \in \bigoplus_{i=1}^4 S_i$ then $s-t \in \bigoplus_{i=1}^8 S_i$. Set $\delta_0 = \beta/2$ then $\sigma'(s-t) \leq \delta_0$ implies $s-t \in S$. Consider S as given in Lemma 8 and let $T = \bigoplus_{i=1}^4 S_i$.

Following the notation of Theorem 8.1.1 [3] we define

$B(S, \delta_0) = \bigcup_{s \in S} B(s, \delta_0)$ where $B(s, \delta)$ denotes an open ball of radius

δ in K' with respect to the σ' metric. Let $s, t \in B(S, \delta_0) \cap T$, we show that for $\delta \leq \delta_0$, $B(s, \delta) \cap T = A_1$ and $B(t, \delta) \cap T = A_2$ are translates of each other, i.e. if $u \in A_1$ then $u + t - s \in A_2$. To do this we need only show that $u + t - s \in T$. Since $t \in B(S, \delta_0)$ there exists a $t' \in S$ such that $\sigma(t-t') < \delta_0$. Set

$$u + t - s = t' + (t-t') + (u-s).$$

Since $t, t' \in T = \bigoplus_{i=1}^4 S_i$, by Lemma 8, $t-t' \in S$. Similarly $u-s \in S$ and since $t' \in S$ we have $u + t - s \in T$.

Consider the process

$$(28) \quad Y'(x) = \sum_{\gamma \in A} g_{\gamma} \operatorname{Re}(a_{\gamma} \gamma(x)) + \sum_{\gamma \in A} g_{\gamma} \operatorname{Im}(a_{\gamma} \gamma(x)), \quad x \in K'.$$

This is a real valued stationary Gaussian process with

$(E|Y'(x) - Y'(y)|^2)^{1/2} = \sigma'(x-y)$ and an equivalent of Lemma 8.1.2 [3]

holds for this process.

Assume that the series (28) converges uniformly a.s. on K' . By the Landau, Shepp, Fernique theorem (Corollary 2.4.6 [5]) we have $E(\sup_{x \in K'} Y'(x)) < \infty$. We refer to the second paragraph of 8.1.4 [3] with S and T as given above. This shows that there exists a $\delta' > 0$ such that

$$\int_0^{\delta'} (\log N_{\sigma'}(S, u))^{1/2} du < \infty$$

and since S is compact we also have $J(S, \sigma') < \infty$. Finally, since K' is compact, there exists a constant $C > 0$ such that $N_{\sigma'}(S, u) \geq C N_{\sigma'}(K', u)$. Therefore $J(K', \sigma') < \infty$. To obtain Theorem 7 for $Y(x)$, $x \in K$ we note that the series (27) and (28) either both converge uniformly a.s. or neither does. Furthermore

$$E(\sup_{x \in K'} Y'(x)) = E(\sup_{x \in K} Y(x))$$

and $N_{\sigma'}(K', u) = N_{\sigma'}(K, u)$. Therefore we obtain Theorem 7.

Let $\{\gamma_k, k = 1, 2, \dots\}$ be an ordering of $A \subset \Gamma$. Our main result on necessary conditions for the convergence of random Fourier series is contained in the following lemma.

Lemma 9: In the notation established above we have

$$(29) \quad \left(E \sup_n \left\| \sum_{k=1}^n a_k \epsilon_k \xi_k \gamma_k \right\|^2 \right)^{1/2} \\ \leq C \left(\sup_k E |\xi_k|^2 \right)^{1/2} \left(E \sup_n \left\| \sum_{k=1}^n a_k g_k \gamma_k \right\|^2 \right)^{1/2}$$

and

$$(30) \quad (E \sup_n \|\sum_{k=1}^n a_k g_k \gamma_k\|^2)^{1/2} \leq C' (E \sup_n \|\sum_{k=1}^n a_k \epsilon_k \gamma_k\|^2)^{1/2}.$$

In particular, if $\sum_{\gamma \in A} a_\gamma \epsilon_\gamma \gamma(x)$ converges uniformly a.s. then so does $\sum_{\gamma \in A} a_\gamma g_\gamma \gamma(x)$ and

$$(31) \quad (E \|\sum_{\gamma \in A} a_\gamma g_\gamma \gamma\|^2)^{1/2} \leq C'' (E \|\sum_{\gamma \in A} a_\gamma \epsilon_\gamma \gamma\|^2)^{1/2}.$$

(Here C, C' and C'' are finite constants independent of $\{a_k\}$).

Proof: Belyaev's dichotomy [1], states that a stationary Gaussian process on the real line either has continuous sample paths a.s. or else is unbounded on all intervals. This dichotomy also holds for $G(x)$ and is a consequence of results of Ito and Nisio. (A proof can be made using Theorems 3.4.7 and 3.4.9 [5].) Consequently, we have that either $\sum_{k=1}^{\infty} a_k g_k \gamma_k(x)$ converges uniformly a.s. on K or else for all open sets $U \subset K$

$$(32) \quad \sup_n \sup_{x \in U} |\sum_{k=1}^n a_k g_k \gamma_k(x)| = \infty \text{ a.s.}$$

We also note that by Levy's inequality and the Landau, Shepp, Fernique theorem, if $\sum_{k=1}^{\infty} a_k g_k \gamma_k$ converges uniformly a.s. then

$$(33) \quad E(\sup_n \|\sum_{k=1}^n a_k g_k \gamma_k\|^2) < \infty.$$

Inequality (29) is a consequence of the closed graph theorem.

Let B_1 be the Banach space of sequences of complex numbers $\{a\} = \{a_1, a_2, \dots\}$ for which $\sum_{k=1}^{\infty} a_k g_k \gamma_k$ converges uniformly a.s. on K and $\|\{a\}\|_1 = (E \sup_n \|\sum_{k=1}^n a_k g_k \gamma_k\|^2)^{1/2} < \infty$. Let B_2 denote the

Banach space of sequences of complex numbers $\{a\} = \{a_1, a_2, \dots\}$ for which $\sum_{k=1}^{\infty} a_k \epsilon_k \xi_k \gamma_k$ converges uniformly a.s. on K for all sequences of complex random variables $\{\xi_k\}$ satisfying $E|\xi_k|^2 \leq 1$ and

$$\|\{a\}\|_2 = \sup_{\{\xi_k\}} (E \sup_n \|\sum_{k=1}^n a_k \epsilon_k \xi_k \gamma_k\|^2)^{1/2} < \infty. \text{ Then } \|\{a\}\|_1 < \infty \text{ implies}$$

$I(\sigma) < \infty$ by Theorem 7, (33) and (17) and $I(\sigma) < \infty$ implies $\|\{a\}\|_2 < \infty$ by (8)

of Theorem 1 (which we have already proved) and Levy's inequality.

Therefore (29) follows from the closed graph theorem applied to the identity mapping of B_1 onto B_2 .

To obtain (30) we write $g_k = g_k' + g_k''$ where $g_k' = g_k I[|g_k| < N]$ ($I[A]$) is the indicator function of the set A and N is chosen so that $(E|g_k''|^2)^{1/2} = (2C)^{-1}$. Then, for all j

$$\begin{aligned} & E(\sup_{n \leq j} \|\sum_{k=1}^n a_k g_k \gamma_k\|^2)^{1/2} \\ & \leq (E \sup_{n \leq j} \|\sum_{k=1}^n a_k g_k' \gamma_k\|^2)^{1/2} + (E \sup_{n \leq j} \|\sum_{k=1}^n a_k g_k'' \gamma_k\|^2)^{1/2}. \end{aligned}$$

By Theorem 5.3 [12]

$$(E \sup_{n \leq j} \|\sum_{k=1}^n a_k g_k' \gamma_k\|^2)^{1/2} \leq N (E \sup_{n \leq j} \|\sum_{k=1}^n a_k \epsilon_k \gamma_k\|^2)^{1/2}$$

and by (29)

$$(E \sup_{n \leq j} \|\sum_{k=1}^n a_k g_k'' \gamma_k\|^2)^{1/2} \leq \frac{1}{2} (E \sup_{n \leq j} \|\sum_{k=1}^n a_k g_k \gamma_k\|^2)^{1/2}.$$

Putting this together we have

$$(E \sup_{n \leq j} \|\sum_{k=1}^n a_k g_k \gamma_k\|^2)^{1/2} \leq 2N (E \sup_{n \leq j} \|\sum_{k=1}^n a_k \epsilon_k \gamma_k\|^2)^{1/2}.$$

Passing to the limit as $j \rightarrow \infty$ we obtain (30) with $C' = 2N$.

If $\sum_{\gamma \in A} a_\gamma \epsilon_\gamma \gamma(x)$ converges uniformly a.s. the right side of (30) is finite by Kahane's theorem; therefore $\sum_{\gamma \in A} a_\gamma g_\gamma \gamma(x)$ converges uniformly a.s. by (30) and the extended Belyaev dichotomy.

By Lemma 9, and what we have already proved, we have that $I(\sigma) < \infty$ is a necessary and sufficient condition for the uniform convergence a.s. of $\sum_{k=1}^{\infty} a_k \epsilon_k \gamma_k$. This, essentially, is all we need to complete the proof of Theorem 1. For instance, if $\{\xi_k\}$ is also independent (besides satisfying (2)) then by Theorem 5.1 [12], $I(\sigma) < \infty$ is a necessary and sufficient condition for the uniform convergence a.s. of the series in (3). Also, one can easily show that $I(\sigma) = \infty$ implies $E \sup_n \|\sum_{k=1}^n a_k \epsilon_k \xi_k \gamma_k\|^2 = \infty$. For the actual completion of the proof of Theorem 1 we refer the reader to Lemma 2.5 [8] and the brief "Proof of Theorem 1.1" on page 2.11 [8]. This material, although written for the case $G = \mathbb{R}$, extends immediately to the case considered here.

All the results of [8] have versions for the more general class of random series considered in this paper. These include a central limit theorem for $Z(x)$ and the identification of the uniformly convergent series of the type given in (3) with a Banach space of cotype 2. An application of random Fourier series to a non-random problem in the study of lacunary series is given in [10].

Note that it is not necessary to assume that $\sup_{\gamma} E |\xi_\gamma|^2 < \infty$ in

(3). Let $\{\xi_\gamma\}$ be simply a sequence of complex valued random variables on the probability space $(\Omega_1, \mathcal{F}_1, P_1)$. Then a necessary and sufficient condition of the series (3) to converge uniformly a.s. is that

$$(34) \quad I((\sum_{\gamma} |a_{\gamma}|^2 |\xi_{\gamma}(\omega_1)|^2 |\gamma(s)-1|^2)^{1/2}) < \infty \text{ a.s. } (P_1).$$

From (34) we can obtain results even when the $\{\xi_\gamma\}$ do not have second moments. For example, let $\{\xi_\gamma\}$ be independent copies of ξ where $E[e^{it\xi}] = e^{-|t|^p}$. Then we have

$$I((\sum_{\gamma} |a_{\gamma}|^p |\gamma(s)-1|^p)^{1/p}) < \infty$$

implies $\sum_{\gamma} a_{\gamma} \epsilon_{\gamma} \xi_{\gamma}(x)$ converges uniformly a.s. (see Theorem 2.9 [8]).

We plan to elaborate upon these remarks and to give a more detailed proof of Theorem 1 in a later paper.

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