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LOCAL TIMES AND SINGULARITIES OF CONTINUOUS LOCAL MARTINGALES

by

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0. INTRODUCTION.

The first section of this paper is mostly expository. Given a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(\Omega, \mathcal{F}, P)$ satisfying the usual hypotheses — that is, $(\mathcal{F}_t)$ is right continuous and $\mathcal{F}_0$ contains all null sets — we consider some properties of the space $\mathcal{L}^c$ of continuous local martingales over $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ related to the local time processes L_t^a $(a \in \mathbf{R})$. Though most results in Section 1 are known, they do not all seem to be well known, and they set the stage for the results of Section 2 where we study continuous local martingales having a singularity at the time origin. Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ as above, let $\mathcal{L}^c_{\text{open}}$ denote the space of all real processes $(M_t)_{t>0}$ defined on the *open* interval $]0,\infty[$ such that $t \to M_t$ is a.s. continuous and

(0.1) *There exists a decreasing sequence* $\{S_n\}$ of *stopping times such that* $P\{0 < S_n < \infty\} = 1$ *for all* n, *and* $P\{S_n \downarrow\downarrow 0\} = 1$;

(0.2) *for each* n, *the process* $t \to M(S_n + t)$ *is a local martingale over the filtration* $(\mathcal{F}(S_n + t))_{t\geq 0}$.

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It should be emphasized that in general the sequence $\{T_n^k\}_{k\geq 1}$ which reduces $(M(S_n+t))$ — that is, such that $t\to M(S_n+t\wedge T_n^k)\, 1_{\{T_n^k>0\}}$ is a uniformly integrable martingale over $(\mathcal{F}(S_n+t))$ and $T_n^k\uparrow\infty$ a.s. as $k\uparrow\infty$ — depends on n. To illustrate the possibilities, let $(B_t)_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}^d$ $(d\geq 2)$ and let f be harmonic on $\mathbb{R}^d\setminus\{0\}$. Since B_t never hits 0 at a strictly positive time, the Itô calculus shows that $f(B_t)_{t>0}$ is in $\mathcal{L}^c_{open}$ relative to P^0, the law of B starting at 0. The nature of the singularity of f at 0 is reflected in the behavior of $f(B_t)$ at $t\downarrow\downarrow 0$. If the singularity is removable, $\lim_{t\downarrow\downarrow 0} f(B_t)$ exists in $\mathbb{R}$. If f has a pole at 0, $\lim_{t\downarrow\downarrow 0} f(B_t)$ exists in $\bar{\mathbb{R}}=[-\infty,\infty]$, while if f has an essential singularity at 0, $\liminf f(B_t)=-\infty$ and $\limsup f(B_t)=\infty$. Walsh [5] studied conformal local martingales on $]0,\infty[$ and showed that almost surely, either the limit as $t\downarrow\downarrow 0$ exists in the Riemann sphere or the path is dense in the Riemann sphere. We consider here two aspects of the space $\mathcal{L}^c_{open}$. First of all, we shall state and prove the analogue of Walsh's Theorem for real continuous local martingales on $]0,\infty[$, with characterizations of the cases in terms of the quadratic variation and local time at zero. Following that, we consider a generalization of these results to stochastic integrals $\int C_s\, dM_s$, where the stochastic integral is meaningful over any interval bounded away from zero, but may have a singularity at time zero. In this case one may not select one single local martingale on $]0,\infty[$ whose increments give the stochastic integral over an arbitrary interval, so new methods are needed.

1. LOCAL MARTINGALES.

For the basic properties of local martingales we shall use Meyer [4] as a reference, but since we shall consider only continuous local martingales here, little is needed beyond the article of Azema and Yor [1]. Given $M\in\mathcal{L}^c$, the local time process (L_t^a) for M at a is defined to be the unique continuous increasing process with $L_0^a = 0$ such that $|M_t - a| - L_t^a$ belongs to $\mathcal{L}^c$. In addition, the quadratic variation process $\langle M, M\rangle_t$ is the unique continuous increasing process with $\langle M, M\rangle_0 = 0$ such that $M_t^2 - \langle M, M\rangle_t$ belongs to $\mathcal{L}^c$. The following facts are very well known.

(1.1) $\langle M, M\rangle_t$ and M_t have the same intervals of constancy ([3], for example).

(1.2) If $\langle M, M\rangle_t = t$ then $M_t - M_0$ is a standard Brownian motion over $(\mathcal{F}_t)$ (Levy's Theorem [4]).

(1.3) For all $a \in \mathbb{R}$, dL_t^a is carried by $H^a = \{t > 0: M_t = a\}$ and dL_t^a does not charge any interval contained in H^a ([1]).

(1.4) If $M \in \mathcal{L}^c$ and $T_n = \inf\{t: |M_t| \ge n\}$ then $T_n \uparrow \infty$ a.s. and for all n, $t \to M_{t\wedge T_n} 1_{\{T_n > 0\}}$ is a (bounded) martingale over $(\mathcal{F}_t)$.

(1.5) If $M \in \mathcal{L}^c$ and $E\langle M, M\rangle_\infty < \infty$, then $M - M_0$ is a martingale with $E[\sup_{t\ge 0} |M_t - M_0|^2] \le 4E\langle M, M\rangle_\infty$ (Doob's inequality).

(1.6) If $M \in \mathcal{L}^c$, if $M_0 = 0$ and if M is uniformly bounded below then M_t is a supermartingale (Fatou's lemma) and $M_\infty = \lim_{t\to\infty} M_t$ exists and is finite a.s.

(1.7) One may choose the L_t^a so that $(a,t,\omega) \to L_t^a(\omega)$ is jointly measurable ([1], p.10) and then $\langle M, M\rangle_t = \int_{-\infty}^{\infty} L_t^a \, da$. (Assume M bounded, by stopping, so that $\int da(|M_t - a| - L_t^a)$ is a martingale, and consequently $M_t^2 - \int L_t^a \, da$ is a martingale.)

(1.8) $d\langle M, M\rangle_t$ does not charge H^a for any $a \in \mathbb{R}$ (by (1.7)).

(1.9) (Tanaka's formula [1]). If $M \in \mathcal{L}^c$ then

$$|M_t - a| = |M_0 - a| + \int_0^t \operatorname{sgn}(M_u - a)\,dM_u + L_t^a$$

where $\operatorname{sgn} x = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.

The following consequence of Tanaka's formula seems to be known, at least to experts.

(1.10) Proposition. *Let* $M \in \mathcal{L}^c$ *with* $M_0 = 0$, *and set* $W_t = -\int_0^t \operatorname{sgn} M_u dM_u$ *so that by* (1.9), $|M_t| = L_t - W_t$. *Then*

(i) $\langle W, W\rangle_t = \langle M, M\rangle_t$ *for all* $t \geq 0$;

(ii) *for all* $t \geq 0$, $L_t^0 = W_t^m := \max\{W_s : 0 \leq s \leq t\}$.

Proof. Statement (i) comes from the fact that

$$\langle W, W\rangle_t = \int_0^t \operatorname{sgn}^2 M_u \, d\langle M, M\rangle_u$$

which is equal to $\langle M, M\rangle_t$ by (1.8). To prove that $L_t^0 = W_t^m$ for all $t \geq 0$, it suffices to prove that their right continuous inverse processes are indistinguishable. Let $\sigma_r(\omega) = \inf\{t: L_t^0 > r\}$ (with $\inf \phi = \infty$) and $\tau_r(\omega) = \inf\{t: W_t^m > r\}$. If $\sigma_r(\omega) < \infty$, $\sigma_r(\omega)$ is a point of increase of $t \to L_t^0(\omega)$ and so by (1.3), $M_{\sigma_r(\omega)}(\omega) = 0$. Since $|M| = L - W$ and $L_{\sigma_r} = r$ if $\sigma_r < \infty$ we obtain $W_{\sigma_r(\omega)}(\omega) = r$ if $\sigma_r(\omega) < \infty$. It follows that for all $s < r$, $\tau_s(\omega) \leq \sigma_r(\omega)$ and so $\tau_{r-}(\omega) \leq \sigma_r(\omega)$ for all r. On the other hand, if $\tau_r(\omega) < \infty$, $\tau_r(\omega)$ is a point of increase of $t \to W_t^m(\omega)$ so

$W^m_{\tau_r(\omega)}(\omega) = W_{\tau_r(\omega)}(\omega)$. Using $L - W = |M| \geq 0$ this implies $L_{\tau_r(\omega)}(\omega) - r \geq 0$ if $\tau_r(\omega) < \infty$. This implies that for all $s < r$, $\sigma_s(\omega) \leq \tau_r(\omega)$ so $\sigma_{r-}(\omega) \leq \tau_r(\omega)$. By right continuity we obtain $\sigma_r(\omega) = \tau_r(\omega)$ for all $r \geq 0$ a.s. .

As a first application of (1.10), we consider the convergence of $M \in \mathcal{L}^c$ as $t \to \infty$.

(1.11) Theorem. *Let* $M \in \mathcal{L}^c$. *Then, almost surely*

$$\{M_\infty \textit{ exists and is finite}\} = \{\limsup_{t\to\infty} M_t < \infty\} = \{\langle M, M\rangle_\infty < \infty\} = \{L^0_\infty < \infty\} .$$

Proof. We may assume that $M_0 = 0$. The first equality is due to Doob ([2], p.382) but since the proof is a model for the other equalities we indicate its proof. Let $T_n = \inf\{t: M_t \geq n\}$ so that $\cup\{T_n = \infty\} = \{\limsup M_t < \infty\}$. Since $M_{t\wedge T_n}$ is bounded above, it converges a.s. (1.6), hence M_∞ exists a.s. on $\cup\{T_n = \infty\}$. For the next equality, first set $T_n = \inf\{t: |M_t| \geq n\}$ so that $M^{T_n}_t = M_{t\wedge T_n}$ is a uniformly bounded martingale. Since $M_{t\wedge T_n}$ is an L^2 bounded martingale, $E\langle M^{T_n}, M^{T_n}\rangle_\infty \leq n^2$, and in particular $\langle M, M\rangle_{T_n} < \infty$. Since $\cup\{T_n = \infty\} \supset \{M_\infty \text{ exists}\}$, this shows that $\{M_\infty \text{ exists}\} \subset \{\langle M, M\rangle_\infty < \infty\}$. In the same way, $EL^0_{t\wedge T_n} = E|M_{t\wedge T_n}| \leq n$ shows that $\{M_\infty \text{ exists}\} \subset \{L^0_\infty < \infty\}$. Now set $R_n = \inf\{t: \langle M, M\rangle_t \geq n\}$ so that $\cup\{R_n = \infty\} = \{\langle M, M\rangle_\infty < \infty\}$. Since $\langle M^{R_n}, M^{R_n}\rangle \leq n$, M^{R_n} is L^2 bounded (1.5) and so $M^{R_n}_\infty$ exists. Thus $\{\langle M, M\rangle_\infty < \infty\} \subset \{M_\infty \text{ exists}\}$. Finally, let $W = L^0 - |M|$ as in (1.10). Because $L^0_\infty = \sup_t W_t$, $\limsup W_t < \infty$ on $\{L^0_\infty < \infty\}$ so W_t converges on $\{L^0_\infty < \infty\}$. But since $\{W_\infty \text{ exists}\} = \{\langle W, W\rangle_\infty < \infty\} = \{\langle M, M\rangle_\infty < \infty\} = \{M_\infty \text{ exists}\}$, we have shown that M converges on $\{L^0_\infty < \infty\}$.

The connection between convergence of M and the local time of M at zero is not surprising because it is well known that L^0_t can be expressed as a normalized limit of the number of downcrossings of $[0, \varepsilon]$ up to time t. If $M \in \mathcal{L}^c$ and $M_0 = 0$, it is easy to see from (1.10) that $EL_\infty^0 = \sup\{E|M_T|: T$ a finite stopping time$\}$ so that in particular, if M is a martingale, M is L^1 bounded if and only if $EL_\infty^0 < \infty$. Further in this direction, if $M \in \mathcal{L}^c$ and $M_0 = 0$ then setting $W = L^0 - |M|$ so that $\langle W, W\rangle = \langle M, M\rangle$, the Burkholder-Davis-Gundy inequalities imply that for every $p > 1$, there exist absolute constants C_p such that

$$(1.12) \qquad E(L_\infty^0)^p \le C_p \sup\{E|M_T|^p: T \text{ a finite stopping time}\}.$$

Similar arguments show that if M is in BMO then

$$(1.13) \qquad E[\exp(\lambda L_\infty^0)] < \infty \quad \text{for some } \lambda > 0.$$

The inequality (1.12) was obtained in [1] by different (more elementary) methods.

If in the situation of (1.10), M is a standard Brownian motion then $W = L^0 - |M|$ is also a standard Brownian motion since $\langle W, W\rangle_t = \langle M, M\rangle_t = t$. The fact that $L_t^0 = W_t^m$ $(= \sup\{W_s: 0 \le s \le t\})$ sharpens the well known fact that L_t^0 and the one-sided maximal process of Brownian motion have the same law, by actually producing a Brownian motion for which L_t^0 is the one-sided maximal function. Similarly, (1.10) demonstrates why $|M_t|$ and $M_t^m - M_t$ are processes with the same law: one produces a Brownian motion W_t with $|M_t| = W_t^m - W_t$.

In preparation for a number of arguments in the next section, we need the following simple lemma about birthing a local martingale at a stopping time.

(1.14) Lemma. *Let* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *satisfy the usual hypotheses and let* R *be a stopping time over* $(\mathcal{F}_t)$. *Then*

(i) *for any stopping time* T *over* $(\mathcal{F}_t)$, *the random variable* $(T-R)1_{\{T>R\}}$ *is a stopping time over* $(\mathcal{F}_{R+t})$;

(ii) *if* M *is a local martingale over* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *then* $N_t = M_{t+R}\, 1_{\{R<\infty\}}$ *is a local martingale over* $(\mathcal{F}_{R+t})$ *relative to the conditional probability measure* $P\{\cdot\,|R<\infty\}$.

Proof. It is easy to see that $(\mathcal{F}_{R+t})$ satisfies the usual hypotheses. To prove (i), just observe that $\{(T-R)1_{\{T>R\}} > t\} = \{T > R+t\} \in \mathcal{F}_{R+t}$. For (ii) we may assume that $P\{R<\infty\} > 0$, for otherwise $N = 0$ and there is nothing to prove, no matter how $P\{\cdot\,|R<\infty\}$ is defined. If M is uniformly integrable with limit M_∞ then using optional sampling, for $G \in b\mathcal{F}_{R+t}$

$$\begin{aligned} E\{N_\infty G|R<\infty\} &= E\{M_\infty\, 1_{\{R<\infty\}}G\}/P\{R<\infty\} \\ &= E\{M_{R+t}\, 1_{\{R<\infty\}}G\}/P\{R<\infty\} \\ &= E\{N_t G|R<\infty\}\ , \end{aligned}$$

so N is a uniformly integrable martingale relative to $(\Omega, (\mathcal{F}_{R+t}), P\{\cdot\,|R<\infty\})$. In the general case assume that $\{T_n\}$ reduces M, and let $T_n' = (T_n-R)1_{\{T_n>R\}}$. Then $\{T_n'\}$ is an increasing sequence of stopping times over $(\mathcal{F}_{R+t})$ such that $P\{\lim T_n' = \infty|R<\infty\} = 1$. For every n

$$\begin{aligned} M(R+t\wedge T_n')1_{\{T_n'>0\}} &= M(R+t\wedge(T_n-R))1_{\{T_n>R\}} \\ &= M((t+R)\wedge T_n)1_{\{T_n>R\}}. \end{aligned}$$

Since $t \to M_{t\wedge T_n}\, 1_{\{T_n>0\}}$ is a uniformly integrable martingale and $\{T_n > R\} \in \mathcal{F}_R$, it follows that $t \to M(R+t\wedge T_n')\, 1_{\{T_n'>0\}}$ is a uniformly integrable martingale over $(\Omega, (\mathcal{F}_{R+t}), P\{\cdot\,|R<\infty\})$ for every $n \geq 1$.

2. LOCAL MARTINGALES OVER $]0,\infty[$.

Observe to begin with that (1.14) implies $\mathcal{L}^c \subset \mathcal{L}^c_{\text{open}}$. It will turn out (2.16) that if $M \in \mathcal{L}^c_{\text{open}}$ and $M_0 = M_{0+}$ exists and is finite a.s. then $(M_t)_{t\geq 0}$ is in $\mathcal{L}^c$.

(2.1) Proposition. *A continuous process* $(M_t)_{t>0}$ *is in* $\mathcal{L}^c_{\text{open}}$ *if and only if for some sequence* $\{S_n\}$ *of stopping times (not necessarily finite valued) satisfying*

(2.2) $$P\{S_n > 0\} = 1 \quad \textit{and} \quad P\{S_n \textit{ decreases to } 0\} = 1$$

it is the case that

(2.3) *for all* $n \geq 1$, *the process* $N^n_t = M(S_n + t)1_{\{S_n<\infty\}}$ *is a local martingale over* $(\Omega, (\mathcal{F}_{S_n+t}), P\{\cdot|S_n < \infty\})$.

If $M \in \mathcal{L}^c_{\text{open}}$, *then for every sequence* $\{S_n\}$ *satisfying* (2.2) *the condition* (2.3) *holds.*

Proof. Fix one sequence $\{S_n\}$ satisfying (2.2) and (2.3) and let $\{R_n\}$ be a sequence satisfying (2.2). We shall prove then that $\{R_n\}$ satisfies (2.3). Taking each R_n finite valued will show $M \in \mathcal{L}^c_{\text{open}}$, and the last assertion of (2.1) will obtain for general $\{R_n\}$. For $m \geq 1$ and $s > 0$, let $T(m, s) = \inf\{t > s: |M_t| \geq m\}$. Then $T(m, s)$ is a stopping time over $(\mathcal{F}_t)$ and for all $s > 0$, $T(m, s)$ increases in m, say to $T(\infty, s)$. By hypothesis $T(\infty, S_n) = \infty$ a.s. for all n. Since $S_n \downarrow 0$ a.s. and $s \to T(m, s)$ is increasing in s for all $m \geq 1$, it follows that $P\{T(\infty, s) = \infty$ for all $s > 0\} = 1$. Because of (2.3), (1.4) and (1.14), the process

$$t \to M((S_n + t) \wedge T(m, S_n))\, 1_{\{T(m,S_n)>S_n\}}$$

is a bounded martingale over $(\Omega, \mathcal{F}_{S_n+t}, P\{\cdot|S_n<\infty\})$ for all $n\geq 1$ and $m\geq 1$. It follows that as $t\uparrow T(m, S_n)$, $M(t)$ converges a.s. on $\{T(m, S_n) > S_n\}$. Denoting the limit by $M(T(m, S_n))\, 1_{\{T(m,S_n)>S_n\}}$, one has

$$M((S_n+t)\wedge T(m, S_n))\, 1_{\{T(m,S_n)>S_n\}} = E\{M(T(m, S_n))\, 1_{\{T(m,S_n)>S_n\}}|\mathcal{F}_{S_n+t}\} .$$

By optional sampling, with t replaced by $(R_k+t-S_n)\, 1_{\{R_k+t>S_n\}}$ we obtain, a.s. on $\{R_k+t>S_n\}$,

$$M((R_k+t)\wedge T(m, S_n))\, 1_{\{T(m,S_n)>S_n\}} = E\{M(T(m, S_n))\, 1_{\{T(m,S_n)>S_n\}}|\mathcal{F}_{R_k+t}\} .$$

This equality holds in particular on $\{R_k>S_n\}$, and one may then interpret the equality to mean that $t\to M(R_k+t)\, 1_{\{R_k<\infty\}}$ is a local martingale relative to $(\Omega, \mathcal{F}_{R_k+t}, P\{\cdot|R_k<\infty\})$ having reducing times $(T(m, S_n)-R_k)\, 1_{\{T(m,S_n)>S_n\vee R_k\}} = T^k_{m,n}$. (Note that for a fixed k, $P\{\sup_{m,n} T^k_{m,n}=\infty|R_k<\infty\} = 1$.)

The first important result describing the behavior at the time origin of $M\in\mathcal{L}^c_{open}$ is the following.

(2.4) Theorem. *Let* $M\in\mathcal{L}^c_{open}$. *Then for a.a.* ω, *either*

(i) $\lim_{t\downarrow\downarrow 0} M_t(\omega)$ *exists in* $\mathbb{R}$

or

(ii) $\lim_{t\downarrow\downarrow 0} M_t(\omega)=\pm\infty$

or

(iii) $\liminf_{t\downarrow\downarrow 0} M_t(\omega)=-\infty$ *and* $\limsup_{t\downarrow\downarrow 0} M_t(\omega)=\infty$.

This theorem may be deduced from Walsh's Theorem on conformal martingales since every local martingale is the real part of some conformal martingale. Because Walsh's proof is a little obscure at one point, we shall derive (2.4) from scratch. In preparation for this we need a couple of lemmas.

(2.5) Lemma. *Let* $M \in \mathcal{L}^c$ *and suppose that* $M_0 = b$ *a.s.. Then if* $a < b < c$, $P\{M_t$ *hits* a *before it hits* $c \mid \mathcal{F}_0\} \le (c-b)/(c-a)$.

Proof. Let T_a (resp., T_c) denote the first time M_t hits a (resp., c). As we mentioned in (1.4), a uniformly bounded local martingale is in fact a martingale. Since T_a and T_c are stopping times over $(\mathcal{F}_t)$, it follows that $M_{t\wedge(T_a\wedge T_c)}$ is a bounded martingale whose limit at infinity is

$$M(T_a \wedge T_c) \le a\, 1_{\{T_a<T_c\}} + c 1_{\{T_a\ge T_c\}} .$$

Taking conditional expectations leads to

$$b \le a\; P\{T_a < T_c \mid \mathcal{F}_0\} + c\; P\{T_a \ge T_c \mid \mathcal{F}_0\} ,$$

from which the desired inequality obtains.

(2.6) Lemma. *Let* $M \in \mathcal{L}^c$ *and for* $a < b < c$ *and* k *a positive integer, let* $R^k_{[a,b]} = \inf\{t\colon M_s\ (0 \le s \le t)$ *completes* k *upcrossing of* $[a, b]\}$ *and* $T_c = \inf\{t\colon M_t \ge c\}$ (*with* $\inf \phi = \infty$). *Then*

$$P\{R^k_{[a,b]} < T_c\} \le \left(\frac{c-b}{c-a}\right)^{k-1} .$$

Proof. In the course of the proof, we let R^k denote $R^k_{[a,b]}$. On $\{R^k < \infty\}$, $M(R^k) = b$ by definition of R^k. Let P_k denote the conditional probability measure $P_k\{\cdot\} = P\{\cdot \mid R^k < \infty\}$ defined in an arbitrary way if $P\{R^k < \infty\} = 0$. We showed (1.14) that the process $t \to M(R^k + t)\, 1_{\{R^k < \infty\}}$ under $P\{\cdot \mid R^k < \infty\}$ is a continuous local martingale over the filtration $\mathfrak{F}(R^k + t)$. Moreover, if $P\{R^k < \infty\} > 0$, $P_k\{M(R^k) = b\} = 1$. For all $k \geq 2$ we have

$$\begin{aligned} P\{R^k < T_c\} &\leq P\{R^{k-1} < T_c,\ M(R^{k-1} + t) \text{ hits } a \text{ before it hits } c\} \\ &= E\{P\{M(R^{k-1} + t) \text{ hits } a \text{ before it hits } c \mid \mathfrak{F}(R^{k-1})\};\ R^{k-1} < T_c\} \\ &= E\{P_{k-1}\{M(R^{k-1} + t) \text{ hits } a \text{ before it hits } c \mid \mathfrak{F}(R^{k-1})\};\ R^{k-1} < T_c\} \\ &\leq (c-b)/(c-a)\ P\{R^{k-1} < T_c\} \end{aligned}$$

because of (2.5) applied to the filtration $\mathfrak{F}(R^{k-1} + t)$. The conclusion of (2.6) is now clear by induction on k.

Proof of (2.4): Given $a < b$, let

$$\Gamma_{a,b} = \{\omega \in \Omega: \liminf_{t \downarrow\downarrow 0} M_t(\omega) < a,\ \limsup_{t \downarrow\downarrow 0} M_t(\omega) > b\}$$

$$\Gamma = \{\omega \in \Omega: \liminf_{t \downarrow\downarrow 0} M_t(\omega) = -\infty,\ \limsup_{t \downarrow\downarrow 0} M_t(\omega) = \infty\}$$

Obviously $\Gamma = \cap\{\Gamma_{a,b}: a < b \text{ rationals}\}$. In order to prove (2.4) it is enough to prove that $\Gamma \supset \Gamma_{a,b}$ a.s. for any pair of rationals $a < b$. If $c > b$, let $T_c = \inf\{t: M_t \geq c\}$. We shall prove that for all $c > b$, $T_c = 0$ a.s. on $\Gamma_{a,b}$ and this will show that $\limsup_{t \downarrow\downarrow 0} M_t = \infty$ a.s. on $\Gamma_{a,b}$. Applying this result to $-M$, one will then obtain $\Gamma \supset \Gamma_{a,b}$ a.s. . Fix $a < b < c$ and let $\{S_n\}$ satisfy (0.1) (and hence (0.2) also by (2.1)). For each $n \geq 1$, let

$T_n^c = \inf\{t: M(S_n + t) \geq c\}$ and let R_n^k denote the first time $M(S_n + t)$ completes k upcrossing of $[a, b]$. Now fix $k \geq 1$. As n increases, the events $\{T_c > S_n\} \cap \{R_n^k < T_c^n\}$ increase, for on $\{T_c > S_n\}$, $T_c^n = T_c - S_n$. On the other hand, their union over all n contains $\{T_c > 0\} \cap \Gamma_{a,b}$. Thus

$$P\{\{T_c > 0\} \cap \Gamma_{a,b}\} \leq \lim_n P\{R_n^k < T_c^n\}$$

$$\leq \left(\frac{c-b}{c-a}\right)^{k-1},$$

using (2.6). Since k is arbitrary, this proves that $T_c = 0$ a.s. on $\Gamma_{a,b}$, completing the proof.

We turn now to characterizing the cases (i), (ii) and (iii) of (2.4) in terms of the quadratic variation and local times for M, which we now describe.

If M is a continuous local martingale over an arbitrary filtration $(\mathcal{G}_t)$ (satisfying the usual hypotheses) of $(\Omega, \mathcal{F}, P)$ it is easy to see, using (1.14), that for any finite stopping time R the quadratic variation process and the local time at a for $t \to M(R+t)1_{\{R<\infty\}}$ are respectively $\langle M, M\rangle_{R+t} - \langle M, M\rangle_R$ and $L_{R+t}^a - L_R^a$.

The following result obtains by an elementary covering argument.

(2.7) Proposition. *Let* $M \in \mathcal{L}_{open}^c$. *There exist unique random measures* $Q(\omega, dt)$ *and* $\lambda^a(\omega, dt)$ *on* $]0,\infty[$, *such that if* $\{S_n\}$ *are stopping times satisfying* (0.1).

Then for all n

(2.8) $M^2_{S_n+t} - Q(\omega,]S_n(\omega), S_n(\omega)+t])$ *is a local martingale over* $(\mathcal{F}_{S_n+t})$;

(2.9) $|M_{S_n+t} - a| - \lambda^a(\omega,]S_n(\omega), S_n(\omega)+t]$ *is a local martingale over* $(\mathcal{F}_{S_n+t})$.

In general, $Q(\omega, dt)$ and $\lambda^a(\omega, dt)$ blow up at the origin and so they are not always generated by continuous increasing processes normalized to vanish at the origin. However, (2.7) shows that a.s., $Q(\omega, \cdot)$ and $\lambda^a(\omega, \cdot)$ are Radon measures on $]0,\infty[$. We record now two elementary operations which preserve the class $\mathcal{L}^c_{open}$. The proofs are routine and are left to the reader.

(2.10) Proposition. *If* $M \in \mathcal{L}^c_{open}$ *then*

(2.11) *for any stopping time* T, $M_{t\wedge T} 1_{\{T>0\}} \in \mathcal{L}^c_{open}$;

(2.12) *if* $H \in \mathcal{F}_0$ *then* $1_H M \in \mathcal{L}^c_{open}$.

It is evident from (2.7) that if Q and λ^a are the quadratic variation and local time measures for M, then those for $M_{t\wedge T} 1_{\{T>0\}}$ and $1_H M$ are respectively $1_{]\!]0,T]\!]}(t)Q(dt)$, $1_{]\!]0,T]\!]}(t)\lambda^a(dt)$ and $1_H(\omega)Q(\omega, dt)$, $1_H(\omega)\lambda^a(\omega, dt)$. For example, the last case above uses the observation that

$$|1_H M_{s+t}-a| - 1_H \lambda^a(]s, s+t]) = 1_H[|M_{s+t}-a| - \lambda^a(]s, s+t])] + 1_{H^c}|a| .$$

is a local martingale over $(\mathcal{F}_{s+t})$ for all $s>0$.

(2.13) Lemma. *If* $M \in \mathcal{L}^c_{open}$ *and* M *is uniformly bounded, then* $(M_t)_{t>0}$ *is a martingale. Consequently,* $M_0 = \lim_{t\downarrow\downarrow 0} M_t$ *exists a.s. and* $(M_t)_{t\geq 0}$ *is a martingale.*

Proof. Once we prove that $(M_t)_{t>0}$ is a martingale, the assertions of the sentence will follow from the reverse martingale convergence theorem. Let $r_n \downarrow\downarrow 0$. Then by (2.1), $t \to M(r_n + t)$ is a local martingale over $(\mathcal{F}(r_n+t))$, and its boundedness implies that it is in fact a uniformly integrable martingale. It follows that M_∞ exists and for all $t \geq 0$, $E\{M_\infty | \mathcal{F}(r_n+t)\} = M(r_n+t)$. That is, $(M_t)_{t>0}$ is a martingale over $(\mathcal{F}_t)$.

(2.14) Lemma. *Let* $M \in \mathcal{L}^c_{open}$ *and suppose that* $M_t^2 - t \in \mathcal{L}^c_{open}$. *Then* $\lim_{t\downarrow\downarrow 0} M_t = M_0$ *exists and is finite, and* $(M_t - M_0)_{t\geq 0}$ *is a standard Brownian motion over* $(\mathcal{F}_t)$.

Proof. Fix a sequence of constant times $r_n \downarrow\downarrow 0$ so that for all n, $M(r_n + t)$ and $M^2(r_n + t) - (r_n + t)$ are continuous local martingales over $(\mathcal{F}_{r_n+t})$. Lévy's Theorem (1.2) implies that $M(r_n + t) - M(r_n)$ is a standard Brownian motion. It follows that for $0 < u < v$, $M_v - M_u$ has a normal distribution with mean 0 and variance $v - u$, and that the increments of M_t are independent. Therefore the process $t \to M_1 - M_{1-t}$ $(0 \leq t < 1)$ is a continuous martingale which is L^2-bounded, so $\lim_{t\uparrow\uparrow 1} M_1 - M_{1-t}$ exists a.s.. Consequently M_0 exists a.s., and since then $M_t - M_0 = \lim_{u\downarrow\downarrow 0} M_t - M_u$ has independent Brownian increments, the result follows.

Here then is the main result of this section.

(2.15) Theorem. *Let* $M \in \mathcal{L}^c_{open}$, *and let* $\Omega_Q = \{\omega \in \Omega\colon Q(\omega,]0, t]) < \infty$ *for some (and hence all)* $t > 0\}$, $\Omega_a = \{\omega \in \Omega\colon \lambda^a(\omega,]0, t]) < \infty$ *for some (and hence all)* $t > 0\}$. *For each fixed* $a \in \mathbb{R}$, *almost surely*

(i) $\{\omega: \lim_{t\downarrow\downarrow 0} M_t(\omega)$ *exists and is finite*$\} = \Omega_Q$;

(ii) $\{\omega: \lim_{t\downarrow\downarrow 0} M_t(\omega) = \pm\infty\} = \Omega_a \backslash \Omega_Q$;

(iii) *for all* $\omega \notin (\Omega_Q \cup \Omega_a)$, $\liminf_{t\downarrow\downarrow 0} M_t = -\infty$ *and* $\limsup_{t\downarrow\downarrow 0} M_t(\omega) = \infty$.

Proof. Let $\Lambda = \{\omega: \lim_{t\downarrow\downarrow 0} M_t(\omega)$ exists and is finite$\}$. Since $\Lambda \in \mathfrak{F}_{0+} = \mathfrak{F}_0$, $N_t = 1_\Lambda M_t \in \mathfrak{L}^c_{\text{open}}$ by (2.12). For all $\omega \in \Omega$, $\lim_{t\downarrow\downarrow 0} N_t = N_0$ exists and is finite. For each $k \geq 1$ let $T_k = \inf\{t: |N_t| \geq k\}$. Then a.s. $\cup\{T_k > 0\} = \Omega$. The process $N_{t\wedge T_k} 1_{\{T_k > 0\}}$ is uniformly bounded and in $\mathfrak{L}^c_{\text{open}}$ by (2.10). According to (2.13), $(N_{t\wedge T_k} 1_{\{T_k>0\}})_{t\geq 0}$ is a bounded martingale over $(\mathfrak{F}_t)_{t\geq 0}$. Thus $(N_{t\wedge T_k} 1_{\{T_k>0\}})$ has a finite quadratic variation process A^k_t. On the other hand, by the remarks following (2.12), $A^k_t(\omega) = 1_\Lambda 1_{\{T_k(\omega)>0\}} Q(\omega,]0, t\wedge T_k(\omega)])$ a.s.. Since $\cup\{T_k > 0\} = \Omega$ a.s., it follows that $\Omega_Q \supset \Lambda$ a.s.. In order to show that $\Lambda \supset \Omega_Q$ a.s., we define now $Z_t = 1_{\Omega_Q} M_t$ for $t > 0$. Since $\Omega_Q \in \mathfrak{F}_{0+} = \mathfrak{F}_0$, $Z \in \mathfrak{L}^c_{\text{open}}$ and Z has a finite quadratic variation process $A_t(\omega) = 1_{\Omega_Q}(\omega)Q(\omega,]0, t])$. It suffices to prove that $Z \in \mathfrak{L}^c_{\text{open}}$ having a finite quadratic variation process implies that Z_{0+} exists and is finite a.s.. To this end, we may adjoin to the underlying space by the usual product construction a standard Brownian motion (B_t) independent of $\mathfrak{F}$. The quadratic variation process for Z remains the same over the augmented filtration $(\overline{\mathfrak{F}}_t)$ since it is given by a limit of quadratic variational sums of Z without conditioning. Replacing Z_t by $\overline{Z}_t = Z_t + B_t$ affects neither the limiting behavior at time zero nor the finiteness of the quadratic variation. Let $\overline{A}_t$ be the quadratic variation process for $\overline{Z}_t$. Then $\overline{A}_t$ is continuous, strictly increasing, and $\overline{A}_\infty = \infty$ a.s. If $\tau_t = \inf\{s: A_s > t\}$, then τ_t is strictly increasing, continuous, and $\tau_t < \infty$ for all $t < \infty$. In addition, $\tau_t \uparrow \infty$

as $t\uparrow\infty$. It is easy to see then that $\overline{Z}(\tau_t)_{t>0}$ is a continuous local martingale on $]0,\infty[$ relative to $(\overline{\mathfrak{F}}_{\tau_t})$. Since $\overline{Z}_t^2 - A_t \in \mathcal{L}^c_{open}$, it is also the case that $\overline{Z}^2(\tau_t) - A(\tau_t) = \overline{Z}^2(\tau_t) - t$ is a continuous local martingale on $]0,\infty[$ relative to $(\overline{\mathfrak{F}}_{\tau_t})$. Then (2.14) shows that $\lim_{t\downarrow\downarrow 0} \overline{Z}_{\tau_t}$ exists a.s., and hence $\lim_{t\downarrow\downarrow 0} Z_t$ exists a.s.. We have now proven (i). We now turn to (ii). On $\{M_{0+} = \pm\infty\}$, $\inf\{t: M_t = a\} > 0$ for all $a \in \mathbf{R}$. Since $\lambda^a(\omega, \cdot)$ is carried by $\{t: M_t(\omega) = a\}$ and $\lambda^a(\omega, \cdot)$ is a Radon measure on $]0,\infty[$, it follows that $\{M_{0+} = \pm\infty\} \subset \Omega_a$. On the other hand, since $\Omega_a \in \mathfrak{F}_{0+} = \mathfrak{F}_0$, $(1_{\Omega_a} M_t)_{t>0}$ is in $\mathcal{L}^c_{open}$ and so, by the remarks following (2.12), if we set $L^a_t(\omega) = 1_{\Omega_a}(\omega)\lambda^a(\omega,]0, t])$, then

$$N_t = |1_{\Omega_a} M_t - a| - L^a_t \in \mathcal{L}^c_{open} .$$

Obviously $\liminf_{t\downarrow\downarrow 0} N_t \geq 0$ so by (2.4), $\lim_{t\downarrow\downarrow 0} N_t$ exists a.s. in $[0, \infty]$. Because of the alternatives (2.4) for M_t, it is clear that on Ω_a M_{0+} must exist a.s. in $[-\infty, \infty]$. That is $\Omega_a \subset \{M_{0+}$ exists in $\overline{\mathbf{R}}\}$. Using (2.4) again, we see that (2.15) has been proven.

(2.16) Corollary. *If* $(M_t)_{t>0} \in \mathcal{L}^c_{open}$ *and if* $\lim_{t\downarrow\downarrow 0} M_t = M_0$ *exists and is finite a.s., then* $(M_t)_{t\geq 0}$ *is in* $\mathcal{L}^c$.

Proof. The first part of the proof of (2.15) shows that if $T_n = \inf\{t \geq 0: |M_t| \geq n\}$ then $M_{t\wedge T_n} 1_{\{T_n>0\}}$ is a bounded martingale over $(\mathfrak{F}_t)$. Since $T_n \uparrow \infty$ a.s. $(M_t)_{t\geq 0} \in \mathcal{L}^c$.

3. LOCAL MARTINGALE INCREMENTS.

The situation described in §2 does not cover the possible ways a singularity at the time origin can manifest itself. Consider the following examples.

(3.1) Let $M \in \mathcal{L}^c$ (a genuine continuous local martingale) and let C be a predictable process such that for all $t > 0$, $\int_t^{t+h} C_s^2 \, d\langle M, M\rangle_s < \infty$ for all $h > 0$ but $\int_0^1 C_s^2 \, d\langle M, M\rangle_s = \infty$ with positive probability. One may then define the stochastic integral $\int_t^{t+h} C_s \, dM_s$ as a local martingale on $[t,\infty[$ for all $t > 0$, but it is not possible in general to find one single normalization at $t = 0$ which makes $\int_t^{t+h} C_s \, dM_s$ the increment over $]t,t+h]$ of one local martingale on $]0,\infty[$, simultaneously for all $t > 0$.

(3.2) Let $M \in \mathcal{L}^c_{\text{open}}$ and let C be a bounded predictable process. Then $\int_t^{t+h} C_s \, dM_s$ is well defined for all $t > 0$ and $n \geq 0$ but, as in (3.1), there is no way to define $N \in \mathcal{L}^c_{\text{open}}$ such that $\int_t^{t+h} C_s \, dM_s = N_{t+h} - N_t$.

(3.3) Let $M \in \mathcal{L}^c_{\text{open}}$ and let Q be its quadratic variation measure. Though we can define the process $M_t^2 - M_s^2 - Q(\omega,]s, t])$ for $t \geq s$, there is no $N \in \mathcal{L}^c_{\text{open}}$ having the same increments on $[s, \infty]$ for all $s > 0$.

These examples motivate the following definition.

(3.4) Definition. *A local martingale increment process* $(M_{s,t})$ *is a family of real random variables indexed by pairs* $0 < s \leq t$ *such that*

(3.5) *for all* $s > 0$, $t \to M_{s,t}$ $(t \geq s)$ *is a local martingale relative to* $(\Omega, (\mathcal{F}_t)_{t \geq s}, P)$;

(3.6) *for all triples* $0 < r \le s \le t$, $M_{r,t} = M_{r,s} + M_{s,t}$.

Note that (3.6) forces $M_{t,t} = 0$ for all $t > 0$. The examples (3.1) – (3.3) obviously fit into the above scheme. Let $\mathcal{L}^c_{inc}$ denote the space of all local martingale increment processes relative to $(\Omega, \mathcal{F}_t, P)$ such that for all $s > 0$, $t \to M_{s,t}$ is a.s. continuous on $[s,\infty[$. If $M \in \mathcal{L}^c_{inc}$ then for all $s > 0$, $t \to M_{s,t}$ $(t \ge s)$ has an associated quadratic variation process $\langle M_{s,.}, M_{s,.}\rangle_t$ $(t \ge s)$. If $0 < r < s$, then since $t \to M_{r,t}$ and $t \to M_{s,t}$ have the same increments over intervals in $[t,\infty[$, one has

$$\langle M_{r,.}, M_{r,.}\rangle_t - \langle M_{r,.}, M_{r,.}\rangle_s = \langle M_{s,.}, M_{s,.}\rangle_t.$$

It follows that there is a well defined random measure $Q(\omega, dt)$ defined on $\mathbb{R}^{++}$ such that if $0 < s < t$, then

$$Q(\cdot,]s,t]) = \langle M_{r,.}, M_{r,.}\rangle_t - \langle M_{r,.}, M_{r,.}\rangle_s$$

for all $r \in]0,s]$. The random measure Q will be called the quadratic variation measure for M. For obvious reasons it is not in principle possible to define local times for $M \in \mathcal{L}^c_{inc}$.

(3.7) Lemma. *Let* $M \in \mathcal{L}^c_{inc}$ *and let* R *be a stopping time with* $P\{0 < R < \infty\} = 1$. *Then* $Y_t = M_{R,R+t}$ $(t \ge 0)$ *is a local martingale over the filtration* $(\mathcal{F}_{R+t})$.

Proof. Let $s_n \downarrow\downarrow 0$. If we show that for every n, $1_{\{R \ge s_n\}} Y_t$ is a local martingale over $(\mathcal{F}_{R+t})$. Then since $P\{R \ge s_n\} \uparrow 1$ as $n \to \infty$, the claimed result will follow from the following argument. Let $\Lambda_n = \{R \geq s_n\} \in \mathcal{F}_R$ and for $k \geq 1$ let $T_k = \inf\{t: |Y_t| \geq k\}$. Since $1_{\Lambda_n} Y_t$ is a continuous local martingale, $1_{\Lambda_n} Y(t \wedge T_k)\, 1_{\{T_k>0\}}$ is a uniformly bounded martingale for all $k \geq 1$ and $n \geq 1$. Now let $n \to \infty$ to see that $Y(t \wedge T_k)\, 1_{\{T_k>0\}}$ is a bounded martingale for all k. That is, $(Y_t)_{t\geq 0}$ is a local martingale. Fix now $s > 0$ and let $\Lambda = \{R > s\}$. Then $t \to M_{s,s+t}$ is a continuous local martingale over the filtration $(\mathcal{F}_{s+t})$, and hence because $R \vee s - s$ is a stopping time over $(\mathcal{F}_{s+t})$ we see from (1.14) that $M_{s,s+(R\vee s-s)+t} = M_{s,R\vee s+t}$ is a local martingale over $(\mathcal{F}_{R\vee s+t})$. But since $M_{R\vee s,R\vee s+t} = M_{s,R\vee s+t} - M_{s,R\vee s}$ and $M_{s,R\vee s} \in \mathcal{F}_{R\vee s}$, it follows that $M_{R\vee s,R\vee s+t}$ is a local martingale over $(\mathcal{F}_{R\vee s+t})$. However, $1_\Lambda Y_t = 1_\Lambda M_{R\vee s,R\vee s+t}$ is therefore a local martingale over $(\mathcal{F}_{R\vee s+t})$, and since the trace of $\mathcal{F}_{R\vee s+t}$ on Λ is equal to the trace of $\mathcal{F}_{R+t}$ on Λ, we are done.

(3.8) Proposition. *Suppose that* $M \in \mathcal{L}^c_{inc}$ *and that for some* $r > 0$, $M_{0,r} = \lim_{s\downarrow\downarrow 0} M_{s,r}$ *exists and is finite almost surely. For arbitrary* $t \geq 0$, *define* $M_t = (M_{0,r} - M_{t,r})1_{[0,r]}(t) + (M_{0,r} + M_{r,t})1_{]r,\infty[}(t)$. *Then* $M \in \mathcal{L}^c$ *and for all* $0 < s < t$, $M_{s,t} = M_t - M_s$.

Proof. The fact that $M_{s,t} = M_t - M_s$ for all $0 < s < t$ is evident, as is the fact that $M_t = \lim_{s\downarrow\downarrow 0} M_{s,t}$ for all $t > 0$. Because the increments $M_{s,t}$ form a local martingale in $t \geq s$ for $s > 0$ it follows that $(M_t)_{t>0} \in \mathcal{L}^c_{open}$. Since $M_t \to 0$ a.s. as $t \downarrow\downarrow 0$, the result follows from (2.16).

We turn now to a criterion which guarantees that $\lim_{s\downarrow\downarrow 0} M_{s,r}$ exists for some (and hence all) $r > 0$.

(3.9) Proposition. *Suppose that* $M \in \mathcal{L}^c_{inc}$ *has quadratic variation measure* Q *such that* $Q(\omega,]0, t]) < \infty$ *for all* $t > 0$ a.s. . *Then there exists* $N \in \mathcal{L}^c$ *such that for all* $0 < s < t$, $M_{s,t} = N_t - N_s$.

Proof. Let $A_t(\omega) = Q(\omega,]0, t])$ for $t \geq 0$, so that $A_0 = 0$ and A is a continuous increasing process adapted to $(\mathcal{F}_t)$. We show that $\lim_{s \downarrow\downarrow 0} M_{s,1}$ exists in $\mathbb{R}$ almost surely. To this effect we may assume that A is strictly increasing and $A_\infty = \infty$ a.s., for if this is not so, adjoin an independent Brownian motion B_t so that $M_{s,t}$ is replaced by $\overline{M}_{s,t} = M_{s,t} + (B_t - B_s)$. See the proof of (2.15). With the above assumption on A in force, let $\tau_t = \inf\{s: A_s > t\}$. Just as in the proof of (2.15), for every $s > 0$ the process M_{τ_s,τ_t} is a local martingale increment process over the filtration $(\mathcal{F}_{\tau_t})$. We are also using (3.7) at this point. For every $s > 0$, the process M_{τ_s,τ_s+t} is a local martingale over $(\mathcal{F}_{\tau_s+t})$ and so is $M^2_{\tau_s,\tau_s+t} - (A(\tau_s + t) - A(\tau_s))$. By time change, it follows that $M^2_{\tau_s,\tau_t} - (t - s)$ is a local martingale for $(t \geq s)$ over the filtration $(\mathcal{F}_{\tau_t})_{t \geq s}$. It follows from Lévy's theorem (1.2) that $t \to M_{\tau_s,\tau_t}$ $(t \geq s)$ is a standard Brownian motion relative to $(\mathcal{F}_{\tau_t})$. In particular, for $s \leq t$, M_{τ_s,τ_t} has a normal distribution with mean zero and variance $t - s$. Consequently $s \to M_{\tau_{1-s},\tau_1}$ $(0 \leq s < 1)$ is an L^2-bounded martingale so $\lim_{s \uparrow\uparrow 1} M_{\tau_{1-s},\tau_1}$ exists and is finite almost surely. In other words, $\lim_{u \downarrow\downarrow 0} M_{u,\tau_1}$ a.s. exists and is finite, and one concludes that $M_{0,1} = \lim_{s \downarrow\downarrow 0} M_{s,1}$ exists and is finite.

(3.10) Theorem. *Let* $M \in \mathcal{L}^c_{inc}$ *with quadratic variation measure* Q. *Then* $\Lambda = \{\omega \in \Omega: \lim_{s \downarrow\downarrow 0} M_{s,1}(\omega)$ *exists and is finite*$\}$ *and* $\Gamma = \{\omega \in \Omega: Q(\omega,]0, 1]) < \infty\}$ *are almost surely equal.*

Proof. Since $\Gamma = \{Q(\omega,]0, \varepsilon] < \infty\}$ for every $\varepsilon > 0$, $\Gamma \in \mathcal{F}_{0+} = \mathcal{F}_0$. In addition, $\Lambda = \{\omega \in \Omega: \lim_{s\downarrow\downarrow 0} M_{s,r}(\omega)$ exists and is finite$\}$ for every $r > 0$ so $\Lambda \in \mathcal{F}_{0+} = \mathcal{F}_0$. Obviously $1_\Lambda M_{s,t} \in \mathcal{L}^c_{inc}$ and since $\lim_{s\downarrow\downarrow 0} 1_\Lambda M_{s,1}$ exists and is finite almost surely, (3.8) shows that $1_\Lambda M_{s,t}$ is obtained from the increments of a genuine continuous local martingale N. Since the quadratic variation of $1_\Lambda M$ is given on one hand by $1_\Lambda Q$ and on the other hand by $\langle N, N\rangle$, this proves that $\Lambda \subset \Gamma$ a.s.. Going the other way, $1_\Gamma M_{s,t} \in \mathcal{L}^c_{open}$ has quadratic variation measure $1_\Gamma Q$, which is a.s. finite near zero. Then (3.9) shows that $\lim_{s\downarrow\downarrow 0} 1_\Gamma M_{s,t}$ exists a.s. and is finite, so $\Gamma \subset \Lambda$ almost surely.

We show next that all continuous local martingale increment processes may be obtained as increments of stochastic integrals in the manner of example (3.1).

(3.11) Theorem. *Let* $M \in \mathcal{L}^c_{inc}$. *Then there exists* $N \in \mathcal{L}^c$ *and a predictable process* C *such that for all* $0 < s < t$

(3.12) $$\int_s^t C_u^2 \, d\langle N, N\rangle_u < \infty \quad \text{a.s. ;}$$

(3.13) $$M_{s,t} = \int_s^t C_u \, dN_u$$

Proof. Let Q be the quadratic variation measure for M. Fix a sequence $t_n \downarrow\downarrow 0$ and let D_t denote the predictable process

$$D_t(\omega) = \sum_{n\geq 1} 2^{-n} 1_{[t_n, t_{n-1}[}(t) \, e^{-Q(\omega,]t_n, t])}$$

where t_0 is set equal to $+\infty$. Obviously $D_t(\omega) > 0$ for all $t > 0$.

We have then, since $A_t^n = Q(]t_n, t])$ is continuous and $A_{t_n}^n = 0$

$$\int_0^\infty D_t^2 \, Q(dt) = \sum_{n\geq 1} 2^{-2n} \int_{t_n}^{t_{n-1}} e^{-Q(]t_n,t])} \, Q(dt)$$

$$= \sum_{n\geq 1} 2^{-2n} \int_{t_n}^{t_{n-1}} e^{A_t^n} \, dA_t^n$$

$$= \sum_{n\geq 1} 2^{-2n} (1 - e^{-A^n(t_{n-1})})$$

$$\leq 1 \, .$$

For any $s > 0$, the stochastic integral

$$N_{s,t} = \int_s^t D_u \, dM_{s,u}$$

is therefore defined, and has quadratic variation process

$$\int_s^t D_u^2 \, d\langle M_{s,\cdot}, M_{s,\cdot}\rangle_u = \int_s^t D_u^2 \, Q(du)$$

bounded by one. If $0 < r < s < t$

$$N_{r,t} = \int_r^t D_u \, dM_{r,u} = \int_r^s D_u \, dM_{r,u} + \int_s^t D_u \, dM_{r,u}$$

$$= N_{r,s} + N_{s,t} + \int_s^t D_u \, d(M_{r,u} - M_{s,u})$$

$$= N_{r,s} + N_{s,t} \, .$$

That is, $N \in \mathcal{L}^c_{inc}$. The quadratic variation measure for N is the measure $D_u^2 \, Q(du)$ which is bounded by one. Consequently, (3.10) shows that $N_{0,t} = \lim_{s\downarrow\downarrow 0} N_{s,t}$ exists a.s. and defines a local martingale. It is clear then that since $M_{s,t} = \int_s^t D_u^{-1} \, dN_u$, one obtains (3.12) and (3.13), setting $C_u = D_u^{-1}$.

Our final results on $\mathcal{L}^c_{inc}$ concerns the behavior of $M_{s,t}$ as $s \downarrow\downarrow 0$ when it is known that convergence does not occur. The result here is rather less precise than either (2.4) or (2.15). Given $M \in \mathcal{L}^c_{inc}$, we define the maximal increment process $W_{s,t} (0 < s < t)$ for M by

$$\text{(3.14)} \qquad W_{s,t} = \sup\{|M_{u,v}| : s \le u < v \le t\} \, .$$

It is clear that for $0 < s \le t$, $W_{s,t} \in \mathcal{F}_t$, and for fixed $s > 0$, $t \to W_{s,t}$ is continuous and increasing, while for fixed $t > 0$, $s \to W_{s,t}$ is continuous on $]0,t]$ and it increases as s decreases. The quantity

$$\text{(3.15)} \qquad W_0 = \lim_{t\downarrow 0} \sup_{s<t} W_{s,t}$$

is in $\mathcal{F}_{0+} = \mathcal{F}_0$. It is clear that on $\{W_0 = 0\}$, $\lim_{s\downarrow\downarrow 0} M_{s,t}$ exists and is finite. On $\{W_0 > 0\}$, $\lim_{s\downarrow\downarrow 0} M_{s,t}$ does not exist in $\mathbb{R}$, though it may exist in $\overline{\mathbb{R}}$.

(3.16) Theorem. *Let* $M \in \mathcal{L}^c_{inc}$ *and let* W_0 *be the limiting oscillation of* M, *defined in* (3.15). *Then* $P\{0 < W_0 < \infty\} = 0$.

Proof. As in the proof of (3.9) we may assume, adding an independent Brownian motion of M if necessary, that the quadratic variation measure Q for M has the property that a.s., for all $s > 0$, $t \to Q(]s, t])$ is strictly increasing on $[s, \infty[$ and tends to infinity as $t \to \infty$. We shall prove that for all $a > 0$, $W_0 \geq 2a$ a.s. on $\{W_0 > a\}$, and from this the assertion follows trivially. Observe first that on $\{W_0 > a\}$, for every $t > 0$ there exist $0 < u < v < t$ with $|M_{u,v}| > a$. For $c > 0$ and $s > 0$ let

$$\begin{aligned} R^1(c, s) &= \inf\{t > s: |M_{u,t}| = c \text{ for some } u \in [s, t[\} \\ &= \inf\{t > s: \max_{s \le u \le t} M_{s,u} \geq \min_{s \le u \le t} M_{s,u} + c\}. \end{aligned}$$

Recursively, for $k \geq 1$ set

$$\begin{aligned} R^{k+1}(c, s) &= R^1(c, R^k(c, s)) \\ &= \inf\{t > R^k(c, s): |M_{u,t}| = c \text{ for some } u \in [R^k(c, s), t[\}. \end{aligned}$$

On $\{W_0 > a\}$ since M must have oscillations of size $> a$ in arbitrarily small time intervals, it must be that for every $k \geq 1$

$$R^k(a, s) \to 0 \quad \text{as} \quad s \downarrow\downarrow 0 .$$

On $\{W_0 < 2a\}$ there exists $t > 0$ such that $|M_{u,v}| < 2a$ for all $0 < u < v \leq t$. Consequently, on $\{W_0 < 2a\} \cap \{W_0 > a\}$, for each fixed $k \geq 1$, $R^k(a, s) < R^1(2a, s)$

for all sufficiently small $s > 0$. We prove that there exists a sequence $\varepsilon_k(a)$, independent of $s > 0$, such that

$$(3.17) \qquad P\{R^k(a, s) < R^1(2a, s)\} \le \varepsilon_k(a) \text{ and } \varepsilon_k(a) \to 0 \text{ as } k \to \infty.$$

Once we prove (3.17), since k is arbitrary, it will follow that $P\{a < W_0 < 2a\} = 0$. In order to prove (3.17), we set $A_t = Q(]s, s+t])$ and let $\tau_t = \inf\{u: A_u > t\}$. Then the process $B_t = M_{s,s+\tau_t}$ is a standard Brownian motion. Since $M_{s,t}$ $(t \ge s)$ and B_t $(t \ge 0)$ run through the same points in the same order, it is enough to prove that $P\{R_a^k < R_{2a}^1\}$ is dominated by a suitable sequence $\varepsilon_k(a)$, where R_c^k denotes $R^k(c, 0)$ for the process (B_t) $(t \ge 0)$. By definition of the R_a^k, $|B(R_{2a}^1)| \le 2a$, and the discrete parameter process $Y_k = B(R_a^k)$ $(k \ge 1)$ is a random walk. Then

$$P\{R_a^k < R_{2a}^1\} \le P\{\sup_{j \le k} |Y_j| \le 2a\} .$$

Letting $\varepsilon_k(a) = P\{\sup_{j \le k} |Y_j| \le 2a\}$, the fact that the law of Y_1 is not degenerate shows that $\varepsilon_k(a) \to 0$ as $k \to \infty$, completing the proof.

(3.18) <u>Corollary</u>. *Let* $M \in \mathcal{L}^c$ *and let* C *be a predictable process such that* $\int_s^t C_u^2 \, d\langle M, M\rangle_u < \infty$ *a.s. for all* $0 < s < t < \infty$. *Then for any* $t > 0$, $\Lambda = \{\omega: \int_s^t C_u(\omega) dM_u(\omega)$ *converges to a finite limit as* $s \downarrow\downarrow 0\}$ *is almost surely equal to* $\Gamma = \{\omega: \int_0^\varepsilon C_u^2(\omega) d\langle M, M\rangle_u < \infty$ *for some* $\varepsilon > 0\}$, *and a.s. on* Γ^c, $s \to \int_s^t C_u(\omega) dM_u$ *has unbounded oscillations as* $s \downarrow\downarrow 0$.

Proof. The first assertion is just (3.10) applied to $N_{s,t} = \int_s^t C_u dM_u$ and the second assertion follows from (3.16) since $\Lambda^c = \{W_0(N) > 0\}$.

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