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A note on L_2 maximal inequalities

by

Jim Pitman*

1. Introduction

According to the L_2 maximal inequality of Doob [3], for a martingale $X_1,\dots,X_n$

$$(1.1) \qquad E(\max_k |X_k|)^2 \le 4EX_n^2 .$$

And an inequality of Newman and Wright [9] states that (1) holds (with constant 2 instead of 4) if $X_k = D_1 + \dots + D_k$ where $D_1,\dots,D_n$ is a collection of mean zero random variables which are associated, meaning that for every two coordinatewise non-decreasing functions f_1 and f_2 on R^n such that the variance of $f_j(D_1,\dots,D_n)$ is finite for $j = 1$ and 2, the covariance of these two random variables is non-negative. (See Esary, Proschan and Walkup [5], Fortuin, Kastelyn and Ginibre [6], and other references in Newman and Wright [9] for uses of this concept of association in statistical mechanics and other contexts.)

This note offers a simple general method for obtaining L_2 maximal inequalities of this kind. Amongst other things, it is shown that Doob's inequality (1.1) admits the following improvement: the random variable $\max_k |X_k|$ can be replaced by the larger random variable

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$$(1.2) \qquad \max_k X_k^+ \; - \; \min_k X_k^-$$

where $X^+ = \max(X,0)$, $X^- = \min(X,0)$.

This is a little surprising in view of the observation of Dubins and Gilat [4] that the constant 4 in Doob's inequality is best possible. Still, it turns out that even with this refinement, equality can never be attained in either (1.1) or its extension to continuous time martingales except in the trivial case of a martingale which is identically zero.

2. Inequalities in Discrete Time

Given a sequence of random variables $X_1,\ldots,X_n$, define

$$\Delta X_k = X_k - X_{k-1} \; , \; k = 1,\ldots,n \; ,$$

where $X_0 = 0$ by convention, so $\Delta X_1 = X_1$, and

$$X_n = \sum_{k=1}^n \Delta X_k \; .$$

The following Lemma is just an algebraic identity for sequences of real numbers, expressed for convenience in terms of random variables:

Lemma. Let $X_1,\ldots,X_n$ and $M_1,\ldots,M_n$ be sequences of random variables such that

$$(2.1) \qquad M_k = X_k \text{ whenever } \Delta M_k \neq 0 \; .$$

Then

$$(2.2) \qquad X_n^2 = (M_n - X_n)^2 + 2 \sum_{k=2}^n M_{k-1} \, \Delta X_k + \sum_{k=1}^n (\Delta M_k)^2 \; .$$

Remark. Here is another way of expressing condition (2.1): viewing k as a time parameter, there are random times

$$0 = T_0 \le T_1 \le T_2 \le \cdots$$

such that if

$$L_k = \max \{T_j : T_j \le k\}$$

then

$$M_k = X_{L_k} , \quad k = 1,\dots,n .$$

That is, M_k is the value of the process X at the last time T_j before time k, with $M_k = 0$ for $k < T_1$. The most important example is

$$M_k = \max_{1 \le j \le k} X_j ,$$

in which case the times T_k are ladder indices.

Proof of the Lemma. For two arbitrary sequences $X_1,\dots,X_n$ and $M_1,\dots,M_n$ there is the product difference rule

$$\Delta(M_k X_k) = M_{k-1}\, \Delta X_k + X_k\, \Delta M_k . \tag{2.3}$$

In particular

$$\Delta M_k^2 = M_{k-1}\, \Delta M_k + M_k\, \Delta M_k ,$$

whence

$$\Delta(2M_k X_k - M_k^2) = 2M_{k-1}\, \Delta X_k + (2X_k - M_k - M_{k-1})\, \Delta M_k . \tag{2.4}$$

If (2.1) holds the last term in (2.4) reduces to $(\Delta M_k)^2$, and (2.2) results from adding (2.4) from $k = 1$ to n .

Theorem. Let $X_1, \ldots, X_n$ be a sequence of random variables with $EX_k^2 < \infty$, and suppose $M_1, \ldots, M_n$ is a sequence with $M_k = X_k$ whenever $\Delta M_k \neq 0$. If

$$(2.5) \qquad E\textstyle\sum_{k=2}^{n} M_{k-1}\, \Delta X_k \geq 0 \; .$$

then

$$(2.6) \qquad E(M_n - X_n)^2 \leq EX_n^2 \quad , \text{ and}$$

$$(2.7) \qquad EM_n^2 \leq 4EX_n^2 \; .$$

Proof. Integrate (2.2).

It seems that in most cases of interest (2.5) is a consequence of the stronger condition

$$(2.8) \qquad EM_{k-1}\, \Delta X_k \geq 0 \quad \text{for} \quad 2 \leq k \leq n \; .$$

Suppose for example that $(X_k, \mathscr{F}_k)$ is a martingale or positive submartingale. Then (2.8) holds if M_k is $\mathscr{F}_k$-measurable, that is if the random times T_j in the remark above are $(\mathscr{F}_k)$ stopping times. For $M_k = \max_{1 \leq j \leq k} X_j$ the resulting inequality (2.7) is Doob's inequality (1.1). The inequality (2.6) in this case seems to be new, though of course it could be obtained with constant 4 instead of 1 from Doob's inequality. To obtain the improvement (1.2) of Doob's inequality for a martingale (X_k) , let $M_k^+ = \max_{1 \leq j \leq k} X_j$, $M_k^- = \min_{1 \leq j \leq k} X_j^-$, so

$$(2.9) \qquad (M_n^+ - M_n^-)^2 \le 2(M_n^+ - X_n)^2 + 2(M_n^- - X_n)^2 ,$$

and use (2.6) twice.

Considering a square integrable martingale (X_k) , from (2.2) it is plain that the process Y defined by

$$(2.10) \qquad Y_k = X_k^2 - (M_k - X_k)^2 - \sum_{j=1}^k \Delta M_j^2 = 2 \sum_{j=2}^k M_{j-1} \Delta X_j$$

is a martingale. It follows that the inequalities (2.6) and (2.7) for the maximum of a martingale are sharp but not attained in discrete time except in the trivial case when $X_n = 0$. In continuous time the situation is different. As will be seen in the next section, equality obtains in the continuous time analogue of (2.6) if and only if the maximal process increases continuously, while there can never be equality in (2.7) except for the zero martingale.

Condition (2.8) also holds if $M_k = \max_{1 \le j \le k} X_j$ and $\Delta X_1, \ldots, \Delta X_n$ is a sequence of associated random variables with $E\Delta X_k \ge 0$, $1 \le k \le n$. In this case (2.7) holds with constant 1 instead of 4 , which can be seen by applying (2.6) after reversing the order of the increments. This is the inequality of Newman and Wright [9].

3. Inequalities in Continuous Time

To obtain an analogue in continuous time of the formula (2.2), let $(X_t, t \ge 0)$ be a semimartingale with right continuous paths adapted to a filtration $(\mathscr{F}_t)$ satisfying the usual conditions (see for example Meyer [7]) and suppose that

(3.1) (M_t) is a process of locally bounded variation such that $\{t : M_t = X_t\}$ a.s. contains the support of the random measure dM_t,

for example $M_t = \sup_{0\leq s\leq t} X_s$.

Then by following the steps used to derive (2.4) using stochastic differential calculus one obtains the formula

$$X_t^2 = (M_t - X_t)^2 + 2\int_0^t M_{s-}\, dX_s + [M,M]_t, \tag{3.2}$$

where $[M,M]_t = \sum_{0<s\leq t} (\Delta M_s)^2$, and a continuous time analogue of the theorem of the previous section follows immediately.

Suppose now that X is a square integrable martingale. Then the process Y defined by

$$Y_t = X_t^2 - (M_t - X_t)^2 - [M,M]_t = 2\int_0^t M_{s-}\, dX_s \tag{3.3}$$

is a martingale. This observation extends a result of Azéma and Yor [2], who showed that for $M_t = \sup_{0\leq s\leq t} X_s$ the process Z defined by

$$Z_t = X_t^2 - (M_t - X_t)^2 \tag{3.4}$$

is a martingale if X has continuous paths. Indeed, we see from (3.3) that Z is a submartingale for any square integrable martingale X and any process M satisfying (3.1), and that Z is a martingale if and only if M has continuous paths.

Considering again the maximum process M, Azéma and Yor [1] gave a characterization of the increasing process $\langle X,X\rangle$ associated with the sqaure-integrable martingale X as the dual predictable projection of the (non-adapted) increasing process

$$A_t = (I_0 - M_\infty)^2 - (I_t - M_\infty)^2$$

where $I_t = \sup_{s \geq t} X_s$. As a consequence of these remarks concerning the process Z of (3.4), this characterization of $\langle X,X \rangle$ now extends to all square intégrable martingales whose trajectories have no upward jumps.

Consider now the continuous time version of the improvement (1.2) of Doob's inequality (1.1) : for a square integrable martingale $(X_t, 0 \leq t \leq \infty)$,

$$E(\sup_t X_t^+ - \inf_t X_t^-)^2 \leq 4EX_\infty^2. \tag{3.5}$$

To partly confirm a conjecture of Dubins and Gilat [4], let us show that equality obtains in (3.5) only in the trivial case when $X_\infty = 0$ a.s..
Indeed, by inspection of (2.9) and (3.3) it is plain that equality in (3.5) implies that both the process M^+ and M^- are continuous, where

$$M_t^+ = \sup_{0 \leq s \leq t} X_s^+, \quad M_t^- = \sup_{0 \leq s \leq t} X_s^-,$$

and moreover that

$$M_t^+ - X_t = X_t - M_t^- \quad \text{a.s.,} \quad t \geq 0, \text{ i.e.}$$

$$M_t^+ + M_t^- = 2X_t \qquad \text{a.s.,} \quad t \geq 0. \tag{3.6}$$

But the left side of (3.6) is a continuous process with bounded variation, while the right side is a martingale. This forces $X = c$ for a constant c, and then obviously $c = 0$.

(I thank Marc Yor for suggesting this argument, which simplifies considerably my earlier one).

4. Concluding remarks

The importance of the property (3.1) of the maximal process seems first to have been appreciated by Azéma and Yor [2].
Using their method one can neatly obtain Doob's maximal inequality in L_p for any $p > 1$ (see Dellacherie [10]), but it is still not clear how to obtain the right extension to L_p of the refinements described here for L_2.

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