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SOME REMARKS ON PROCESSES WITH INDEPENDENT INCREMENTS

by WANG Jia-Gang

Let $(\Omega,\underline{\underline{F}},P)$ be a complete probability space. In this note we consider processes $X=(X_t,\ t\in\mathbb{R}_+)$ with (non homogeneous) independent increments, which have no fixed discontinuities :

i. $P\{X_0=0\}=1$

ii. For $0\leq t_1<t_2\ldots<t_n$, $X_{t_n}-X_{t_{n-1}},\ldots,X_{t_2}-X_{t_1}$, X_{t_1} are independent random variables

iii. Every sample function of X is right continuous with left hand limits, and $P\{\Delta X_t\neq 0\}=0$ for every $t\in\mathbb{R}_+$.

$\underline{\underline{F}}^o_t$ is the σ-field $\sigma(X_s,s\leq t)$, and $\underline{\underline{F}}_t$ the σ-field generated by $\underline{\underline{F}}^o_t$ and all sets of measure 0 in $\underline{\underline{F}}$.

1. In this section we are going to prove that <u>the filtration</u> $(\underline{\underline{F}}_t)$ <u>satisfies the usual conditions and is quasi-left continuous</u>, a fact which is essentially well-known, but difficult to find in the literature in the non homogeneous case. We also give some auxiliary results.

We need some notations concerning martingales. First of all, we denote by $M_t(u,r,s)$ the following martingale, for $u\in\mathbb{R}$, $r\leq s$

(1)
$$M_t(u,r,s)=E[e^{iu(X_s-X_r)}|\underline{\underline{F}}^o_t] = e^{iu(X_s-X_r)} \text{ if } t\geq s$$
$$= e^{iu(X_t-X_r)}\varphi_{t,s}(u) \text{ if } r\leq t<s$$
$$= \varphi_{r,s}(u) \text{ if } t\leq r$$

where $\varphi_{a,b}$ is the characteristic function of X_b-X_a $(a\leq b)$. This process is right continuous with left-hand limits, and jumps only at jump times of X . Next, consider the set H of all random variables

(2)
$$Z = e^{i(u_1X_{t_1}+u_2(X_{t_2}-X_{t_1})+\ldots+u_n(X_{t_n}-X_{t_{n-1}}))}$$

with $u_1,\ldots,u_n\in\mathbb{R}$, $0\leq t_1<\ldots<t_n$. Then the linear span of H is dense in L^1 and we have

(3)
$$Z_t= E[Z|\underline{\underline{F}}^o_t] = M_t(u_1,0,t_1)\ldots M_t(u_n,t_{n-1},t_n)$$

and therefore Z_t has the same continuity properties as M_t above.

PROPOSITION 1[1]. For every $t\in\mathbb{R}_+$ we have $\underline{\underline{F}}_t=\underline{\underline{F}}_{t+}$ ($=\underline{\underline{F}}_{t-}$ if $t>0$).

Proof. Since Z_t is bounded and right continuous, $E[Z|\underline{\underline{F}}^o_{t+}]=Z_{t+}=Z_t=E[Z|\underline{\underline{F}}^o_t]$ a.s. This extends to all random variables in L^1. In particular, any r.v.

in $L^1(\underline{\underline{F}}^o_{t+})=L^1(\underline{\underline{F}}_{t+})$ is a.s. equal to a r.v. in $L^1(\underline{\underline{F}}^o_t)$. Hence $\underline{\underline{F}}_t=\underline{\underline{F}}_{t+}$. The reasoning is the same on the left side.

PROPOSITION 2. If T is a stopping time of $(\underline{\underline{F}}_t)$, then

$$\underline{\underline{F}}_{T-} = \sigma(T, X^{T-}, \underline{\underline{N}})$$

where X^{T-} is X stopped at $T-$, and $\underline{\underline{N}}$ is the class of all negligible sets.

Proof. This result is true for any process X which is continuous with left hand limits.

It is obvious that the σ-field $\underline{\underline{K}}$ on the right is contained in $\underline{\underline{F}}_{T-}$. To prove the reverse inclusion it suffices to show that $\underline{\underline{K}}$ contains $A\cap\{t<T\}$ for $t\in\mathbb{R}_+$, $A\in\underline{\underline{F}}^o_t=\sigma(X_s, s\leq t)$. Hence it suffices to show that, for $s\leq t$, any function $f(X_s)1_{\{t<T\}}$ is $\underline{\underline{K}}$-measurable. This is true since $X_s=X_s^{T-}$ on $\{t<T\}$.

PROPOSITION 3. If T is a stopping time of $(\underline{\underline{F}}_t)$, then

$$\underline{\underline{F}}_T = \sigma(T, X^T, \underline{\underline{N}})$$

Proof. It is obvious that the σ-field $\underline{\underline{L}}$ on the right is contained in $\underline{\underline{F}}_T$. To prove the reverse inclusion it suffices to show that for any random variable $Z\in L^1$, $E[Z|\underline{\underline{F}}_T]$ is $\underline{\underline{L}}$-measurable, and it suffices to prove it for the random variables (2). Because of (3) it suffices to prove that $M_T(u,r,s)$ is $\underline{\underline{L}}$-measurable, which is obvious.

COROLLARY. If T is a.s. finite, $\underline{\underline{F}}_T=\sigma(\underline{\underline{F}}_{T-}, X_T)$.

THEOREM 1. The filtration $(\underline{\underline{F}}_t)$ is quasi-left continuous.

Proof. It is well known ([2], chap. 3) that all jump times of X are totally inaccessible. Therefore at any bounded predictable time T we have $X_T=X_{T-}$ a.s., and from the corollary above we have $\underline{\underline{F}}_T=\underline{\underline{F}}_{T-}$.

2. Assume now that X has only finitely many jumps in every finite interval, and is constant between jumps. Then we may consider X (or rather its jump process) as a multivariate point process, and it is natural to ask which kind of conditions on X , <u>considered as a multivariate point process with values in</u> $\mathbb{R}$, express that X has independent increments, with Lévy measure $\nu(dt,dx)$.

Let us introduce some notation : we denote by $T_1(\omega)$, $T_2(\omega)$... the successive jump times (if $T_m(\omega)$ is the last finite jump, we set $T_n(\omega)=+\infty$ for $n>m$) ; $\Delta_1(\omega), \Delta_2(\omega)$... are the successive jump sizes (if $T_n(\omega)=+\infty$, we make the convention that $\Delta_n(\omega)=0$). We denote by $n_t(\omega)$ the total number of jumps of $X_.(\omega)$ on $[0,t]$. Given the Lévy measure ν , we set $\nu([0,t]\times\mathbb{R})=\Lambda(t)$, a non decreasing, continuous function with $\Lambda(0)=0$. Using the existence of regular conditional distributions, we may deduce that there exists a transition probability $N(t,dx)$ from $]0,\infty[$ to $\mathbb{R}$ such that $N(t,\{0\})=0$ and we have for any Borel set A

(4) $$\nu([0,t]\times A) = \int_{[0,t]} N(u,A)\,d\Lambda(u).$$

We extend this definition to $t=+\infty$, setting then $N(+\infty,.)=\varepsilon_0$. We are going to prove (assuming always X is a multivariate point process).

THEOREM 2. _If_ X _is a process with independent increments, then it satisfies the following two properties,_

A) (n_t) _is a Poisson process relative to_ $(\underline{\underline{F}}^\circ_t)$, _with expectation_ $E[n_t]=\Lambda(t)$.

B) _Conditional to the event_ $T_1=t_1$, $T_2=t_2,\dots,T_n=t_n$ (i.e., conditional to the sample path of the process (n_t)), _the random variables_ $\Delta_1,\Delta_2,\dots$ _are independent, the law of_ Δ_n _being_ $N(t_n,.)$.

Conversely, if X _satisfies the slightly weaker properties_

A') (n_t) _is a Poisson process_ (w.r. to its natural filtration) _and_ $E[n_t]=\Lambda(t)$.

B') $P\{\Delta_{n+1}\in dx\,|\,T_1,\Delta_1,\dots,T_n,\Delta_n,T_{n+1}\} = N(T_{n+1},dx)$

Then X _has independent increments, and its Lévy measure is given by_ (4).

REMARK. Like all statements concerning conditional distributions, it must be understood that property B) holds for almost every path of (n_t). In particular, different choices of $N(u,.)$ in (4) will differ only for a set of values of u which has $d\Lambda$-measure 0, and the conditional distributions will be the same for almost every path, but not for every path.

Proof. We first show that A') and B') imply X has independent increments. We begin by assuming that $\Lambda(t)=t$. Then (n_t) is a homogeneous Poisson process with parameter 1, so all differences $T_{n+1}-T_n$ are independent exponential random variables. We then can compute

$$G_n(dt,dx) = P\{T_{n+1}\in dt,\ \Delta_{n+1}\in dx\,|\,T_1,\Delta_1,\dots,T_n,\Delta_n\} = 1_{\{t>T_n\}}e^{-(t-T_n)}N(t,dx)\,dt$$

and also $\quad H_n(]t,\infty[) = P\{T_{n+1}>t\,|\,T_1,\Delta_1,\dots,T_n,\Delta_n\} = e^{-(t-T_n)}$ for $t>T_n$

According to [2], p. 86, prop. 3.41, we can compute the predictable compensating measure of the process X as

$$\nu(dt,dx) = \Sigma_{n=0}^{\infty}\ \frac{G_n(dt,dx)}{H_n(]t,\infty[)}\ 1_{\{T_n<t\leqq T_{n+1}\}}$$

According to the above computations, this is a _deterministic_ measure, which implies ([2], p. 91, theorem 3.51) that X has independent increments. The Lévy measure of this process is $\nu(dt,dx)=N(t,dx)dt$.

The case of arbitrary $\Lambda(t)$ reduces to the preceding one by a deterministic change of time. If Λ is unbounded the reduction is trivial, while if $\Lambda(\infty)=a$, we are reduced to a homogeneous Poisson process on the finite interval $[0,a[$. But then we may extend X by an independent Poisson process on $[a,\infty[$, and apply the preceding reasoning. So finally A') and B') are sufficient conditions.

Conversely, we want to show that the process X with independent increments and Lévy measure ν satisfies A) and B). Property A) is well known. On the other hand, it is simple to construct a multivariate point process $\overline{X}$ such that 1) the corresponding process $(\overline{n}_t)$ is Poisson with expectation $E[\overline{n}_t]=\Lambda(t)$ and 2) the conditional law of the jump sizes given $(\overline{n}_t)$ is given by B). Since this process satisfies A') and B), it also satisfies A') and B'), hence from the first part of the proof it is a process with independent increments and Lévy measure ν, and finally it has the same law as X. This implies that, inversely, X has the same conditional law given (n_t) as $\overline{X}$ given $(\overline{n}_t)$. That is, X satisfies B).

3. We have used martingale theory to prove theorem 2, but we may interpret it in a way which doesn't use the order structure of the time set. Let us denote by $(E,\underline{\underline{E}})$ the measurable space $(\mathbb{R}_+,\underline{\underline{B}}(\mathbb{R}_+))$, and by $(F,\underline{\underline{F}})$ the space $(\mathbb{R}\setminus\{0\}, \underline{\underline{B}}(\mathbb{R}\setminus\{0\}))$. We are given a measure $d\Lambda$ on $(E,\underline{\underline{E}})$, which ascribes finite mass to the sets $A_n=[n,n+1[$ whose union in E, and we construct the Poisson random measure dn_t with expectation measure $d\Lambda$.

On the other hand, we are given a transition probability $N(t,du)$ from E to F , and construct a new random measure on $E\times F$ as follows : for each sample $\eta = \Sigma_n \varepsilon_{t_n}$ of the measure dn , we choose independently elements Δ_n in F, each one according to the law $N(t_n,.)$, and set $\xi=\Sigma_n \varepsilon_{t_n,\Delta_n}$. Then theorem 2 asserts that ξ <u>is again a Poisson random measure</u>, <u>with expectation measure</u> $\nu(dt,dx)=N(t,dx)\Lambda(dt)$.

Of course, we have proved it only for particular spaces, but on the other hand, if we start from the above description, assuming just that $(E,\underline{\underline{E}})$ and $(F,\underline{\underline{F}})$ are <u>Lusin</u> measurable spaces, we may identify A_n to a Borel subset of $[n,n+1[$ ([4], chapter III, 20), F to a Borel subset of $\mathbb{R}\setminus\{0\}$, and it is very easy to see that theorem 2 for $\mathbb{R}_+$, $\mathbb{R}\setminus\{0\}$ will imply that the above statement applies to E and F.

This statement is well known in the theory of Poisson random measures, the oldest version being probably that of Doob's book ([3], chapter VIII, § 5 , p. 404-406), and in this form it is true for general measurable spaces, without the Lusin restriction. But still it is interesting to see that a theorem like theorem 2 which seems very special contains fairly general " abstract" results.

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