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A Local Time Inequality For Martingales

by S.D. Jacka*

M.T. Barlow and M. Yor [1] have established the existence of universal constants $c_p, C_p > 0$ such that, for all continuous martingales M, with $M_0 = 0$:

$$c_p \, \| \langle M \rangle_\infty^{\frac{1}{2}} \|_p \leq \| \sup_a L_\infty^a(M) \|_p \leq C_p \, \| \langle M \rangle_\infty^{\frac{1}{2}} \|_p \, . \tag{A}$$

One is naturally led to consider possible extensions of these inequalities involving the term $\sup_a \sup_t | L_t^a(M) - L_t^a(N) |$ and in this paper we establish the existence of a universal constant c_p such that

$$\| (\langle M-N \rangle_\infty - \langle M-N \rangle_0)^{\frac{1}{2}} \|_p \leq C_p \, \| \sup_a \sup_t | L_t^a(M) - L_t^a(N) | \, \|_p$$

for all continuous martingales M and N (Theorem 1).

Conversely, Barlow and Yor [2], have recently established the inequality:

$$\| \sup_a \sup_t | L_t^a(M) - L_t^a(N) | \, \|_p \leq C_p \, \| (M-N)_\infty^* \|_p^{\frac{1}{2}} \, \| M_\infty^* + N_\infty^* \|_p^{\frac{1}{2}} \left\{ 1 \vee \ln \left(\frac{\| M_\infty^* + N_\infty^* \|_p}{\| (M-N)_\infty^* \|_p} \right) \right\}^{\frac{1}{2}} \tag{B}$$

We also establish (Theorem 2) the ess sup equality:

$$\operatorname{ess\,sup} \sup_t \, | L_t^a(M) - L_t^a(N) | = \operatorname{ess\,sup} | L_\infty^a(M) - L_\infty^a(N) |$$

for each $a \in \mathbb{R}$.

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t; t\geq 0))$ be a filtered probability space satisfying the usual conditions. For any random variable f and any $p\in(0,\infty)$ we set $\|f\|_p = (E[|f|^p])^{1/p}$ and we set $\|f\|_\infty$ = ess sup$|f|$. For any continuous local $\mathcal{F}_t$-martingale X and any $p\in(0,\infty)$ we set $\|X\|_{H^p} = \|\langle X\rangle_\infty^{\frac{1}{2}}\|_p$ where $\langle X\rangle_t$ is the unique, increasing adapted process such that $\langle X\rangle_0 = X_0^2$ and $X_t^2 - \langle X\rangle_t$ is a local $\mathcal{F}_t$-martingale, and define H^p $= \{X : \|X\|_{H^p}<\infty\}$.

We recall the Burkholder-Davis-Gundy inequalities which state that for each $p\in(0,\infty)$ there exist universal constants $c_p, C_p>0$ such that, for all $X\in H^p$

$$c_p \|X_\infty^*\|_p \leq \|\langle X\rangle_\infty^{\frac{1}{2}}\|_p \leq C_p \|X_\infty^*\|_p$$

where $X_t^* = \sup_{s\leq t}|X_s|$.

Following [5] we define the local time of X by Tanaka's formula:

$$|X_t-a| = |X_0-a| + \int_{0+}^t \operatorname{sgn}(X_s-a)\,dX_s + L_t^a(X) ,$$

we recall that, for each a , $L_t^a(X)$ is increasing in t , [6], and the support of the measure dL_t^a is contained in $\{t : X_t=a\}$. Furthermore, since we are working with continuous local martingales we may take a version of $(L_t^a(X);\ a\in\mathbb{R},\ t\geq 0)$ which is jointly continuous in a and t, [3].

For any $X\in H^p$ set $\hat{X} = X-X_0$. Finally we recall two definitions : if $F : \mathbb{R}^+\to\mathbb{R}^+$ is an increasing function with $F(0)=0$, $F(x)\neq 0$ for $x\neq 0$ we say that F is <u>moderate</u> if there exists an $\alpha>1$ such that

$$\sup_{x>0} \frac{F(\alpha x)}{F(x)} < \infty$$

and that F is <u>slowly increasing</u> if there exists an $\alpha>1$ such that

$$\sup_{x>0} \frac{F(\alpha x)}{F(x)} < \alpha.$$

<u>Theorem 1</u> <u>For each $p>0$ there exists a universal constant c_p such that</u>

$$c_p \left\| \sup_a \sup_t |L_t^a(M) - L_t^a(N)| \right\|_p \geq \left\| (\langle M-N\rangle_\infty - \langle M-N\rangle_0)^{\frac{1}{2}} \right\|_p \qquad (1)$$

<u>for all M and N in H^p.</u>

The proof is obtained via several lemmas.

For $M,N\in H^p$ define, for each $c>0$, the stopping time

$$\tau_c = \inf\{t\geq 0 : |M_t-N_t| \geq |M_0-N_0|+c\}$$

where the infimum of the empty set is taken as $+\infty$.

<u>Lemma 2</u> <u>For M and N in H^1</u>

$$8E\left[\sup_a \sup_t |L_t^a(M)-L_t^a(N)|\, I_{(\tau_c<\infty)}\right] \geq cP(\tau_{2c}<\infty) \qquad (2)$$

<u>Proof</u> Define

$$\sigma_c = \inf\{t\geq\tau_c : |M_t-M_{\tau_c}| \vee |N_t-N_{\tau_c}| \geq \tfrac{1}{2}c\}$$

Now, by the continuity of M and N, $|M_{\tau_c}-N_{\tau_c}|=|M_0-N_0|+c$ on $(\tau_c<\infty)$, and so N_t does not hit M_{τ_c} and M_t does not hit N_{τ_c} on the interval $[\tau_c,\sigma_c]$; therefore

$$\left.\begin{aligned} L_{\sigma_c}^{M_{\tau_c}}(N) &= L_{\tau_c}^{M_{\tau_c}}(N) \\ L_{\sigma_c}^{N_{\tau_c}}(M) &= L_{\tau_c}^{N_{\tau_c}}(M) \end{aligned}\right\} \tag{3}$$

setting

$$U(a,t) = L_t^a(M) - L_t^a(N)$$

$$D_t = \sup_a \sup_{s \le t} U(a,s)$$

we see that

$$4D_{\sigma_c} I_{(\tau_c<\infty)} \ge [U(M_{\tau_c},\sigma_c) - U(M_{\tau_c},\tau_c)] - [U(N_{\tau_c},\sigma_c) - U(N_{\tau_c},\tau_c)]$$

Using (3) we obtain

$$4D_\infty I_{(\tau_c<\infty)} \ge (L_{\sigma_c}^{M_{\tau_c}}(M) - L_{\tau_c}^{M_{\tau_c}}(M)) + (L_{\sigma_c}^{N_{\tau_c}}(N) - L_{\tau_c}^{N_{\tau_c}}(N)) \tag{4}$$

Applying Tanaka's formula we see that the right-hand side of (4) is

$$|M_{\sigma_c} - M_{\tau_c}| + |N_{\sigma_c} - N_{\tau_c}| - \int_{\tau_c}^{\sigma_c} \operatorname{sgn}(M_s - M_{\tau_c})\, dM_s$$
$$- \int_{\tau_c}^{\sigma_c} \operatorname{sgn}(N_s - N_{\tau_c})\, dNs \tag{5}$$

The two stochastic integrals in (5) are martingales in H^1, as M and N are in H^1, and so, applying the optional sampling theorem we obtain

$$4E(D_\infty I_{(\tau_c<\infty)}) \ge E[|M_{\sigma_c} - M_{\tau_c}| + |N_{\sigma_c} - N_{\tau_c}|] \tag{6}$$

Finally, $\sigma_c < \tau_{2c}$, and on $(\sigma_c < \infty)$, $|M_{\sigma_c} - M_{\tau_c}| + |N_{\sigma_c} - N_{\tau_c}| \geq c/2$ so, substituting in (6) we obtain (2) □.

The following is a slight adaptation of lemma 1.4 of [4].

Lemma 3 [4, lemma 1.4] If X is a positive, right-continuous adapted process and B is an increasing, previsible process with $X_0 = B_0 = 0$, such that for all finite stopping times T; $E[X_T] \leq E[B_T]$, then for each slowly increasing function F there exists a constant C_F such that

$$C_F \, E[F(X_\infty^*)] \leq E[F(B_\infty)] \quad .$$

Lemma 4 There exists a universal constant K such that for all $M, N \in H^1$

$$KE\left[\sup_a \sup_t |L_t^a(M) - L_t^a(N)|\right] \geq E[(M-N)_\infty^* - |M_0 - N_0|] \tag{7}$$

Proof Integrating the inequality (2) with respect to c we obtain

$$8E[D_\infty[(M-N)_\infty^* - |M_0 - N_0|]] = 8\int_0^\infty E[D_\infty I_{(\tau_c < \infty)}]dc$$

$$\geq \int_0^\infty c \; P(\tau_{2c} < \infty)dc = \tfrac{1}{2}E[((M-N)_\infty^* - |M_0 - N_0|)^2]$$

which gives, using Hölder's inequality

$$KED_\infty^2 \geq E[((M-N)_\infty^* - |M_0 - N_0|)^2] \tag{8}$$

$(M-N)_t^* - |M_0 - N_0|$ is a positive right-continuous adapted process whilst D is continuous (and so previsible) as a consequence of the joint continuity in (a,s) of $(L_s^a(M)$ and $(L_s^a(N).)$ Applying (8) to the martingales M^T and N^T we see that $[(M-N)_t^* - |M_0 - N_0|]^2$ and KD_t^2 satisfy the conditions of lemma 3 so setting $F(x) = x^{\frac{1}{2}}$ we obtain (7) □.

<u>Lemma 5 There exists a universal constant c such that</u>

$$c\, E[\sup_a \sup_t |L_t^a(M)-L_t^a(N)|] \geq E[\langle \widehat{M-N}\rangle_\infty^{\frac{1}{2}}] \qquad (9)$$

<u>Proof</u> Set

$$\nu = \inf\{t\geq 0 : |M_t-M_0| \vee |N_t-N_0| \geq \tfrac{1}{2}|M_0-N_0|\}.$$

As the ranges of $(M_t ; t\leq\nu)$ and $(N_t; t\leq\nu)$ are disjoint $L_t^a(M)\wedge L_t^a(N)=0$ for each a, for $t\leq\nu$. Thus

$$D_\nu = (\sup_a L_\nu^a(M)) \vee (\sup_a L_\nu^a(N)) \geq \tfrac{1}{2}(\sup_a L_\nu^a(M)) + \tfrac{1}{2}(\sup_a L_\nu^a(N))$$

and so by theorem 3.1 of [1]

$$c\, ED_\infty \geq E(\hat{M}_\nu^* + \hat{N}_\nu^*)$$

which leads to

$$4c\, ED_\infty \geq 4E((\hat{M}_\nu^*+\hat{N}_\nu^*)I_{(\nu<\infty)}) + 4E((\hat{M}_\infty^*+\hat{N}_\infty^*)I_{(\nu=\infty)})$$

$$\geq 2E(|M_0-N_0|I_{(\nu<\infty)}) + E((\hat{M}-\hat{N})_\infty^* I_{(\nu=\infty)}) \qquad (10)$$

Adding (7) and (10) we obtain

$$CED_\infty \geq E((M-N)_\infty^* - |M_0-N_0|) + 2E(|M_0-N_0|I_{(\nu<\infty)}) + E((\hat{M}-\hat{N})_\infty^* I_{(\nu=\infty)})$$

$$= E[((M-N)_\infty^* - |M_0-N_0|)I_{(\nu=\infty)} + ((M-N)_\infty^* + |M_0-N_0|)I_{(\nu<\infty)}]$$

$$+ E((\hat{M}-\hat{N})_\infty^* I_{(\nu=\infty)})$$

$$\geq E[(\hat{M}-\hat{N})_\infty^*]$$

We obtain (9) by observing that $(\hat{M}-\hat{N}) = (\widehat{M-N})$ and by applying the Burkholder-Davis-Gundy inequality with $p=1$ □.

<u>Lemma 6 [4, lemma 1.1]</u> <u>If A and B are increasing, previsible processes and there exist $a,q>0$ such that for all pairs of finite stopping times $S\le T$</u>

$$E[(A_T I_{(T>0)} - A_S I_{(S>0)})^q] \le aE[B_T^q I_{(T>S)}]$$

<u>then for every moderate function F there exists a $c=c(a,q,F)$ such that</u>

$$E[F(A_\infty)] \le c\ E[F(B_\infty)]$$

<u>Proof of theorem 1</u> For $M,N\in H^1$ set

$$m_t = M^T_{(S+t)}$$

$$n_t = N^T_{(S+t)}$$

We see that

$$L^a_t(m) = L^a_{S+t}(M^T) - L^a_S(M^T)$$

$$L^a_t(n) = L^a_{S+t}(N^T) - L^a_S(N^T)$$

and, applying lemma 5 to these $(\mathcal{F}_{S+t})$-martingales we obtain, with some simple manipulation

$$2ED_T I_{(S<T)} \ge E[\sup_{S\le s\le T} \sup_a |(L^a_s(M) - L^a_S(M)) - (L^a_s(N) - L^a_S(N))|]$$

$$\ge cE[\langle\hat{m}-\hat{n}\rangle^{\frac{1}{2}}_{T-S}]$$

$$\ge cE[\langle\hat{M}-\hat{N}\rangle^{\frac{1}{2}}_T - \langle\hat{M}-\hat{N}\rangle^{\frac{1}{2}}_S]$$

So we obtain (1) by lemma 6 with $F(x) = x^p$. To complete the proof in the case $p<1$, we apply the above inequality to M^{S_n} and N^{S_n} , where $S_n = \inf\{t : |M_t| \vee |N_t| \geq n\}$. and then use monotone convergence to obtain the result. □

<u>Corollary 7</u> <u>If F is a moderate function there exists a universal constant C_F such that</u>

$$C_F \; E(F(\sup_a \sup_t |L_t^a(M) - L_t^a(N)|) \geq E(F((\langle M-N\rangle_\infty - \langle M-N\rangle_0)^{\frac{1}{2}}))$$

<u>for all continuous local martingales M and N .</u>

The proof follows immediately from the above.

<u>Remark</u> Inequality (B) [Barlow and Yor] leads one to ask whether there exists a universal c such that

$$c \; E[\sup_a \sup_t |L_t^a(M) - L_t^a(N)|] \geq \|(M-N)_\infty^*\|_1^{1-\varepsilon} \; \|M_\infty^* + N_\infty^*\|_1^{\varepsilon}$$

for some $\varepsilon>0$. The answer is no. For, take a brownian motion B with $B_0=0$, let $T = \inf\{t\geq 0 : |B_t|=1\}$ and take $\delta>0$; setting $M = B^{T+\delta}$ $N = B^T$ we find that

$D_\infty = \sup_a \sup_{T\leq t\leq T+\delta} |L_t^a(B) - L_T^a(B)| = \sup_a (L_{T+\delta}^a(B) - L_T^a(B))$ and so,

by [1] , $ED_\infty \leq c\delta^{\frac{1}{2}}$ whilst $E(M_\infty^* + N_\infty^*) \geq 2$ and $E[(M-N)_\infty^*] \geq C\delta^{\frac{1}{2}}$ so that

$$\frac{\|(M-N)_\infty^*\|_1^{1-\varepsilon} \; \|M_\infty^* + N_\infty^*\|_1^{\varepsilon}}{ED_\infty} \geq K\delta^{-\frac{1}{2}\varepsilon} \longrightarrow \infty \quad \text{as} \quad \delta \downarrow 0 .$$

We now present our second result.

<u>Theorem 8</u> <u>If M and N are in H^1 then, for each $a \in \mathbb{R}$,</u>

$$\|L_\infty^a(M) - L_\infty^a(N)\|_\infty = \|\sup_t |L_t^a(M) - L_t^a(N)|\,\|_\infty$$

<u>Proof</u> Let $\eta = \text{ess sup}|L_\infty^a(M) - L_\infty^a(N)|$, and define

$$\sigma = \inf\{t\geq 0 : L_t^a(M) - L_t^a(N) \geq \eta + 2\varepsilon\}$$

$$\tau = \inf\{t\geq\sigma : L_t^a(M) - L_t^a(N) \leq \eta + \varepsilon\}$$

Since $L_\infty^a(M) - L_\infty^a(N) \leq \eta$ we see that $(\sigma<\infty)=(\tau<\infty)$. Consider

$$|N_\tau-a|-|N_\sigma-a| = (|N_\tau-a|-|N_\sigma-a|)I_{(\sigma<\infty)} =$$

$$L_\tau^a(N)-L_\sigma^a(N) - \int_\sigma^\tau \text{sgn}(N_s-a)dNs \qquad (11)$$

$N\in H^1$ so the stochastic integral in (11) is uniformly integrable so, by the optional sampling theorem

$$E[(|N_\tau-a|-|N_\sigma-a|)I_{(\sigma<\infty)}] = E[L_\tau^a(N)-L_\sigma^a(N)]$$

But on $(\tau<\infty)=(\sigma<\infty)$, $N_\tau=a$ so

$$0 \geq E[(|N_\tau-a|-|N_\sigma-a|)I_{(\sigma<\infty)}] = E[L_\tau^a(N)-L_\sigma^a(N)] \geq 0 \qquad (12)$$

Now

$$[L_\tau^a(N)-L_\sigma^a(N)]I_{(\sigma<\infty)} = [(L_\tau^a(M)-(\eta+\varepsilon))-(L_\sigma^a(M)-(\eta+2\varepsilon))]I_{(\sigma<\infty)}$$

$$\geq \varepsilon I_{(\sigma<\infty)} ,$$

so we conclude from (12) that $0 \geq \varepsilon P(\sigma<\infty)$. As ε is arbitrary

$$\sup_t(L_t^a(M) - L_t^a(N)) \overset{a.s.}{\leq} \eta$$

and we may deduce the same inequality with M and N reversed.□

<u>Corollary 9 If M and N are in H^1 with $M \not\equiv M_0$ then $M=N$ if and only if for each a $L^a_\infty(M)=L^a_\infty(N)$.</u>

<u>Proof</u> The reverse implication is clear. Now suppose $M_0=N_0$ then, since $D_\infty=0$, theorem 1 implies that $E(M-N)^*_\infty=0$ so that $M=N$. Suppose now $M_0 \neq N_0$, set $\nu=\inf\{t\geq 0 : |M_t-M_0| \vee |N_t-N_0| =\frac{1}{2}|M_0-N_0|\}$ then, since the ranges of $(M_t;t\leq\nu)$ and $(N_t;t\leq\nu)$ are distinct we may conclude that $L^a_\nu(M) \wedge L^a_\nu(N)=0$ but $L^a_\nu(M)=L^a_\nu(N)$ so $L^a_\nu(M)=L^a_\nu(N)=0$ $a\in\mathbb{R}$ and so we conclude that $E((M-M_0)^*_\infty)=0$ and so $M_0=M_t$ for $t\leq\nu$ and thus $(\nu=\infty)$ and $M=M_0$ which contradicts the initial assumption. □

<u>Remark</u> In fact, to conclude that $M=N$, it is sufficient that $L^a_\infty(M)=L^a_\infty(N)$ holds for all $a\in$range (M); the proof is left to the reader.

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