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A COUNTEREXAMPLE RELATED TO A_p–WEIGHTS IN MARTINGALE THEORY

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Given a continuous local martingale M, set $Z=\exp(M-\langle M,M\rangle/2)$. Let a(M) be the infimum of the set of p>1 for which the condition

$$(A_p) \qquad E\left[\left(\frac{Z_t}{Z_\infty}\right)^{\frac{1}{p-1}} \Big| F_t\right] \leqq K$$

holds for every $t \geqq 0$, with a constant K depending only on p. We note that the condition (A_p) plays an important role in various weighted norm inequalities for martingales (see [6] for example) and that $BMO = \{ M : a(M) < \infty \}$ (see [3]). Recall that on the space BMO a norm can be defined by $\|M\|_{BMO} = \sup_t \| E[\,|M_\infty - M_t|\,|F_t]\,\|_\infty$. The class L^∞ of all bounded martingales is obviously contained in BMO, but BMO is not just L^∞. Quite recently, it is proved in [4] that, if $p > \max\{ a(M), a(-M)\}$, then $d(M,L^\infty) < 8(\sqrt{p}-1)$ where d(,) denotes the distance on BMO deduced from the norm by the usual process. Now, is it true that $a(M)=a(-M)$ in general ? Unfortunately the author did not know the answer. As is noted above, it turns out that $a(M)=a(-M)=\infty$ for $M \notin BMO$. And so, in order to consider the question, we may assume that $M \in BMO$. The aim of this short note is to exemplify that the answer is negative.

For that purpose, let (Ω,F,Q) be a probability space which carries a one dimensional Brownian motion $B=(B_t,F_t)$ with $B_0=0$, and we use the stopping time $\tau = \min\{t : |B_t|=1\}$. Then B^τ is a bounded martingale with respect to Q, so that $E_Q[\exp(B_\tau - \tau/2)]=1$ where $E_Q[\ ]$ denotes expectation with respect to Q. That is to say, $dP = \exp(B_\tau - \tau/2)\,dQ$ is a probability measure on Ω. By Girsanov's theorem on such a change of law, the process $M = \langle B^\tau, B^\tau\rangle - B^\tau$ is a continuous martingale with respect to P and further $\langle M,M\rangle = \langle B^\tau, B^\tau\rangle$ under either probability measure. Let now $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Noticing $|B^\tau| \leqq 1$, we find that

$$E[(\frac{Z_t}{Z_\infty})^{\frac{1}{p-1}}|F_t] = E_Q[\exp\{q(B_\tau - B_{t\wedge\tau}) - \frac{q}{2}(\tau - t\wedge\tau)\}|F_t] \leqq \exp(2q).$$

This implies $a(M)=1$, since $p>1$ is arbitrary.
Next, to estimate $a(-M)$, let $Z^{(-1)} = \exp(-M - \langle M,M\rangle/2)$. If $1<p\leqq 2$, we have

$$E[\{\frac{Z_t^{(-1)}}{Z_\infty^{(-1)}}\}^{\frac{1}{p-1}}|F_t] = E_Q[\exp\{\frac{p-2}{p-1}(B_\tau - B_{t\wedge\tau}) + \frac{4-p}{2(p-1)}(\tau - t\wedge\tau)\}|F_t]$$
$$\geqq \exp\{-\frac{2(2-p)}{p-1}\} E_Q[\exp\{\frac{4-p}{2(p-1)}(\tau - t\wedge\tau)\}|F_t].$$

On the other hand, we know that $E_Q[\exp(\lambda\tau)]=\infty$ for $\lambda>\pi^2/8$ (see Proposition 8.4 in [5]). Let now $1<p<(16+\pi^2)/(4+\pi^2)$. Then, noticing $p<2$ and $(4-p)/\{2(p-1)\}>\pi^2/8$, we can find that $a(-M) \geqq (16+\pi^2)/(4+\pi^2)$. Thus $a(-M) \neq a(M)$.

Now, when is it true that $a(-M) = a(M)$? In the following, we assume that any martingale adapted to the underlying filtration (F_t) is continuous.

PROPOSITION. If $M \in \overline{L^\infty}$, then $a(-M) = a(M)$.

PROOF. It suffices to show that $p \geqq a(-M)$ whenever $p>a(M)$. First let $\alpha(M)$ be the supremum of the set of α for which

$$\sup_t \|E[\exp\{\alpha|M_\infty - M_t|\}|F_t]\|_\infty < \infty.$$

In [2] Emery proved the following :

$$\frac{1}{4d(M,L^\infty)} \leqq \alpha(M) \leqq \frac{4}{d(M,L^\infty)} .$$

Observe that $M \in \overline{L^\infty}$ if and only if $\alpha(M)=\infty$. Now, let $p>a(M)$. Then, letting $0<\varepsilon<p-a(M)$ and using Hölder's inequality with exponents $(p-1)/\varepsilon$ and $(p-1)/(p-\varepsilon-1)$, we find

$$E[\{\frac{Z_t^{(-1)}}{Z_\infty^{(-1)}}\}^{\frac{1}{p-1}}|F_t] = E[\exp\{\frac{2}{p-1}(M_\infty - M_t)\}(\frac{Z_t}{Z_\infty})^{\frac{1}{p-1}}|F_t]$$
$$\leqq E[\exp\{\frac{2}{\varepsilon}(M_\infty - M_t)\}|F_t]^{\frac{\varepsilon}{p-1}} E[(\frac{Z_t}{Z_\infty})^{\frac{1}{p-\varepsilon-1}}|F_t]^{\frac{p-\varepsilon-1}{p-1}}.$$

So, noticing $\alpha(M)=\infty$, it turns out that the first conditional expectation on the right hand side is bounded by some constant. Furthermore, the second one is also bounded by some constant, since Z satisfies $(A_{p-\varepsilon})$. Thus we have $p \geqq a(-M)$.

From this result it follows that the example given at the beginning of this paper does not belong to $\overline{L^\infty}$. More generally, it is proved in [1] that $BMO \neq \overline{L^\infty}$ if $BMO \neq L^\infty$.

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