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Predictable Representation of Martingale Spaces and Changes of Probability Measure

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ABSTRACT

We study the predictable representation of martingale spaces under a change of probability measure. The canonical decomposition of special semimartingales provides a simple route to the identity and cardinality of a minimal generating subset of martingales under a change of probability.

1. Introduction

Certain "high"- dimensional spaces of martingales can be generated by a fixed vector of martingales via predictable representations. This property has found application, for example, in stochastic control [1], filtering [11], and more recently in the economics of security trading [5]. The classic study by Kunita and Watanabe [9] of square-integrable martingales has been widely extended; a book [8] and paper [7] by Jean Jacod cover much of the theory I am aware of.

Here we characterize an association between martingale subspaces under different probability measures, in particular the identity and cardinality of minimal generating subsets of martingales. This cardinality has been termed *multiplicity* [2], and more generally, *q-dimension* [8]. Under regularity conditions on the change of probability measure, the q-dimension of the space of q-integrable martingales is invariant. A generating vector of local martingales under one probability measure maps to a generating vector of local martingales under a new probability measure via the transformation specified by "Girsanov's Theorem".

This paper was instigated by a study of multiperiod security markets [4], a setting in which the predictable representation property under a change of probability plays an important role [6].

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2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $F = \{\mathcal{F}_t; t \in R_+\}$ be a filtration of sub-tribes of $\mathcal{F}$ satisfying the usual conditions. We work exclusively on the filtered probability space $(\Omega, \mathcal{F}, F, P)$ for this section.

It is well known that any special semimartingale[1] X has a unique decomposition of the form: $X = X^{\triangleleft} + X^{\triangleright}$, where $X^{\triangleleft}$ denotes the local martingale part and $X^{\triangleright}$ denotes the predictable finite variation null-at-zero part. Similarly, if $X = (X_1, \ldots, X_n)$ is an R^n-valued process whose components are special semimartingales, we write $X^{\triangleleft}$ for $(X_1^{\triangleleft}, \ldots, X_n^{\triangleleft})$, and so on. If $\mathcal{X}$ is a set of special semimartingales, we use $\mathcal{X}^{\triangleleft}$ to denote the set of local martingales $\{X^{\triangleleft};\ X \in \mathcal{X}\}$.

The following general conditions for the existence of real-valued stochastic integrals with respect to R^n-valued semimartingales were developed by Jacod [8]. I use a presentation similar to that of Memin [12].

First, let M be an R^n-valued local martingale. Then there is an increasing finite variation real-valued process C and an optional $n \times n$ positive semi-definite matrix valued process $c = (c_{ij})$ such that $[X_i, X_j] = c_{ij} \cdot C$, where as usual $[\cdot, \cdot]$ denotes quadratic variation (optional compensator) and the raised dot notation $A \cdot B$ is used for the path-by-path Stieltjes integral of A with respect to B (and soon for stochastic integrals as well). Let $L(M)$ denote the set of R^n-valued predictable processes $H = (H_1, \ldots, H_n)$ such that $((\sum_{i,j} H_i c_{ij} H_j) \cdot C)^{1/2}$ is locally integrable. If $H \in L(M)$ the *stochastic integral* $H \cdot M$ is defined as the unique local martingale satisfying

$$[H \cdot M, N] = (\sum_i H_i K_i) \cdot C,$$

for every real valued local martingale N, where K_i denotes the optional process satisfying $[M_i, N] = K_i \cdot C$.

In the case of an R^n-valued RCLL finite variation process $A = (A_1, \ldots A_n)$, there is an increasing real-valued finite variation process V and an optional process $v = (v_1, \ldots, v_n)$ such that $A_i = v_i \cdot V$. If A is predictable, we can choose v and V to be predictable. Let $L(A)$ denote the set of R^n-valued predictable processes $H = (H_1, \ldots, H_n)$ such that $| \sum_i H_i v_i | \cdot V$ is a finite variation process. For $H \in L(A)$, the *stochastic integral* $H \cdot A$ is defined as the Stieltjes integral $(\sum_i H_i v_i) \cdot V$.

Finally, let X be an R^n-valued semimartingale. Let $L(X)$ denote the set of R^n-valued predictable processes H such that there exists a decomposition of X as the sum of an R^n-valued local martingale M and an R^n-valued finite variation process A with $H \in L(M) \cap L(A)$. The stochastic integral $H \cdot X$, defined as the sum of $H \cdot M$ and $H \cdot A$

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[1] The definitions of a special semimartingale and other standard concepts used in this paper may be found in Jacod [8], or Dellacherie and Meyer [3].

does not depend on the particular decomposition chosen for X. This definition extends that for the sum of component stochastic integrals $\sum_i H_i \cdot X_i$, which may not exist for all $H \in L(X)$.

For any $q \in [1, \infty)$ define the positive extended real-valued functional $\| \cdot \|_q$ on the space of semimartingales by

$$\| X \|_q \quad = \quad \| sup_{t \geq 0} [X, X]_t^{1/2} \|_{L^q(\Omega, \mathcal{F}, P)}$$

for any semimartingale X. Let $\mathcal{M}^q$ denote the subspace of local martingales M such that $M_0 = 0$ and $\| M \|_q < \infty$. We restricition our attention to the null-at-zero merely for convenience. Extensions of our results to the general case are easily deduced from Lemma 4.8 of Jacod [8]. As is well known, $\mathcal{M}^q$ is a Banach space under the norm $\| \cdot \|_q$, taking an element of $\mathcal{M}^q$ to be an equivalence class of indistinguishable processes. A *stable subspace* of $\mathcal{M}^q$ is a $\| \cdot \|_q$-closed vector subspace $\mathcal{M}$ of $\mathcal{M}^q$ such that $1_A M^T \in \mathcal{M}$ for every $M \in \mathcal{M}$, $A \in \mathcal{F}_0$, and stopping time T. This is equivalent to stability under stochastic integration, in the sense that $\mathcal{M}$ is a stable subspace of $\mathcal{M}^q$ if and only if, for any vector local martingale M whose components are elements of $\mathcal{M}$,

$$\mathcal{L}^q(M) \equiv \{H \cdot M \in \mathcal{M}^q; H \in L(M)\} \subset \mathcal{M}.$$

This is a trivial extension of Jacod [8;(4.3)]. For any set $\mathcal{M}$ of local martingales let $\mathcal{L}^q(\mathcal{M})$ denote the smallest stable subspace of $\mathcal{M}^q$ containing $\mathcal{L}^q(M)$ for all $M \in \mathcal{M}$. In fact, $\mathcal{L}^q(\mathcal{M})$ is the closure of $\bigcup_{M \in \mathcal{M}} \mathcal{L}^q(M)$ [8;(4.5)]. Of course $\mathcal{L}^q(M)$ is itself a stable subspace of $\mathcal{M}^q$ for any vector M of local martingales.

For any set $\mathcal{A}$ of adapted processes let $\mathcal{A}_{loc}$ denote the set of processes which are "locally" in $\mathcal{A}$. That is, $A \in \mathcal{A}_{loc}$ if there is an increasing sequence of stopping times (T_n) such that $T_n \to \infty$ *a.s.* and $A^{T_n} \in \mathcal{A}$ for all n.

For any vector M of local martingales, let $\mathcal{L}(M) = \{H \cdot M; H \in L(M)\}$.

LEMMA 2.1. *For any vector M of local martingales, $\mathcal{L}(M)_{loc} = \mathcal{L}(M)$.*

PROOF: Let $X \in \mathcal{L}(M)_{loc}$ and (T_n) be an increasing sequence of stopping times converging to infinity such that there exist $H_n \in L(M)$ with $X^{T_n} = H_n \cdot M$ for all $n \geq 1$. Define a sequence (Y_n) of processes in $L(M)$ as follows. Let $Y_1 = H_1$. Let $Y_{n+1} = Y_n$ on $[\![0, T_n]\!]$ and $Y_{n+1} = H_{n+1}$ on $]\!]T_n, \infty[\![$. Since $Y_{n+1}^{T_n} = Y_n$, the processes (Y_n) "paste together" to form a process Y, which is predictable since $Y = \lim_n Y_n$. Then $Y \in L(M)$ and $X = Y \cdot M$ since $X^{T_n} = (Y \cdot M)^{T_n}$ for all n. ∎

If $\mathcal{M}$ is a stable subspace of $\mathcal{M}^q$, a *q–generator* of $\mathcal{M}$ is a vector M of local martingales whose components are elements of $\mathcal{M}_{loc}$ such that $\mathcal{L}^q(M) = \mathcal{M}$. If $M = (M_1, \ldots, M_n)$ is a q–generator of $\mathcal{M}$ and there is no q–generator of fewer components, the *q–dimension* of $\mathcal{M}$ is n and M is a *q–basis* for $\mathcal{M}$. Many examples are given by Jacod [7,8]. If $\mathcal{M}$ has no (finite) q–generator, its q–dimension is defined to be infinite. If $\mathcal{M} = \{0\}$, its q–dimension is defined to be zero. This covers all cases, although it is possible to distinguish countably infinite from uncountably infinite q–dimension [8;Chapter 4].

The following result shows that the q–dimension of a stable subspace $\mathcal{M} \subset \mathcal{M}^q$ is in fact the minimum dimension of a vector of martingales M which generates $\mathcal{M}$ (or $\mathcal{M} \subset \mathcal{L}(M)$), *whether or not the components of M are in $\mathcal{M}_{loc}$.*

LEMMA 2.2. *Suppose, for some $q \in [1, \infty)$, that $\mathcal{M}$ is a stable subspace of $\mathcal{M}^q$ and $M = (M_1, \ldots, M_n)$ is a vector of local martingales such that $\mathcal{M} \subset \mathcal{L}(M)$. Then q-dim$(\mathcal{M}) \leq n$.*

PROOF: Choose any vector martingale $N = (N_1, \ldots, N_m)$ whose components are in $\mathcal{M}$, with associated *dimension process* ς^N, as defined by Jacod [8,p.147]. Let ς^M denote the dimension process associated with M. By assumption, there exists an $m \times n$ matrix valued process K whose rows are elements of $L(M)$ such that $N = K \cdot M$, in the obvious sense. Since the components of both M and N are elements of $\mathcal{M}^1_{loc}$, Jacod's Proposition (4.71) applies for $q = 1$ and $\varsigma^N \leq \varsigma^M \leq n$ almost surely. Then Jacod's Theorem (4.74) can be applied to complete the proof. ▮

COROLLARY. *Suppose $1 \leq q \leq p < \infty$. Then p-dim$(\mathcal{M}^p) \leq q$-dim$(\mathcal{M}^q)$.*

PROOF: This follows from the fact that $\mathcal{M}^p \subset \mathcal{M}^q$. ▮

3. Change of Probability

Let Q be any probability measure on $(\Omega, \mathcal{F})$ absolutely continuous with respect to P. When defining concepts under Q we work on the filtered probability space $(\Omega, \mathcal{F}^Q, F^Q, Q)$, where $\mathcal{F}^Q$ and F^Q denote completions for Q. We distinguish definitions for the two filtered probability spaces $(\Omega, \mathcal{F}, F, P)$ and $(\Omega, \mathcal{F}^Q, F^Q, Q)$ by augmenting the notation with "P" or "Q", as in $L_P(X)$ and $L_Q(X)$, $\|\cdot\|_{qP}$ and $\|\cdot\|_{qQ}$, $X^{\triangleleft P}$ and $X^{\triangleleft Q}$, $H \overset{P}{\cdot} X$ and $H \overset{Q}{\cdot} X$, $\mathcal{M}^q_P$ and $\mathcal{M}^q_Q$, and so on. Let $\mathcal{S}(P)$ and $\mathcal{S}_p(P)$ denote the spaces of semimartingales and special semimartingales, respectively, under P, and similarly define $\mathcal{S}(Q)$ and $\mathcal{S}_p(Q)$. We use the facts that $\mathcal{S}(P) \subset \mathcal{S}(Q)$ and that the quadratic variation of a semimartingale under P is a Q-version of its quadratic variation under Q. See, for example, Chapter 7 of [8].

Let the P–martingale ξ denote the *density process* [8, Chapter 7] for the Radon–Nikodym derivative $\frac{dQ}{dP}$, equating $\xi(t)$ with the restriction of $\frac{dQ}{dP}$ to $\mathcal{F}_t$ for all $t \geq 0$. For reference, we identify the mapping $M \mapsto M^{\triangleleft Q}$ and its domain of definition, the P–local martingales in $\mathcal{S}_p(Q)$. This identification is known as Girsanov's Theorem, due to Lenglart [10] in this generality. The following form of the theorem is from [8,(7.29)].

THEOREM 3.1. *Let $M \in \mathcal{M}^1_{P,loc}$. Then $M \in \mathcal{S}_p(Q)$ if and only if $[M, \xi]$ is locally of integrable variation, in which case $M^{\triangleright Q} = (\xi^{-1})_- \cdot \langle M, \xi\rangle^P$ and $M^{\triangleleft Q} = M - M^{\triangleright Q}$.*

The condition on $[M,\xi]$ may be difficult to verify. The following sufficient condition is a trivial consequence of [3;VII.39]. As usual, $\mathcal{M}_P^\infty$ denotes the space of P-essentially bounded martingales.

LEMMA 3.1. *Let $q \in [1,\infty)$ and $q* \in (1,\infty]$ satisfy $\frac{1}{q}+\frac{1}{q*}=1$. If $\xi-\xi_0 \in \mathcal{M}_{P,loc}^{q*}$ then $\mathcal{M}_{P,loc}^{q} \subset S_p(Q)$.*

The following result is due to Memin [12].

PROPOSITION 3.1. *Let X be any R^n-valued P-semimartingale. If $H \in L_P(X)$ then $H \in L_Q(X)$ and $H \overset{P}{\cdot} X$ is a Q-version of $H \overset{Q}{\cdot} X$.*

The next lemma is a technical aid. We write $X^{\triangleleft P \triangleleft Q}$ for $(X^{\triangleleft P})^{\triangleleft Q}$ whenever the operations are defined, and so on for other combinations.

LEMMA 3.2. *For any $M \in \mathcal{M}_{Q,loc}^1 \bigcap S_p(P)$, both $M^{\triangleleft P}$ and $M^{\triangleright P}$ are in $S_p(Q)$, and:*

(a) $M^{\triangleleft P \triangleleft Q} = M$

(b) $M^{\triangleleft P \triangleright Q} = -M^{\triangleright P}$

(c) $M^{\triangleright P \triangleright Q} = M^{\triangleright P}$

(d) $M^{\triangleright P \triangleleft Q} = 0$

PROOF: Since $M^{\triangleright P}$ is predictable, finite variation, and null-at-zero under P, and $Q \prec\!\!\prec P$, the same properties hold under Q, proving $M^{\triangleright P} \in S_p(Q)$ as well as (c) and (d). Since $M^{\triangleleft P} = M - M^{\triangleright P}$ forms the canonical decomposition of $M^{\triangleleft P}$ under Q, the remaining claims follow immediately. ∎

PROPOSITION 3.2. *Suppose the components of $M = (M_1,\ldots,M_n)$ are elements of $\mathcal{M}_{P,loc}^1 \cap S_p(Q)$ and $H \in L_P(M)$. If $H \overset{P}{\cdot} M \in S_p(Q)$ then $H \in L_Q(M^{\triangleleft Q})$ and $(H \overset{P}{\cdot} M)^{\triangleleft Q}$ is a Q-version of $H \overset{Q}{\cdot} M^{\triangleleft Q}$.*

PROOF: For the case $n=1$, [8,(7.26(a))] shows that $H \in L_Q(M^{\triangleleft Q})$ and that $(H \overset{P}{\cdot} M) - (\xi^{-1})_- \cdot \langle H \overset{P}{\cdot} M, \xi\rangle$ is a Q-version of $H \overset{Q}{\cdot} M^{\triangleleft Q}$. A proof of this result for $n > 1$ is a straightforward extension of Jacod's proof of [8, (7.26(a))]. Then the result follows from Theorem 3.1. ∎

We have a preliminary result showing the basic relationship between stable subspaces under a change of probability.

PROPOSITION 3.3. *Suppose $\frac{dQ}{dP}$ is essentially bounded. For any $q \in [1,\infty)$ and any set $\mathcal{M}$ of local martingales, $\mathcal{L}_P^q(\mathcal{M})^{\triangleleft Q} \subset \mathcal{L}_Q^q(\mathcal{M}^{\triangleleft Q})$. If, in addition, $\frac{dP}{dQ}$ exists and is essentially bounded, then $\mathcal{L}_P^q(\mathcal{M})^{\triangleleft Q} = \mathcal{L}_Q^q(\mathcal{M}^{\triangleleft Q})$.*

PROOF: Only the first assertion is proved here. The proof of the second is clear given the proof below.

Let $X \in \mathcal{L}^q_P(\mathcal{M})$, implying a sequence (X_n) converging to X in $\|\cdot\|_{qP}$, and thus also converging in $\|\cdot\|_{qQ}$, such that, for all n, $X_n = H_n \overset{P}{\cdot} M_n \in \mathcal{M}^q_P$, where $M_n \in \mathcal{M}$, and $H_n \in L_P(M_n)$. By Proposition 3.2 and Lemma 3.1, $H_n \in L_Q(M_n)$ and $(H_n \overset{P}{\cdot} M_n)^{\triangleleft Q} = H_n \overset{Q}{\cdot} M_n^{\triangleleft Q}$. By Dellacherie and Meyer [3], (VII.95, Remark (c)),

$$\| X^{\triangleleft Q} - H_n \overset{Q}{\cdot} M_n^{\triangleleft Q} \|_{qQ} \quad \leq \quad K_q \| X - H_n \overset{Q}{\cdot} M_n \|_{qQ} \tag{A}$$

for a given constant K_q depending only on q. Thus $H_n \overset{Q}{\cdot} M_n^{\triangleleft Q} \to X^{\triangleleft Q}$ in $\|\cdot\|_{qQ}$. Since $\mathcal{L}^q_Q(\mathcal{M}^{\triangleleft Q})$ is $\|\cdot\|_{qQ}$–closed, $X^{\triangleleft Q} \in \mathcal{L}^q_Q(\mathcal{M}^{\triangleleft Q})$. ∎

The following may be considered the main result. As a reminder, necessary and sufficient conditions for a local martingale to be a special semimartingale under a change of measure are given by Theorem 3.1, with convenient sufficient conditions for a q–generator given by Lemma 3.1. It may be worth noting that a semimartingale with locally bounded jumps is a special semimartingale under any absolutely continuous change of probability measure.

THEOREM 3.2. *Suppose $M = (M_1, \ldots, M_m)$ is a q-generator of $\mathcal{M}^q_P$ whose components are Q-special semimartingales, and $\frac{1}{\xi}$ is locally (P-essentially) bounded, then:*

(a) $\mathcal{L}^q_Q(M^{\triangleleft Q}) = \mathcal{M}^q_Q$

(b) $q\text{-dim}(\mathcal{M}^q_Q) \leq q\text{-dim}(\mathcal{M}^q_P) \leq m$.

PROOF: *[Part (a)] By definition, $\mathcal{L}^q_Q(M^{\triangleleft Q}) \subset \mathcal{M}^q_Q$. Suppose $X \in \mathcal{M}^q_Q$. Let (T_n) be an increasing sequence of stopping times such that $T_n \to \infty$ P-almost surely and $(\frac{1}{\xi})^{T_n}$ is (P-essentially) bounded for all n. Then $X^{T_n} \in S_p(P)$ for all n, and by [3, VII.26], $X \in S_p(P)$. For any n, the quadratic variation $[X,X]^Q$ is a P-version of $[X,X]^P$ on $[0, T_n]$. Thus, for any n,*

$$\| (X^{\triangleleft P})^{T_n} \|_{qP} \quad \leq \quad K_q \| X^{T_n} \|_{qP} \quad \leq \quad B \| X^{T_n} \|_{qQ} < \infty,$$

where B is an essential upper bound on $(\frac{1}{\xi})^{T_n}$ and K_q is as given in relation (A). (We have not assumed $P \ll Q$, but for the last claim we can restrict ourselves to $(\Omega, \mathcal{F}^Q_{T_n})$ and apply the results of Jacod [8], Chapter 7, in particular Theorem 7.2.) Thus $X^{\triangleleft P} \in \mathcal{M}^q_{P,loc}$, and by Lemma 2.2 there exists $H \in L(M)$ such that $X^{\triangleleft P} = H \overset{P}{\cdot} M$. By Proposition 3.2 and Lemma 3.2 we have $X = H \overset{Q}{\cdot} M^{\triangleleft Q}$, proving part (a).

[Part (b)] This follows from Lemma 2.2. ∎

Remark: For the case $q = 1$, the assumption that $1/\xi$ is locally bounded may be replaced by the assumption that $\mathcal{M}^1_Q \subset S_p(P)$.

The following corollary is verified by applying symmetry and the bound K_q used in expression (A).

COROLLARY 1. *Suppose Q and P are equivalent and ξ as well as $\frac{1}{\xi}$ are locally (essentially) bounded. Then q–dim$(\mathcal{M}^q_Q)$ = q–dim$(\mathcal{M}^q_P)$. If M is a q–generator (q–basis) for $\mathcal{M}^q_P$ then $M^{\triangleleft Q}$ is a q–generator (q–basis) for $\mathcal{M}^q_Q$.*

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