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SOME ADDITIONAL REMARKS ON FOCK SPACE STOCHASTIC CALCULUS

by K.R. Parthasarathy

These remarks refer to P.A. Meyer's article << Quelques remarques au sujet du calcul stochastique sur l'espace de Fock >>, in this volume, and use essentially the same notations.

Let Φ be Fock space over $L_2[0,\infty)$, $\mathcal{K}=\{u\in L_2[0,\infty),\ \mathrm{supp}(u)\ \text{compact}\}$ and $\mathcal{E}$ = linear manifold generated by $\{\mathcal{E}(u),\ u\in\mathcal{K}\}$.

PROPOSITION 1. Let $M=\{M(t)\}$, $M^*=\{M^*(t)\}$ be martingales with domain $\mathcal{E}$ and adjoint to each other on $\mathcal{E}$. Suppose

(1) $\sup_t\ (\|M(t)\mathcal{E}(u)\|+\|M^*(t)\mathcal{E}(u)\|) < \infty$ for $u\in\mathcal{K}$.

Then there exist operators $M(\infty)$, $M^*(\infty)$ with domain $\mathcal{E}$, adjoint to each other on $\mathcal{E}$, satisfying for $u\in\mathcal{K}$

(2) $M(\infty)\mathcal{E}(u) = \lim_{t\to\infty} M(t)\mathcal{E}(u)$, $M^*(\infty)\mathcal{E}(u) = \lim_{t\to\infty} M(t)\mathcal{E}(u)$

<u>Proof</u>. Let $\mathrm{Supp}(u)\subset[0,a]$. Then the random variables $\{M(t)\mathcal{E}(u),\ t\geqq a\}$ and $\{M^*(t)\mathcal{E}(u),\ t\geqq a\}$ constitute classical martingales in Wiener space with bounded mean square norms. Hence the limits on the right sides of (2) exist. The linear independence of coherent vectors ensures that the operators $M(\infty)$ and $M^*(\infty)$ are well defined by (2) and extension to $\mathcal{E}$ by linearity. []

PROPOSITION 2 . There exist sequences $\{X_j\}$, $\{Y_j\}$ of operators of rank $(\leqq)$ one on Φ such that

(3) $M(\infty) = \Sigma_1^\infty X_j$, $M^*(\infty) = \Sigma_1^\infty Y_j$ on $\mathcal{E}$.

where the right hand sides converge strongly on $\mathcal{E}$.

<u>Proof</u>. Since $\mathcal{E}$ is dense in Φ we can choose in $\mathcal{E}$ a complete orthonormal basis $\{\xi_j,\ j=1,2,\dots\}$ of Φ . Thus by proposition 1

$$M(\infty)\mathcal{E}(u) = \Sigma_j < \xi_j,\ M(\infty)\mathcal{E}(u)>\xi_j = \Sigma_j <M^*(\infty)\xi_j,\mathcal{E}(u)>\xi_j$$

for every $u\in\mathcal{K}$. Using Dirac's notation and putting $X_j=|\xi_j><M^*(\infty)\xi_j|$ (and similarily $Y_j=|\xi_j><M(\infty)\xi_j|$) .

<u>Remark</u>. Y_j is not necessarily the adjoint of X_j in the above proof !

<u>Definition</u>. An adapted process M is said to be <u>of rank</u> $\leqq m$ if $M(t)|_{\Phi_{t]}}$ is an operator of rank $\leqq m$ for all t.

PROPOSITION 3. Let $\{M(t)\}$, $\{M^*(t)\}$ be martingales satisfying the conditions of proposition 1. Then there exists a sequence $M_j=\{M_j(t)\}$ of martingales of rank ≤ 1 such that

(4) $$M(t) = \Sigma_j M_j(t) \text{ on } \mathcal{E} \text{ for all } t ,$$

where the right hand side converges strongly on $\mathcal{E}$.

Proof. X_j denoting the same operator as in Proposition 2, for $u\in K$ define the operator $M_j(t)$ by

$$M_j(t)\varepsilon(u) = \{E_{t]}X_j\varepsilon(u_{t]})\}\varepsilon(u_{[t})$$

where $E_{t]}$ is conditional expectation given the brownian motion up to time t . Then $M_j = \{M_j(t)\}$ is a martingale, and

$$M_j(t)|_{\Phi_{t]}} = |E_{t]}\xi_j\rangle \langle E_{t]}M^*(\infty)\xi_j|$$

with the notations of the proof of proposition 2. Now the Proposition is immediate from (3) and the definitions. []

PROPOSITION 4. Let X be an operator of rank one on Φ and let $\{X(t)\}$ be the martingale of rank ≤ 1 defined by

(5) $$X(t)\varepsilon(u) = \{E_{t]}X\varepsilon(u_{t]})\}\varepsilon(u_{[t})$$

Then there exist adapted processes K,L of rank ≤ 1 such that

(6) $$dX = Kda^+ - Xda^\circ + Lda^- .$$

Remark. Since X is a bounded operator, $X(t)$ can be defined outside $\mathcal{E}$. However, we shall use only (5).

Proof. Suppose $X = |\xi\rangle\langle\eta|$. There exists a square integral adapted process ρ_ξ such that

$$E_{t]}\xi = E\xi + \int_0^t \rho_\xi(s)dB(s) \quad \text{where } E=E_{0]} \text{ and } B \text{ is brownian motion,}$$

and a similar representation for η . For any $v\in L_2[0,\infty)$ we have

(7) $$< \varepsilon(v_{t]}),\xi > = < \varepsilon(v_{t]}), E_{t]}\xi > = E\xi + \int_0^t \overline{v}(s)\langle\varepsilon(v_{s]}),\rho_\xi(s)\rangle\, ds .$$

Then for $u,v \in L_2[0,\infty)$ we have by (5)

$$< \varepsilon(v),X(t)\varepsilon(u)> = < \varepsilon(v_{t]}),E_{t]}\xi> \langle E_{t]}\eta,\varepsilon(u_{t]})\rangle\, e^{\int_t^\infty \overline{v}(s)u(s)ds}$$

Using (7) we get for $\frac{d}{dt}< \varepsilon(v),X(t)\varepsilon(u)>$ the expression

$$-\overline{v}(t)u(t)\langle\varepsilon(v),X(t)\varepsilon(u)\rangle +$$
$$e^{\int_t^\infty \overline{v}u\,ds}\{\overline{v}(t)\langle\varepsilon(v_{t]}),\rho_\xi(t)\rangle\langle E_{t]}\eta,\varepsilon(u_{t]})\rangle + u(t)\langle\varepsilon(v_{t]}),E_{t]}\xi\rangle\langle\rho_\eta(t),\varepsilon(u_{t]})\rangle$$

On other words,

$$dX = -Xda^\circ + Kda^+ + Lda^-$$

where

$$K(t) = (|\rho_\xi(t)\rangle \langle E_{t]}\eta|) \otimes I_{[t}$$

$$L(t) = (|E_{t]}\xi\rangle \langle \rho_\eta(t)|) \otimes I_{[t}$$

and $I_{]t}$ is the identity operator in $\Phi_{[t}$.

COROLLARY. Let X be a Hilbert-Schmidt operator on Φ and let $\{X(t)\}$ be the Hilbert-Schmidt martingale defined by

$$X(t)\varepsilon(u) = \{E_{t]}X\varepsilon(u_{t]})\}\varepsilon(u_{[t}) , \quad u \in L_2[0,\infty) .$$

Then there exist Hilbert-Schmidt adapted processes K,L such that

$$dX = Kda^+ - Xda^\circ + Lda^- .$$

If $\operatorname{rank} X = m < \infty$, then K and L are of rank $\leq m$.

Proof. Immediate.

THEOREM. Let M, M^* be martingales satisfying the conditions of Proposition 1. Then there exist martingales M_j ($j=1,2,\dots$) satisfying the following conditions

(i) $M(t) = \Sigma_j M_j(t)$ on $\mathcal{E}$, in the sense of strong convergence on $\mathcal{E}$.

(ii) For every j , $M_j(\infty) = \operatorname{slim}_{t\to\infty} M_j(t)$ is an operator of rank one.

(iii) For every j, $dM_j = K_j da^+ - M_j da^\circ + L_j da^-$, where K_j, L_j are adapted processes of rank ≤ 1 .

Proof. Immediate from Propositions 1-4 .

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