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A COMPARISON THEOREM FOR SEMIMARTINGALES

AND ITS APPLICATIONS

by YAN Jia-an

We work on a filtered probability space $(\Omega,\underline{\underline{F}},P\ ;\ (\underline{\underline{F}}_t))$ satisfying the usual conditions. Let X be a semimartingale such that $\Sigma_{0<s\leqq t}|\Delta X_t|<\infty$ for $t<\infty$ (as usual, we allow an evanescent exceptional set in our inequalities without mentioning it) : this is the class of semimartingales for which Yor (Astérisque 52-53 , Temps Locaux, p.23-35) has shown the existence of local times $L_t^a(X)$ continuous in t, and cadlag in a . On the other hand, X has a unique decomposition

$$X = X_0 + M + A$$

where M is a continuous local martingale, and A is of finite variation. We denote by A^c the continuous part of A .

LEMMA 1. Assume the following conditions

(i) $L^0(X)=0$ (ii) $\int_0^{\cdot} I_{\{X_{s-}>0\}}dA_s^c \leqq 0$ (iii) $\Delta X\leqq 0$.

Then we have $X\leqq 0$ on the set $\{X_0\leqq 0\}$.

<u>Proof</u>. We have from Tanaka-Meyer's formula

$$X_t^+ = X_0^+ + \Sigma_{0<s\leqq t} I_{\{X_{s-}\leqq 0\}}X_s^+ + \tfrac{1}{2}L_t^0 + \Sigma_{0<s\leqq t} I_{\{X_{s-}>0\}}X_s^- + \int_0^t I_{\{X_{s-}>0\}}dX_s$$

On $\{X_0\leqq 0\}$ the first term vanishes. The second one vanishes because of (iii) and the third one because of (i). Therefore on $\{X_0\leqq 0\}$

$$X_t^+ = \Sigma_{0<s\leqq t} I_{\{X_{s-}>0\}}(X_s^-+\Delta X_s) + \int_0^t I_{\{X_{s-}>0\}}(dM_s+dA_s^c)$$

We have $X^-+\Delta X = X^+-X_- \leqq 0$ on $\{X_->0\}$ by (iii) and $\int_0^t I_{\{X_{s-}>0\}}dA_s^c \leqq 0$ by (ii). Therefore

$$\int_0^t I_{\{X_{s-}>0\}}dM_s \geqq 0 \text{ on } \{X_0\leqq 0\}$$

Since this is a continuous local martingale starting from 0, it must be equal to 0, and from this we deduce $X_t^+\leqq 0$, and finally $X\leqq 0$.

We apply this lemma to a generalization of the comparison lemma given by Ikeda-Watanabe ([1], p.352). One might extract from the proof a slightly more general version of lemma 1, but we shall not give it explicitly.

THEOREM 1. Let X^1, X^2 be solutions of two stochastic differentials equations

$$X_t^i = X_0^i + \int_0^t \sigma(s,X_{s-}^i)dM_s + \int_0^t b^i(s,X_{s-}^i)dB_s + \int_0^t c^i(s,X_{s-}^i)dC_s \quad (i=1,2)$$

where M is a continuous local martingale, B is a continuous increasing process and C an increasing process (B and C adapted). We assume

- $\sigma(s,x)$ is Borel measurable, $|\sigma(s,x)-\sigma(s,y)| \leq \rho(|x-y|)$, where ρ is an increasing function on $\mathbb{R}_+$ such that $\int_{0+} \rho^{-2}(u)du = +\infty$.
- $b^i(s,x)$, $c^i(s,x)$ are continuous on $\mathbb{R}_+ \times \mathbb{R}$ given the product of the right topology on $\mathbb{R}_+$ and the ordinary topology on $\mathbb{R}$.
- $b^1(s,x) < b^2(s,x)$ and $c^1(s,x) < c^2(s,x)$
- $x < y \Rightarrow c^1(s,x) \leq c^2(s,y)$.

Then we have $X^1 \leq X^2$ on the set $\{X_0^1 \leq X_0^2\}$.

<u>Proof</u>. We may assume $X_0^1 \leq X_0^2$ everywhere. Consider the stopping time

$$T = \inf\{ t>0 : X_t^1 - X_t^2 > 0 \}$$

We assume $P\{T<\infty\}>0$ and derive a contradiction. First of all, we have $X_T^1 \geq X_T^2$ (on $\{T<\infty\}$), and $X_{T-}^1 \leq X_{T-}^2$ on $\{0<T<\infty\}$. We cannot have $X_{T-}^1 < X_{T-}^2$ on $\{0<T<\infty\}$, because $\Delta X_T^1 \leq \Delta X_T^2$ (last hypothesis) would then imply $X_T^1 < X_T^2$. Therefore $X_{T-}^1 = X_{T-}^2$ on $\{0<T<\infty\}$. On $T=0$, we have by convention $X_{T-}^i = X_T^i$, and it is clear that $X_T^1 = X_T^2$ on this set.

Let X be the semimartingale $(X^1-X^2)_{T+t}$ on $\{T<\infty\}$, relative to the family $(\underline{\underline{F}}_{T+t})$. From the above, we have $X_0=0$. X belongs to the class of semimartingales considered at the beginning, and we set $X=M+A$ as before. There is an interval $[0,U(\omega)[$ on which $\Delta X \leq 0$, $\int_0^{\cdot} I_{\{X_{s-}>0\}} dA_s^c \leq 0$, due to the third hypothesis, and the right continuity of $b^i(T+s,X_{T+s}^i)$, $c^i(T+s,X_{T+s}^i)$. Finally, the first hypothesis will imply, exactly as in LeGall's paper [2], that $L^0(X)=0$ (this is the key point of the proof).

Then we apply lemma 1, not to X, but to X stopped at $U-$, where

$$U = \inf\{t>0 : \Delta X_t > 0 \text{ or } \int_0^t I_{\{X_{s-}>0\}} dA_s^c > 0 \}$$

which is a.s. $>T$ due to the above : we deduce that $X \leq 0$ on $[0,U[$, which contradicts the definition of T .

REMARKS. 1) The first hypothesis can be weakened as

- $\sigma(s,x)$ is Borel measurable, and for any x there is a $\delta(x)>0$ such that

$$|\sigma(s,x)-\sigma(s,y)| \leq \rho(|x-y|) \text{ for } y \in [x-\delta(x), x+\delta(x)] .$$

In fact, if we set $V=\inf\{t>0 : |\sigma(t,X_{t-}^1)-\sigma(t,X_{t-}^2)| > \rho(|X_{t-}^1 - X_{t-}^2|)$ $L^{\circ}(X^{V-})=0$ and we may apply lemma 1 to $X^{(U\wedge V)-}$.

2) As we mentioned, the key point of the proof is to check $L^0(X)=0$, and we deduced this from our first hypothesis as in [2]. Similar conditions ensuring that $L^0(X)=0$ (see [2], Corollaire 1.2) will lead to the same conclusion $X^1 \leq X^2$.

Similarly, we can prove the following theorem.

THEOREM 2. Let X^i be solutions of the following stochastic differential equations

$$X^i_t = X^i_0 + \int_0^t \sigma(s,X^i_{s-})dW_s + \int_0^t b^i(s,X^i_{s-})ds + \int_0^t\int_{U_0} f(s,X^i_{s-},u)\hat{N}_p(ds,du)$$
$$+ \int_0^t\int_{U\setminus U_0} g^i(s,X^i_{s-},u)N_p(ds,du) .$$

Here (W_t) is a Wiener process, N_p is the counting measure of a quasi-left continuous point process p on a standard measurable space U, $U_0 \subset U$ is a measurable subset such that $E[N_p(t,U\setminus U_0)] < \infty$ for t finite, and $\hat{N}_p = N_p - \tilde{N}_p$ ($\sim$ denoting compensation as usual).

We may assert that $X^1 \leq X^2$ on $\{X^1_0 \leq X^2_0\}$ if the following hypotheses are satisfied :

- σ and b^i are as in the preceding theorem.
- f^i, g^i are measurable functions on $\mathbb{R}_+ \times \mathbb{R} \times U$, and for any fixed $u \in U$, $f^i(s,x,u)$ and $g^i(s,x,u)$ are continuous on $\mathbb{R}_+ \times \mathbb{R}$ in the same topology as in theorem 1.
- $(x \leq y) \Rightarrow (f^1(s,x,u) \leq f^2(s,y,u)$ and $g^1(s,x,u) \leq g^2(s,y,u)$.

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