SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

GREGORY J. MORROW
MARTIN L. SILVERSTEIN

Two parameter extension of an observation of Poincaré

Séminaire de probabilités (Strasbourg), tome 20 (1986), p. 396-418
<http://www.numdam.org/item?id=SPS_1986__20__396_0>

TWO PARAMETER EXTENSION OF AN OBSERVATION OF POINCARÉ

by

Gregory J. Morrow

and

Martin L. Silverstein

Summary

The infinite dimensional Ornstein-Uhlenbeck process is derived as the weak limit of processes $x^n(t,s)$ constructed from first hitting position of spheres $S^{n-1}(e^{t/2})$ for standard Brownian motion in $\mathbb{R}^n$ starting at the origin.

0. Introduction

Poincaré (1912) observed that relative to normalized uniform measure on the sphere $S^{n-1}(\sqrt{n})$ of radius $\sqrt{n}$, any fixed set of coordinate variables $y_1, \ldots, y_m$ converges in law as $n \uparrow \infty$ to independent standard normal variables. (See McKean (1973) for an interesting discussion of this from the "modern" point of view.) An equivalent statement is that on the unit sphere $S^{n-1}(1)$ with the random process $x^n(s)$, $0 \leqq s \leqq 1$ defined by

$$(0.1) \qquad x^n(s) = \begin{cases} 0 & \text{for} \quad s = 0 \\ y_1 + \ldots + y_k & \text{for} \quad s = k/n \end{cases}$$

and linear interpolation otherwise, the finite dimensional distributions converge to those of one dimensional Brownian motion starting at $s = 0$. (In fact it is not hard to establish weak convergence relative to the usual uniform topology.)

Our main result is that if the processes (0.1) for all spheres $S^{n-1}(r)$ are normalized by (division by) $\sqrt{r}$ and linked up by using the first hitting positions of n-dimensional Brownian motion starting at the origin and if $t = 1/2 \log r$ is used as the second time parameter, then again there is a limit, the infinite dimensional Ornstein-Uhlenbeck process of Malliavin (1976), Stroock (1981), Williams (1981) and Meyer (1981).

Our convergence result splits naturally into two parts. The first is

Theorem A. The finite dimensional distributions converge to those of the infinite dimensional Ornstein-Uhlenbeck process.

The proof depends on careful analysis of "Laplace's method", carried out in Section 2.

The second part of the result is weak convergence relative to the Skorohod topology on the set of càdlag function on any bounded interval with values in the usual space C of continuous functions. This is carefully formulated, stated as Theorem B and then proved in Section 3. We rely on martingale maximal inequalities and estimates of $f(h,n) = E|x^n(t+h,1) - x^n(t,1)|^p$ and again Laplace's method plays an important role. An interesting consequence of our calculations is that $\text{Lim sup}_{n\to\infty} f(h,n) = o(h)$ as $h \downarrow 0$ but $\text{Lim inf}_{h\to 0} \text{Sup}_n f(h,n) > 0$. This suggests that the weak convergence result is somewhat delicate.

1. Preliminaries

$B^n(u)$, $u \geq 0$ is standard n-dimensional Brownian motion starting at the origin. For $-\infty < t < +\infty$

$$T^n(t) = \inf\{u > 0 : |B^n(u)|^2 = e^t\}$$

$$y^n(t) = e^{-t/2} B^n(T^n(t))$$

$$y^n(t,j) = j^{th} \quad \text{component of} \quad y^n(t) \quad .$$

It is well known and easy to prove using rotational invariance of B^n , that the distribution of $y^n(t)$ is normalized uniform surface area on the unit sphere $S^{n-1}(1)$. Define $x^n(t,s)$ for $0 \leq s \leq 1$ as follows:

(1.1) $$x^n(t,0) = 0$$

$$x^n(t\,,\,k/n) = \sum_{j=1}^{k} y^n(t,j) \qquad 1 \leq k \leq n$$

and for $k/n < s < (k+1)/n$, interpolate:

$$x^n(t,s) = n((k+1)/n - s)x(t,k/n) + n(s - k/n)x^n(t,(k+1)/n) \tag{1.2}$$

The limiting infinite dimensional Ornstein-Uhlenbeck process $x(t,s)$ can be described as follows. For each t the one parameter process $x(t,s)$, $0 \leqq s \leqq 1$ is standard one dimensional Brownian motion starting at 0.

If $0 = s_0 < s_1 < \ldots < s_\ell = 1$ are fixed then the one parameter processes

$$x(t,s_j) - x(t,s_{j-1}) \qquad -\infty < t < \infty$$

for $j = 1, \ldots, \ell$ are independent stationary Gaussian diffusions and, in particular, Markovian. If $t_1 < t_2$ then the s-increment at time t_2 can be represented

$$\begin{aligned} x(t_2,s_j) - x(t_2,s_{j-1}) &= e^{-(t_2-t_1)/2}\{x(t_1,s_j) - x(t_1,s_{j-1})\} \\ &\quad + ((s_j - s_{j-1})(1 - e^{-(t_2-t_1)}))^{1/2} N_j \end{aligned} \tag{1.3}$$

with N_j a mean 0 variance 1 Gaussian variable independent of all $x(t,s)$ for $t \leqq t_1$. Because of the Markov property in t, this suffices to determine the finite dimensional distributions of x. In fact x has a version which is everywhere jointly continuous in t and s.

In Section 2 we will use the Fourier transforms of the finite dimensional distributions of the x^n. The corresponding Fourier transforms for x are easily calculated using (1.3) and independence of s-increments.

$$\begin{aligned} &E\exp\{\sum_{j=1}^{\ell} ia_j(x(t_1,s_j) - x(t_1,s_{j-1})) + ib_j(x(t_2,s_j) - x(t_2,s_{j-1}))\} \\ &= \exp\{-1/2 \sum_{j=1}^{\ell}(s_j - s_{j-1})(a_j^2 + b_j^2 + 2a_jb_je^{-(t_2-t_1)/2})\}. \end{aligned} \tag{1.4}$$

In the rest of this section we collect some formulae for integration on spheres. Good references are the first few pages of Chapter IX in Vilenkin (1968) and of Chapter 1 in Muller (1961). The symbols $d\sigma(\xi)$ or $d\sigma(\eta)$ will always denote normalized (total mass = 1) uniform surface area on $S^{n-1}(1)$. To show explicit dependence on two coordinates ξ_1, ξ_2

(1.5)
$$\int d\sigma_{n-1}(\xi) f(\xi_1, \xi_2, \tilde{\xi})$$

$$= \beta^{-1}(1/2, (n-1)/2)\beta^{-1}(1/2, (n-2)/2) \int_{-1}^{+1} dx (1-x^2)^{(n-3)/2} \int_{-1}^{+1} du (1-u^2)^{(n-4)/2}$$

$$\times \int d\sigma_{n-3}(\tilde{\xi}) \; f(x, \sqrt{1-x^2}\; u, \sqrt{1-x^2}\; \sqrt{1-u^2}\; \tilde{\xi}) \; .$$

Where $\tilde{\xi} = (\xi_3, \ldots, \xi_n)$ and subscripts $n-1$, and $n-3$ distinguish $d\sigma$ on $S^{n-1}(1)$ and $S^{n-3}(1)$. Also $\beta(p,q)$ denotes the Beta Function

$$\beta(p,q) = \int_0^1 dx\, x^{p-1} (1-x)^{q-1} = (p-1)!\,(q-1)!/(p+q-1)!$$

We take for granted the simpler formulae obtained from (1.5) by integrating out $\tilde{\xi}$ and/or u.

The joint distribtuion of $y^n(t_1)$, $y^n(t_2)$ involves the <u>spherical Poisson kernel</u> (p. 145 in Stein and Weiss (1971)) as follows:

(1.6)
$$E(F(y^n(t_1), y^n(t_2)))$$

$$= \iint d\sigma(\xi) d\sigma(\eta) (1-r^2)(1+r^2-2r\xi\cdot\eta)^{-n/2} F(\xi,\eta)$$

with $r = e^{-|t_2 - t_1|/2}$.

2. Convergence of Finite Dimensional Distributions

In this section we prove Theorem A. From Section 1, this amounts to verifying convergence of

$$(2.1)\qquad E\exp\{i\sum_{j=1}^{\ell} a_j(x^n(t_1,s_j)-x^n(t_1,s_{j-1}))+b_j(x^n(t_2,s_j)-x^n(t_2,s_{j-1}))\}$$

to the right hand side of (1.4).

We begin by replacing the x^n increments in (2.1) by uninterpolated ones. For n sufficiently large we can choose integers $0=i_0<i_1<\ldots<i_\ell=n$ so that $|(i_j/n)-s_j|<1/n$ and then define

$$W^n(t_i,j)=\sum_{i_{j-1}<p\leqq i_j} y^n(t_i,p) \quad .$$

To justify replacement we need only show that $M(t_i)=\max_{1\leqq\ell\leqq n}|y^n(t_i,\ell)|$ converges to 0 in probability. But this follows from

$$P(M(t_i)>a)\leqq nP(|y^n(t_i,1)|>a)$$

$$=n\beta^{-1}(1/2,(n-1)/2)\,2\int_a^1 dx(1-x^2)^{(n-3)/2}=0(n^{1/2}(1-a^2)^{n/2}) .$$

Here and below we use Stirling's approximation to estimate

$$(2.2)\qquad \beta(1/2,(n-k)/2)\sim(2\pi/n)^{1/2} \quad .$$

Next we reduce the problem to one coordinate at t_1 and two at t_2. The two sums

$$\sum_{j=1}^{\ell} a_jW^n(t_1,j) , \qquad \sum_{j=1}^{\ell} b_jW^n(t_2,j)$$

have the same joint distribution as

$$A_ny^n(t_1,1) , \quad B_ny^n(t_2,1)+C_ny^n(t_2,2)$$

where

(2.3) $$A_n = \Big(\sum_{j=1}^{\ell} a_j^2 (i_j^n - i_{j-1}^n)\Big)^{1/2}, \qquad B_n = \Big(\sum_{j=1}^{\ell} a_j b_j (i_j^n - i_{j-1}^n)\Big) / A_n$$

$$C_n = \Big(\sum_{j=1}^{\ell} b_j^2 (i_j^n - i_{j-1}^n) - B_n^2\Big)^{1/2} .$$

Clearly

(2.4) $$A_n/\sqrt{n}\ ,\ \ B_n/\sqrt{n}\ ,\ C_n/\sqrt{n} \to A, B, C$$

with the latter obtained by replacing in (2.3) each occurence of

$$i_j^n - i_{j-1}^n \quad \text{by} \quad s_j - s_{j-1} .$$

All this reduces our problem to showing that

(2.5) $$E \exp\{iA_n y^n(t_1,1) + iB_n y^n(t_2,1) + iC_n y^n(t_2,2)\}$$

converges to the right side of (1.4).

By (1.5) and (1.6) we can write (2.5) as a 6-fold integral consistent with the decomposition $\xi = (\xi_1, \xi_2, \tilde{\xi})$ and the corresponding one for η. The integrand depends on $\tilde{\xi}$ and $\tilde{\eta}$ only through the inner product $\tilde{\xi}\cdot\tilde{\eta}$. This allows us to make $\tilde{\eta}$ constant, do the corresponding integral and represent (2.5) as the following 5-fold integral:

(2.6)

$$\Delta_n \iiiint\!\!\int_{\{-1 \leqq x,y,u,v,z \leqq 1\}} dxdydudvdz\, \mu(x,y,u,v) \exp\{iA_n x + iB_n y + iC_n \sqrt{1-y^2}\ v\}$$

$$\times\ (1-z^2)^{(n-5)/2} (1 + r^2 - 2ra - 2rbz)^{-n/2}$$

where

$$r = e^{-(t_2 - t_1)/2}$$

$$\Delta_n = (1-r^2)\beta^{-2}(1/2\,,(n-1)/2)\beta^{-2}(1/2\,,(n-2)/2)\beta^{-1}(1/2\,,(n-3)/2)$$

$$\mu = (1 - x^2)^{(n-3)/2}(1 - y^2)^{(n-3)/2}(1 - u^2)^{(n-4)/2}(1 - v^2)^{(n-4)/2}$$

$$a = xy + \sqrt{1 - x^2}\ \sqrt{1 - y^2}\ uv$$

$$b = \sqrt{1 - x^2}\ \sqrt{1 - y^2}\ \sqrt{1 - u^2}\ \sqrt{1 - v^2}$$

The parameter z here represents $\tilde{\xi} \cdot \tilde{\eta}$ in the notation of (1.5). The first step in our asymptotic evaluation of (2.6) is to restrict the variables. In the rest of this section we fix $1/2 < r < 1$. Let $r^* > 0$ be determined by

$$1 - r^{*2} = (1 - r)^3 .$$

If any one of $|x|$, $|y|$, $|u|$, $|v|$, $|z| > r^*$ then

$$\mu \cdot (1 - z^2)^{(n-5)/2} \leqq (1 - r^{*2})^{(n-5)/2} = (1 - r)^{(3n-15)/2} .$$

By two applications of the Cauchy-Schwarz inequality, $|a| + |b| \leqq 1$ and so

$$(1 + r^2 - 2ra - 2rbz)^{-n/2} \leqq (1 - r)^{-n} .$$

Thus we can restrict the integration in (2.6) to $|x|$, $|y|$, $|u|$, $|v|$, $|z| \leqq r^*$ with an error $O(n^{5/2}(1 - r)^{n/2})$ and this $\to 0$ as $n \to \infty$. (By Stirling's approximation, $\Delta_n = O(n^{5/2})$.)

Keeping these restrictions in mind, we apply Laplace's method to the inner z integral

$$I_n = \int_{-r^*}^{r^*} dz\,(1 - z^2)^{(n-5)/2}(1 + r^2 - 2rz - 2rbz)^{-n/2} .$$

An appropriate Lagrangian is

$$L(z) = \log(1 - z^2) - \log(2r) - \log(Q - a - bz)$$

where we have introduced an additional notation

$$Q = (1 + r^2)/2r .$$

The relevant derivatives are

$$L'(z) = (1+z)^{-1} - (1-z)^{-1} + b(Q-a-bz)^{-1}$$

$$L''(z) = -(1+z)^{-2} - (1-z)^{-2} + b^2(Q-a-bz)^{-2}$$

$$L'''(z) = 2(1+z)^{-3} - 2(1-z)^{-3} + 2b^3(Q-a-bz)^{-3} .$$

The Euler equation $L'(z) = 0$ has unique solution in $[-1,1]$

$$z^* = (Q-a-R)/b$$

with one more new notation

$$R = ((Q-a)^2 - b^2)^{1/2} .$$

We can write

$$z^* = \sigma - (\sigma^2 - 1)^{1/2} \tag{2.7}$$

with $\sigma = (Q-a)/b$ from which it is clear that $z^* > 0$. To see that $z^* < r^*$ and therefore is in the range of the integral, we argue as follows. The inequality $a + b < 1$ implies

$$\sigma - 1 = (Q-a-b)/b \geqq Q - 1 = (1-r)^2/2r \geqq (1-r)^2/2$$

and so

$$1 - z^* = \sqrt{\sigma^2 - 1} - (\sigma - 1)$$

$$= 2\sqrt{\sigma - 1}\,(\sqrt{\sigma+1} + \sqrt{\sigma - 1})^{-1}$$

$$\geqq \sqrt{2}\,(1-r)(\sqrt{\sigma+1} + \sqrt{\sigma-1})^{-1}$$

In (2.7) z^* decreases to 0 as σ increases to $+\infty$. Thus $z^* < 2 - \sqrt{3}$ or else $\sigma \leqq 2$ and the last estimate yields $1 - z^* > \sqrt{2}\,(1+\sqrt{3})^{-1}(1-r)$. For $r > 1/2$ the first inequality implies the second and so

$$1 - z^* > \sqrt{2}\,(1 + \sqrt{3}\,)^{-1}(1 - r) > (1 - r)^3 \tag{2.8}$$

whence $z^* < r^*$.

A little algebra including

$$-(1+z)^{-2} - (1-z)^{-2} = -2(1+z^2)/(1-z^2)^2 \tag{2.9}$$

$$Q - a - bz^* = R, \quad 1 - z^{*2} = 2R(Q - a - R)/b^2, \quad 1 + z^{*2} = 2(Q - a)(Q - a - R)/b^2$$

yields

$$L''(z^*) = -2/(1 - z^{*2}) \tag{2.10}$$

and in particular

$$L''(z^*) \leqq -2 . \tag{2.11}$$

With the help of (2.8) we conclude that in the range

$$-1/2 < z < z^* + (1 - z^*)/2 \tag{2.12}$$

we can uniformly estimate

$$L'''(z) = O(1) . \tag{2.13}$$

Let $0 < \epsilon < 1$ arbitrary and choose $\alpha > 0$ such that $\alpha|L''(z)| < \epsilon$ in the range (2.12). For ϵ small, $|z - z^*| < \alpha$ implies (2.12) and therefore $|L''(z) - L''(z^*)| < \epsilon$. Also $L''(z) < -1$ and so $L(z^* \pm \alpha) \leqq L(z^*) - \alpha^2/2$. Since z^* is the only critical point, $L(z)$ is increasing for $z < z^*$ and decreasing for $z > z^*$ and so $L(z) \leqq L(z^*) - \alpha^2/2$ in the entire range $|z - z^*| \geqq \alpha$. All this allows us to expand $L(z) = L(z^*) + L''(\tilde{z})(z - z^*)^2/2$ with $\tilde{z}$ between z and z^* and obtain

$$I_n = e^{nL(z^*)/2} \int_{z^* - \alpha}^{z^* + \alpha} dz\,(1 - z^2)^{-5/2}\, e^{n\{L''(z^*) + \epsilon\theta(z)\}(z - z^*)^2/4} \tag{2.14}$$

$$+ O(e^{nL(z^*)/2 - \alpha^2/4}) .$$

<u>Convention. Here and below we use θ and φ to represent quantities which are undetermined except for the estimate $|\theta|$, $|\varphi| \leqq 1$. They may be constant or functions depending on the context. Also they vary from one occurrence to another - although sometimes we use prime ($'$) to emphasize the difference.</u>

From (2.14) we immediately get the preliminary estimate

$$I_n = e^{nL(z^*)/2} 0\{\int_{z^*-\alpha}^{z^*+\alpha} dz e^{-n(z-z^*)^2/4} + e^{-n\alpha^2/4}\} .$$

The integral converges to $(4\pi/n)^{1/2}$ and so

$$I_n = 0\left(n^{-1/2} e^{nL(z^*)/2}\right) . \tag{2.15}$$

Evaluating, we get

$$e^{L(z^*)} = \frac{(1 - z^{*2})}{2r(Q - a - bz^*)} = (Q - a - R)/b^2 r . \tag{2.16}$$

We write

$$(n/2\pi)^{1/2} I_n = K_n e^{-nL(z^*)/2}$$

and we will now show that $K_n \to K$. Note first that $J_n \leqq K_n \leqq L_n$ where, by (2.14)

$$J_n = (1 - (z^* - \alpha)^2)^{-5/2} (n/2\pi)^{1/2} \int_{z^*-\alpha}^{z^*+\alpha} dz e^{n\{L''(z^*) - \epsilon\}(z - z^*)^2/4} + o(1)$$

$$L_n = (1 - (z^* + \alpha)^2)^{-5/2} (n/2\pi)^{1/2} \int_{z^*-\alpha}^{z^*+\alpha} dz e^{n\{L''(z^*) + \epsilon\}(z - z^*)^2/4} + o(1) .$$

Substituting $w = \sqrt{n}\,(z - z^*)$ into both integrals, letting $n \to \infty$ and then ϵ (and therefore α) $\downarrow 0$, we see that

$$K_n \to (1 - z^*)^{-5/2} \sqrt{2}\, |L''(z^*)|^{-1/2} .$$

Thus (2.6) is asymptotically equivalent to

$$(2.17)\qquad (n/2\pi)\iiiint_{-r^* \leqq x,y,u,v \leqq r^*} dxdydudv(1-x^2)^{-3/2}(1-y^2)^{-3/2}$$

$$\times (1-u^2)^{-2}(1-v^2)^{-2}\exp\{iA_n x + iB_n y + iC_n \sqrt{1-y^2}\ v\}$$

$$\times (1-z^{*2})^{-2}(Q-a-\sqrt{(Q-a)^2-b^2})^{n/2}/r^{n/2}$$

where we have used (2.16) and the relation $\mu = b^n$ and have applied Stirling's approximation to Δ_n.

Our attention centers now on the expression which is raised to the power $n/2$ in (2.17). Write $b = 1-\beta$ and expand

$$(2.18)\qquad \{Q-a-\sqrt{(Q-a)^2-b^2})\}/r = (Q-a)/r$$

$$-(\sqrt{Q^2-1}/r)\{1-(2aQ-2\beta-a^2+\beta^2)/(Q^2-1)\}^{1/2}\ .$$

As above, let $0<\epsilon<1$ arbitrary and choose $\alpha>0$ such that $\beta<\alpha$ (and therefore $|a| \leqq \beta < \alpha$) guarantees validity of the expansion

$$\{1-(2aQ-2\beta-a^2+\beta^2)/(Q^2-1)\}^{1/2}$$

$$= 1+(1+\epsilon\theta)(Q^2-1)^{-1}(-aQ+\beta+a^2/2-\beta^2/2)\ .$$

Also α can be chosen so that $a^2<\epsilon a$ and $\beta^2<\epsilon\beta$ and with ϵ sufficiently small we can rewrite the expansion as

$$\{1-(2aQ-2\beta-a^2+\beta^2)/(Q^2-1)\}^{1/2} = 1+(Q^2-1)^{-1}(-aQ(1+\theta\epsilon)+\beta(1+\varphi\epsilon))\ .$$

Now (2.18) can be expressed

$$(2.19)\qquad 1-pa-q\beta$$

with two more notations

$$p = (1/r)(1-(1+\epsilon\theta)Q(Q^2-1)^{-1/2}),\quad q = (1+\varphi\epsilon)r^{-1}(Q^2-1)^{-1/2}\ .$$

(In establishing (2.19) we have used the relations $\sqrt{Q^2 - 1} = (1 - r^2)/2r$ and $Q - \sqrt{Q^2 - 1} = r$.) Notice that $\beta < \alpha$ implies $x^2 < 2\alpha$ since $1 - x^2/2 > \sqrt{1 - x^2} \geqq 1 - \beta > 1 - \alpha$. Of course the same is true for y, u, v and so for α sufficiently small

$$a = xy + uv - \theta \varepsilon uv$$

$$b = 1 - (1/2 + \varphi\varepsilon)(x^2 + y^2 + u^2 + v^2)$$

in the region $\beta < \alpha$.

The above arguments allow sharp approximation in the region $\beta < \alpha$. In order to "estimate away" the contribution from $\beta \geqq \alpha$, we show that

$$F(a, \beta) = Q - a - \{(Q - a)^2 - (1 - \beta)^2\}^{1/2}$$

has a strict maximum at $a = 0$, $\beta = 0$ in the relevant range $0 \leqq \beta < 1$ and $|a| \leqq \beta$. First, $\frac{\partial F}{\partial a} = -1 + (Q - a)((Q - a)^2 - (1 - \beta)^2)^{1/2} > 0$ since $Q > 1$ and $a + b < 1$. Thus $F(a, \beta) \leqq F(\beta, \beta)$. Also

$$\frac{d}{d\beta} F(\beta, \beta) = -1 + (Q - 1)(Q^2 - 1 - 2\beta)Q - 1))^{-1/2}$$

$$= -1 + (Q - 1)(Q^2 - 2Q + 1 + (2 - 2\beta)(Q - 1))^{-1/2}$$

$$= -1 + (1 + (2 - 2\beta)/(Q - 1))^{-1/2} < 0$$

uniformly since $\beta \leqq 1 - (1 - r^{*2})^2$. But $F(0, 0) = r$ and so

$$F(a, \beta)/r = (Q - a - \{(Q - a)^2 - b^2\}^{1/2})/r < 1 - \delta\beta .$$

for some $\delta > 0$. Now we proceed as for I_n and write (2.17) as

$$(2.20)\qquad (n/2\pi)^2(1-r^2)\iiiint_{\beta\leqq\alpha} dxdydudv\,(1-z^{*2})^{-2}$$

$$\times(1-x^2)^{-3/2}(1-y^2)^{-3/2}(1-u^2)^{-2}(1-v^2)^{-2}\exp(iA_nx+iB_ny+iC_n\sqrt{1-y^2}\,v)$$

$$\times(1-pa-q\beta)^{n/2}+0(1-\delta\alpha)^{n/2}$$

and from now on we can ignore the last (error) term. Finally, we replace x, y, u, v by $x/\sqrt{n}$, $y/\sqrt{n}$, $u/\sqrt{n}$, $v/\sqrt{n}$, noting that the set $\beta\leqq\alpha$ contains an open ball about the origin and therefore with the new variables will expand to the full space R^4 in the limit $n\to\infty$. The expression $(1-pa-q\beta)^{n/2}$ transforms into

$$(2.21)\qquad \{1-(p/n)(xy+uv-\theta\varepsilon uv)-(q/n)(1/2+\varphi\varepsilon)(x^2+y^2+u^2+v^2)\}^{n/2}$$

which converges to

$$(2.22)\qquad \exp\{-1/2\ q(1/2+\varphi\varepsilon)(x^2+y^2+u^2+v^2)-1/2\,p(xy+uv+\theta\varepsilon uv)\}$$

$$=\exp\{-(1-r^2)^{-1}(1/2+\varphi\varepsilon)(x^2+y^2+u^2+v^2)$$

$$+(1-r^2)^{-1}((r-\varepsilon\theta(1+r^2)/(1-r^2))(xy+uv+\theta'\varepsilon uv)\}\ .$$

The transform of $1-z^{*2}$ converges to

$$2\sqrt{Q^2-1}\,(Q-\sqrt{Q^2-1}\,)=(1-r^2)r^{-1}\{(1+r^2)/2r\}$$

$$=(1-r^2)\ ,$$

the transforms of $(1-x^2)$, $(1-y^2)$, $(1-u^2)$, $(1-v^2)$ converge to 1 and $\exp(iA_nx/\sqrt{n}+iB_ny/\sqrt{n}+iC_nv\sqrt{1-y^2/n}\,/\sqrt{n}\,)\to\exp(iAx+iBy+iCv)$ by (2.4).

Finally, (2.21) is dominated by (2.22) which is integrable for ϵ sufficiently small, allowing us to pass to the limit under the integral sign in (2.20). Thus after letting $n \to \infty$ and then $\epsilon \downarrow 0$, we get

(2.23)

$$(2\pi)^{-2}(1-r^2)^{-1}\iiiint dxdydudv \exp\{-1/2\ (1-r^2)^{-1}(x^2+y^2) - 2rxy + u^2 + v^2 - zruv) + i(Ax+By+Cv)\},$$

a Gaussian integral which is easily computed. Make an orthogonal change of variables to remove the cross terms:

$$x = (w-z)/\sqrt{2} \qquad y = (x+z)/\sqrt{2}$$

$$u = (p-q)/\sqrt{2} \qquad v = (p+q)/\sqrt{2}$$

and rewrite (2.23) as

$$(2.24) \qquad (2\pi)^{-2}(1-r^2)^{-1}\iiiint dwdzdpdq \exp\{-1/2\ (1-r^2)^{-1}\{w^2+z^2$$

$$-r(w^2-z^2)+p^2+q^2-r(p^2-q^2)\}+i(w(A+B)/\sqrt{2}+z(B-A)/\sqrt{2}+pC/\sqrt{2}+qC/\sqrt{2})\}.$$

The expression inside the exponential can be regrouped

$$-1/2\ (1+r)^{-1}(w^2 - i\sqrt{2}\ (1+r)(A+B)w - (1+r)^2(A+B)^2/2)$$

$$-1/4\ (1+r)(A+B)^2$$

$$-1/2\ (1-r)^{-1}(z^2 - i\sqrt{2}\ (1-r)(B-A)z - (1-r)^2(B-A)^2/2)$$

$$-1/4\ (1-r)(B-A)^2$$

$$-1/2\ (1+r)^{-1}(p^2 - i\sqrt{2}\ (1+r)Cp - (1+r)^2\ C^2/2) - 1/4\ (1+r)C^2$$

$$-1/2\ (1-r)^{-1}(q^2 - i\sqrt{2}\ (1-r)Cq - (1-r)^2\ C^2/2) - 1/4\ (1-r)C^2$$

and then (2.24) is a product of 4 one-dimensional integrals which are easily computed to give

$$\exp\{-1/4\ (1+r)(A+B)^2 - 1/4\ (1-r)(B-A)^2 - 1/4\ (1+r)C^2 - 1/4\ (1-r)C^2\}$$

$$= \exp\{-1/2\ (A^2+B^2+C^2) - rAB\}$$

$$= \exp\{-1/2 \sum_{j=1}^{\ell} (s_j - s_{j-1})(a_j^2 + b_j^2 + 2ra_jb_j)\},$$

in complete agreement with (1.4). Theorem A is now proved.

3. Weak Convergence

Let C be the collection of continuous functions w on $[0,1]$ with $w(0) = 0$. We equip C with the usual uniform norm, denoted $\|w\|$, making C into a Banach space. For $T > 0$ let $D(T)$ be the collection of C valued functions φ with $\varphi(t)$ defined for $-T \leqq t \leqq T$ and satisfying

Condition D.1. $\varphi(t)$ is right continuous in C for $0 \leqq t \leqq 1$.

Condition D.2. The left hand limits $\varphi(t)$ exist in C for $0 < t \leqq 1$.

Condition D.3. φ is left continuous in C at $t = 1$.

We equip $D(T)$ with a version of the Skorohod topology (see Skorohod (1965)) as follows. Let Λ be the collection of strictly increasing homeomorphisms λ from $[-T, T]$ onto itself such that $\|\lambda\| < \infty$ where

$$\|\lambda\| = \sup_{t \neq u} \left| \log\left\{ \frac{\lambda(t) - \lambda(u)}{t - u} \right\} \right| .$$

(Roughly speaking λ belongs to Λ if $\lambda(-T) = -T$, $\lambda(T) = T$, λ is strictly increasing and its slope is bounded away from 0 and ∞.) The metric d is defined on $D(T)$ by

$$d(\varphi, \psi) :=$$

$$\inf\{\epsilon > 0 : \lambda \in \Lambda \text{ exists with } \|\lambda\| \leqq \epsilon \text{ and } \sup_t \|\varphi(t) - \psi(\lambda(t))\| \leqq \epsilon\} .$$

We mentioned in Section 1 that the Ornstein-Uhlenbeck process $x(t,s)$ has a jointly continuous version and so for each $T>0$ determines a $D(T)$ valued random variable. The latter statement can be proved also for the approximating processes x^n, using properties such as "quasi left-continuity" of the original Brownian motion in R^n, even though the x^n are certainly not continuous in t. Of course it is lack of continity of x^n which forces us to deal with $D(T)$ instead of the simpler space of continuous C-valued functions.

We prove in this section,

Theorem B. *For each* $T>0$ *the processes* $x^n \to x$ *weakly in* $D(T)$.

Billingsley (1968) proves in Section 14, using the real line R in place of C, that $D(T)$ is a complete separable metric space. His arguments extend routinely to our situation and we take the extension for granted here. Thus Theorem B will follow from Theorem A if we can establish tightness of the distributions of the x^n. We also take for granted the following adaption of Theorem 15.5 in Billingsley (1968) which gives sufficient conditions for tightness. Thus we view Theorem B as proved once we establish the following two conditions.

Condition T1. *For each* $\alpha>0$ *there exists* $K\subset C$ *compact such that* $P(x^n(-T,\cdot)\in K)>1-\alpha$ *for all* n.

Condition T2. *For each* $\varepsilon>0$ *there exists* $h>0$ *and a positive integer* n_0 *such that*

$$P(\sup_{|t-u|<h} \|x^n(t,\cdot)-x^n(u,\cdot)\|>\varepsilon)<\varepsilon \quad \text{for} \quad n\geqq n_0 .$$

We begin with the easier Condition T.1. Let $h>0$ be such that h^{-1} is integer and consider only $n\geqq 4/h$. Choose integers $0=i_0<i_1<\dots<i_\ell=n$ such that

$$|(i_j/n)-jh|\leqq h/4 ,$$

for $j=0,\dots,\ell$. To simplify the writing below we put $s(j)=i_j/n$. Let

$$M^n(h) = \max_{|v-s| \leqq h/2} |x^n(-T,v) - x^n(-T,s)| \ .$$

If $|v-s| \leqq h/2$ then s and v belong to the same interval $[s(j), s(j+1)]$ for some j and so

$$M^n(h) \leqq 3 \max_{j=0}^{\ell-1} \max_{s(j) \leqq s \leqq s(j+1)} |x^n(-T,s) - x^n(-T,s(j))|$$

$$\leqq 3 \max_{j=0}^{\ell-1} \max_{i_j < k \leqq i_{j+1}} |x^n(-T,k/n) - x^n(-T,s(j))| \ ,$$

the second inequality following from the definition by interpolation from integer multiples of n^{-1} in Section 1. Let $\epsilon > 0$ arbitrary. For each j, $s(j+1) - s(j) \leqq r$ where $r \leqq 2nh$ and so

$$(3.1) \qquad P(\max_{i_j < k \leqq i_{j+1}} |x^n(-T,k/n) - x^n(-T,s(j))| > \epsilon)$$

$$\leqq P(\max_{1 \leqq i \leqq r} |x_1 + \ldots + x_i| > \epsilon) \ ,$$

where $x_1, \ldots, x_r$ are the coordinates of a random vector uniformly distributed on the unit sphere $S^{n-1}(1)$. Also the partial sums of these coordinates form a martingale sequence and so by Doob's submartingale inequality (3.1) is less than

$$\epsilon^{-4} E|x_1 + \ldots + x_r|^4 \ .$$

Putting all this together we get

$$(3.2) \qquad P(M^n(h) > \epsilon) \leqq 3\ell\epsilon^{-4} E|x_1 + \ldots + x_r|^4 \ .$$

where $\ell = h^{-1}$ and $r \leqq 2nh$. To estimate the right hand side, note first that $x_1 + \ldots + x_r$ has the same distribution as $\sqrt{r}\, x_1$, and using the formulae in Section 1,

$$E|x_1 + \ldots + x_r|^4 = r^2 \beta^{-1}(1/2, (n-1)/2) \int_{-1}^{1} dx (1-x^2)^{(n-3)/2} x^4 \ .$$

For $n \geqq 6$ certainly $(n-3)/2 \geqq n/4$ and replacing x by $x/\sqrt{n}$ gives an estimate

$$r^2\beta^{-1}(1/2\,,(n-1)/2)\,n^{-5/2}\int_{-\sqrt{n}}^{\sqrt{n}} dx\,(1-x^2/n)^{n/4}x^4 = O(r^2n^{-2}) = O(n^2)\ ,$$

after treating the integral as in Section 2. Thus (3.2) is replaced by

$$P(M^n(h) > \epsilon) \leqq Ah^2 \tag{3.3}$$

for $n \geqq n_0$ depending on h and $A > 0$ independent of h. Let $\alpha > 0$ arbitrary and for a sequence $\epsilon_m \downarrow 0$ choose h_m so that $Ah_m < \alpha 2^{-m}$ in (3.3). Since $x^n(-T,s)$ is always uniformly continuous in s we can shrink h_m to accomodate the finite number of $n < n_0$ and thus guarantee

$$P(M^n(h_m) > \epsilon_m) \leqq \alpha 2^{-m} \tag{3.4}$$

for all $n \geqq 1$. The set K of $w \in C$ satisfying for all m the condition $|w(s)-w(v)| < \epsilon_m$ whenever $|s-v| < h_m/2$ is compact in C by the Arzèla-Ascoli theorem and (3.4) implies $P(x^n(-T,\cdot) \in K) > 1-\alpha$ for all n. This establishes Condition T.1.

Certainly every component of the original process $B^n(T^n(t))$ is a martingale and so for $j = 1,\ldots,n$

$$e^{(u-t)/2}x^n(u,\,j/n) - x^n(t,\,j/n) \qquad t \leqq u \leqq t+h$$

is a martingale. Taking the maximum over j we conclude that

$$\|e^{(u-t)/2}\,x^n(u,\cdot) - x^n(t,\cdot)\|\ , \qquad t \leqq u \leqq t+h$$

is a submartingale in u and so we can estimate

$$P(\max_{t\leqq u\leqq t+h}\|e^{(u-t)/2}\,x^n(u,\cdot) - x^n(t,\cdot)\| > \epsilon) \tag{3.5}$$

$$\leqq \epsilon^{-m}E\|e^{h/2}\,x^n(t+h,\cdot) - x^n(t,\cdot)\|^m$$

for $m > 1$. Using the submartingale property along the lattice in s as above we estimate the right hand side by the same moment at $s = 1$ and replace (3.5) by

$$(3.6) \qquad P(\max_{t \leqq u \leqq t+h} \| e^{(u-t)/2} x^n(u,\cdot) - x^n(t,\cdot)\| > \epsilon)$$

$$\leqq A_m \epsilon^{-m} E\,|e^{h/2} x^n(t+h,1) - x^n(t,1)|^m$$

with A_m the constant occuring in the submartingale inequality for the m^{th} moment. All this suggests that the key to verifying Condition T.2 is the right kind of estimate for $E|x^n(t+h,1) - x^n(t,1)|^m$. Indeed the main result in this section is that for $h > 0$ sufficiently small

$$(3.7) \qquad E|x^n(t+h,1) - x^n(t,1)|^m \leqq Kh^2 \quad .$$

with $m,K > 0$ independent of h and for $n \geqq n_0$ depending on h . We turn now to the proof of this inequality.

The moment is represented as an integral and then estimated via "Laplace's method" as in Section 2, but now we will keep more careful track of the error terms. Again $x^n(t,1)$ and $x^n(t+h,1)$ have the same joint distribution as $\sqrt{n}\, y^n(t,1)$ and $\sqrt{n}\, y^n(t+h,1)$. Thus

$$(3.8) \qquad E|x^n(t,1) - x^n(t+h,1)|^m =$$

$$= \beta^{-2}(1/2, (n-1)/2)\beta^{-1}(1/2, (n-2)/2) n^{m/2}$$

$$\iiint dz\,dy\,dz\,\mu(x,y)|x-y|^m (1-z^2)^{(n-4)/2}(1+r^2-2ra-2rbz)^{-n/2}$$

where now

$$r = e^{-h/2}$$

$$\mu = (1-x^2)^{(n-3)/2}(1-y^2)^{(n-3)/2}(1-r^2)$$

$$a = xy$$

$$b = \sqrt{1-x^2}\,\sqrt{1-y^2} \quad .$$

The restriction to $|x|$, $|y|$, $|z| \leqq r^*$ leaves an error term

$$2^m(1 - r^{*2})^{(n-r)/2}(1 - r^2)(1 - r)^{-n}$$

$$\leqq 2^{m+1}(1 - r)^{n/2 - 6}$$

which can be handled.

We turn our attention now to the inner z integral I_n defined as in Section 2 but with $(1 - z^2)^{(n-4)/2}$ in place of $(1 - z^2)^{(n-5)/2}$. The main point is to get better control in (2.13). The argument there shows that

$$L(z) = \log(1 - z^2) - \log 2r - \log(Q - a - bz)$$

has a unique maximum $z^* = (Q - a - R)/b$ with Q and R defined as before. Also

$$L''(z^*) = -2/(1 - z^{*2}) \leqq -2$$

and in the range (2.12) we have

$$|L'''(z)| = |z(1 + z)^{-3} + 2(1 - z)^{-3} + 2b^3(Q - a - bz)^{-3}| \leqq 16 + 16(1 - z^*)^{-3} + 2(Q - 1)^{-3} .$$

The function $f(w) = (1 - w^2)/2$ is concave with $f(1) = 0$ and $f'(1) = -1$. Thus $f(w) \leqq 1 - w$ for all w and we can estimate

$$1 - z^* > 1 - r^* > (1 - r^{*2})/2 = (1 - r)^3/2 .$$

More simply,

$$Q - 1 = (1 - r)^2/2r \geqq (1 - r^2)$$

assuming as we can that $r > 1/2$ always. Thus for z in the range (2.12) we can sharpen (2.13) to

$$|L'''(z)| \leqq 200(1 - r)^{-9} . \tag{3.9}$$

Now let $\alpha = (1-r)^9/200$. Then $|z - z^*| < \alpha$ implies (2.12) and therefore (3.9) and finally $L''(z) \leqq L''(z^*) + 1 < -1$ and so

$$L(z) \leqq L(z^*) - (z - z^*)^2/2 \qquad \text{for} \quad |z - z^*| < \alpha$$

$$L(z) \leqq L(z^*) - \alpha^2/2 \qquad \text{for} \quad |z - z^*| \geqq \alpha$$

(since z^* is the only critical point). Then using the estimate $(1 - z^2) \geqq (1 - r^{*2}) \geqq (1-r)^3$ we obtain finally instead of (2.14),

$$I_n \leqq (1-r)^{-8} e^{nL(z^*)/2} \left\{ \int_{z^*-\alpha}^{z^*+\alpha} dz e^{-n(z-z^*)^2/4} + e^{-n\alpha^2/4} \right\} . \tag{3.10}$$

The integral is $0(n^{-1/2})$ and $\sqrt{n}\, e^{-n\alpha^2/4} \leqq (\sqrt{2}/\alpha) e^{-1/2}$ and since $\alpha^{-1} = 0((1-r)^{-9}$ we can replace (3.10) by

$$I_n = 0((1-r)^{-17} n^{-1/2} e^{nL(z^*)/2}) . \tag{3.11}$$

using our estimates so far, we get for $m > 1$

$$E|x^n(t,1) - x^n(t+h,1)|^m = 0(n^{m/2+1} (1-r^2) \times \tag{3.12}$$

$$\left\{ \iint_{|x|,\,|y| \leqq r^*} dxdy\mu(x,y) |x-y|^m (1-r)^{-17} e^{nL(z^*)/2} + 0((1-r)^{n/2-6} n^{1/2}) \right\}) .$$

Treating the integral in the same way that we treated (2.17), we get for $n \geqq n_0$ depending on r,

$$E|x^n(t,1) - x^n(t+h,1)|^m \tag{3.13}$$

$$= 0((1-r)^{-16} \iint dxdy \exp\{-1/2(1-r^2)^{-1}(x^2+y^2-2rxy)\} |x-y|^m$$

$$+ 0(n^{(m+3)/2} (1-r)^{n/2-5}) .$$

The integral can be estimated using an appropriate version of (1.3) or it can be calculated directly with the orthogonal change of variables

$$x = \frac{1}{\sqrt{2}} (z - w) \qquad y = \frac{1}{\sqrt{2}} (z + w)$$

to get

$$\iint dzdw \exp\{-1/2\ (1-r^2)^{-1} (z^2(1-r) + w^2(1+r))\}\ |w|^m 2^{m/2}$$

$$= \int dz \exp\{-1/2\ (1+r)^{-1} z^2\} \int dw \exp\{-1/2\ (1-r)^{-1} w^2\}\ |w|^m 2^{m/2}$$

$$= 0((1-r)^{m/2}), \quad r \to 1$$

and substitution into (3.13) gives (3.7).

Because of the exponential factors occurring in (3.6) we need the elementary result

(3.14) $$\sup_n E|x^n(t,1)|^m < \infty$$

a consequence of applying Stirling's approximation to the exact formula

$$E|x^n(t,1)|^m = \beta((m+1)/2, (n-1)/2)/\beta(1/2, (n-1)/2) .$$

Also arguments which are now well known and routine (see Garsia (1973)) allow us to replace (3.6) by the corresponding norm inequality (since $m > 1$) which together with (3.14) implies

(3.15) $$\sup_n E \max_{t \leqq u \leqq t+h} \|x(u,\cdot)\|^m < \infty .$$

By (3.15)

$$E \max_{t \leqq u \leqq t+h} \|x^n(u,\cdot) - x^n(t,\cdot)\|^m$$

$$\leqq E \max_{t \leqq u \leqq t+h} \|e^{(u-t)/2} x^n(u,\cdot) - x^n(t,\cdot)\|^m + 0((e^{h/2} - 1)^m)$$

and by (3.14) for $t = t+h$,

$$E\left|e^{h/2}x^n(t+h,1) - x^n(t,1)\right|^m$$

$$\leqq E\left|x^n(t+h,1) - x^n(t,1)\right|^m + 0((e^{h/2}-1)^m) \ .$$

These two estimates combine with (3.6) and (3.7) to give

$$P(\max_{t \leqq u \leqq t+h} \|x^n(u,\cdot) - x^n(t,\cdot)\| \geqq \epsilon) \leqq A\epsilon^{-m} \cdot h^2$$

for h sufficiently small. Now Condition T.2 is verified by partitioning the t axis and arguing as before for the s-axis. Theorem B is now proved.

References

1. Billingsley, P. (1968). Convergence of Probability Measures. Wiley, New York.

2. Garsia, A. (1973). Martingale Inequalities. W.A. Benjamin, Reading, Mass.

3. McKean, H.P. (1973). Geometry of Differential Space. Annals of Prob. 1, pp. 197-206.

4. Malliavin, P. (1976). Stochastic Calculus of Variations and Hypoelliptic Operators. Proc. Intern. Conf. on Stoch. Diff. Eqs. Kyoto 1976, pp. 195-263. New York, Wiley.

5. Meyer, P.A. (1981). Note sur les Processus d'Ornstein-Uhlenbeck. Sem. de Prob. XIV, pp. 95-132. Lect Notes in Math. 920, Springer-Verlag.

6. Müller, C. (1966). Spherical Harmonics. Lecture Notes in Mathematics 17, Springer-Verlag.

7. Poincarè, H. (1912). Calcul des Probabilitès. Gauthier-Villars, Paris.

8. Skorokhod, A.V. (1965). Studies in the Theory of Random Processes. Addison-Wesley. Reading, Mass.

9. Stein, E.M. and Weiss, G. (1971). Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press.

10. Stroock, D. (1981). The Malliavan Calculus and its Applications to 2nd Order Parabolic Differential Equations. Math. Systems Th. 13, 1981.

11. Vilenkin, N.J. (1968). Special Functions and the Theory of Group Representations. Translations of Mathematical Monographs, V. 22. American Mathematics Society.

12. Williams, D. (1981). Notes.

Washington University
St. Louis, Missouri 63130
U.S.A.