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REMARK ON THE CONDITIONAL GAUGE THEOREM

K. L. Chung

This is a sequel to my note "The gauge and conditional gauge theorem" in the last volume of this Séminaire (XIX, 1983/4). That note, being prepared *in extremis*, contains a misleading error of writing, as well as some trivial misprints. The serious corrections are as follows.

p. 502, l.7: add "for $x \in \bar{D}_2$" after "that".

p. 502,(16): delete "$\sup\limits_{x\in\bar{D}_2}$", "$\inf\limits_{x\in\bar{D}_2}$", and "$a_3$".

p. 502,(18): delete "a_3".

In fact, only the second inequality in (15) is needed with $x = x'$. (These obvious errors were overlooked by Falkner and Zhao, as well as the author. They were discovered when I lectured on the result in Beijing in May, 1985.)

The details of the Remark at the end of the cited note will now be supplied, with continued numbering of the displayed formulas and reference. M. Cranston informed me that the argument given below can be extended to obtain similar results for certain Lipschitz domains, by using more elaborate analysis of the domain.

We consider conditions for the validity of the conditional gauge theorem for a bounded Borel function q. In this case (5) reduces to the following:

$$(20) \qquad \lim_{m(C)\to 0} \sup_{\substack{x\in C \\ z\in\partial D}} E_z^x\{\tau_C\} = 0 .$$

It follows from my previous note that this is a sufficient condition. For $d = 2$, the result by Cranston and McConnell (see my simplified proof in [8]) of course implies (20), provided that the Poisson kernel function is replaced by the Martin kernel function. For $d \geq 3$, and a bounded C^1 domain D, we can prove (20) by using the following results communicated to me by Carlos Kenig.

Let $x_0 \in D$, and $H(D)$ be the class of (strictly) positive functions which are harmonic in D with $h(x_0) = 1$. Then there exists a constant $C_1(D,x_0)$ such that for all $h \in H(D)$ we have in D

$$G_D 1 \leq C_1 h \ , \tag{21}$$

where G_D is the Green's operator for D. Next, for each $\varepsilon \in (0,1)$ there exists a constant $C_2(D,\varepsilon)$ such that for all $(x,z) \in D \times \partial D$ we have

$$K(x,z) \leq C_2/|x-z|^{d-1+\varepsilon} . \tag{22}$$

The proof of (21) seems rather hard; that of (22) apparently follows from Widman's inequalities for $C^{1,\alpha}$ domains. Since $K(x_o,\cdot)$ is continuous on ∂D, we have $c = \inf_{z \in \partial D} K(x_o,z) > 0$. Hence we may apply (21) with $h = K(\cdot,z)/K(x_o,z)$ to see that it holds for all $K(.,z)$, $z \in \partial D$, provided we replace C_1 by C_1/c there. We shall do so without changing the notation.

<u>Lemma 3</u>. Let D be a bounded C^1 domain in R^d, $d \geq 3$. There exists a constant $C_o(D)$ such that

$$\sup_{\substack{x \in C \\ z \in \partial D}} E_z^x\{\tau_C\} \leq C_o(D) m(C)^{1/d(d+2)} \tag{23}$$

for every domain C such that $C \subset D$ and $\partial D \subset \partial C$.

<u>Proof</u>: Write h for $K(.,z)$. It is well known that

$$E_z^x\{\tau_C\} = \frac{1}{h(x)} G_C h(x) \ , \tag{24}$$

where G_C is the Green's operator for C. Proceeding as in [4], we have for any $s > 0$,

$$G_C h = \int_o^s P_t^C h \, dt + G_C(P_s^C h) \leq sh + G_C(P_s^C h) \ , \tag{25}$$

where $\{P_t^C\}$ is the semigroup for the (unconditioned) Brownian motion killed outside C. We know that

$$P_s^C h \leq \frac{1}{(2\pi s)^{d/2}} \int_C h(y) m(dy) . \tag{26}$$

Using (22) with $\varepsilon = 1/2$, we obtain for any $\delta > 0$,

$$\int_C h(y)m(dy) \leq C_2(D)\{\delta^{\frac{1}{2}} + \delta^{-d+\frac{1}{2}} m(C)\} \leq C_2(D)m(C)^{1/2d} , \tag{27}$$

by integrating over $C \cap B(z,\delta)$ and $C\backslash B(z,\delta)$ respectively, and then putting $\delta = m(C)^{1/d}$. Hence by (26) and (27),

$$P_s^C h \leq C_2(D)s^{-d/2}\, m(C)^{1/2d} . \tag{28}$$

It follows by (28) and (21), since $G_C 1 \leq G_D 1$ (which requires no smoothness of C), that

$$\begin{aligned} G_C(P_s^C h) &\leq C_2(D)s^{-d/2}\, m(C)^{1/2d}\, G_C 1 \\ &\leq C_2(D)s^{-d/2}\, m(C)^{1/2d}\, C_1(D)h , \end{aligned} \tag{29}$$

and consequently by (25)

$$\frac{1}{h} G_C h \leq s + C_3(D)s^{-d/2}\, m(C)^{1/2d} . \tag{30}$$

Choosing $s = m(C)^{1/d(d+2)}$, we see that the right member of (30) becomes that of (23). Hence (23) is proved in view of (24).

Needless to say, if we use a smaller ε than 1/2 in the step leading to (27), we get a sharper estimate. This does not matter here, though it may be interesting to determine the best estimate of this kind.

Reference

[8] Kai Lai Chung, The lifetime of conditional Brownian motion in the plane, Ann. Inst. Henri Poincaré, vol. 30, no. 4 (1984), 349-351. Errata: in the Theorem, read "only on d" for "only on D"; under (2): Y is an h-supermartingale; in (3) and (4): read "E_h^x" for "E^x" four times.
I am indebted to J. L. Doob for pointing out the last few obvious misprints.