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RICHARD F. BASS

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L_p INEQUALITIES FOR FUNCTIONALS OF BROWNIAN MOTION

Richard Bass

1. Introduction

Let M_t be a continuous martingale. Let $\langle M\rangle_t$ be the quadratic variation process, let $M_t^* = \sup_{s\leq t} |M_s|$, let $L(t,x)$ be local time at x, and let $L_t^* = \sup_x L(t,x)$. Barlow and Yor [3,4] showed that in addition to the well-known equivalence in L_p norm between M_t^* and $\langle M\rangle_T^{1/2}$, one also had equivalence in L_p norm between L_T^* and $\langle M\rangle_T^{1/2}$. That is, if $p \in (0,\infty)$, there exist constants c_p and C_p depending only on p such that if T is any stopping time,

$$(1.1) \qquad c_p\, E\langle M\rangle_T^{p/2} \leq EL_T^{*p} \leq C_p\, E\langle M\rangle_T^{p/2}.$$

Many other functionals of M have been found to be dominated in L_p norm by $\langle M\rangle_T^{1/2}$. These include various ratios of M^* and $\langle M\rangle^{1/2}$ [4,6,9]; moduli of continuity of M and $L(t,x)$ [4]; and number of upcrossings [1,2]. For example, if $U_t(a,a+\epsilon)$ is the number of upcrossings of the interval $[a,a+\epsilon]$ by M up to time t, and $V_t = \sup_\epsilon \sup_a \epsilon U_t(a,a+\epsilon)$, the main result of [2] is that

$$(1.2) \qquad EV_T^p \leq C_p\, E\langle M\rangle_T^{p/2},$$

C_p a constant depending only on p.

The main purpose of this paper is to give some quite general and easily verifiable conditions for increasing functionals and ratios of increasing functionals of Brownian motion to dominate or be dominated in L_p norm by $T^{1/2}$. We state our results for Brownian motion, but these translate immediately via a time change argument to results for arbitrary continuous martingales. The results on L^*, ratios of M^* and $\langle M\rangle^{1/2}$, moduli of continuity, and upcrossings mentioned above then become special cases of our general theorems. In particular, our proofs of Theorems 1 and 2 give a new and very simple demonstration of the main results of [3], while the proof of Theorem 4 gives a very simple demonstration of the result of [2].

As another application of Theorem 4, we also prove a new inequality. Let $N_t(a,\epsilon)$ be the number of excursions of Brownian motion at level a of length longer than ϵ that are completed by time t. Let $S_t = \sup_{\epsilon} \sup_{a} \epsilon^{1/2} N_t(a,\epsilon)$. We then show that there exists C_p depending on $p \in (0,\infty)$ such that

(1.3) $\quad ES_T^p \leq C_p\, E\, T^{p/2}.$

Section 2 contains the results on increasing continuous functionals of Brownian motion plus some examples, while Section 3 contains the results on ratios of increasing continuous functionals. To handle upcrossings, we also need to consider discontinuous functionals, and this is done in Section 4.

I would like to thank Marc Yor for suggesting this problem and for his continued interest.

2. Increasing functionals

Suppose (B_t, P^x, θ_t) is canonical Brownian motion. That is, $\Omega = C([0,\infty], \mathbb{R})$, the continuous functions from $[0,\infty)$ to $\mathbb{R}$, and $B_t(\omega) = \omega(t)$, the coordinate map. P^x is Wiener measure on Ω with $P^x(B_0 = x) = 1$. When $x = 0$, we will usually write just P. Denote the natural filtration by $\mathcal{F}_t$. Finally $\theta_t : \Omega \to \Omega$ are the translation operators defined by $(\theta_t(\omega))(s) = \omega(s+t)$.

Suppose Φ is increasing, continuous, $\Phi(0) = 0$, and of moderate growth:

(2.1) $$\sup_{\lambda>0} \frac{\Phi(a\lambda)}{\Phi(\lambda)} \le a^p \quad \text{for all } a > 2, \text{ for some } p \in (0,\infty).$$

The functions x^p, $p \in (0,\infty)$ obviously satisfy these hypotheses.

Suppose F_t is a continuous adapted nondecreasing functional of ω satisfying

(2.2) (i) (Uniform scaling near ∞) $\sup_{x,\lambda} P^x (F_{\lambda^2} > b\lambda) \to 0$ as $b \to \infty$;

(ii) (Subadditivity) There exists a constant K_1 such that for all s,t,

$$F_t - F_s \le K_1 F_{t-s} \circ \theta_s.$$

Suppose G_t is a nondecreasing adapted functional of ω satisfying

(2.3) (i) (Uniform scaling near 0) $\sup_{x,\lambda} P^x (G_{\lambda^2} < b\lambda) \to 0$ as $b \to 0$;

(ii) There exists a constant K_2 such that for all s,t,

$$G_{t-s} \circ \theta_s \le K_2 G_t .$$

Note we do not require G to be continuous. A consequence of (2.3) (i) is that $G_t > 0$, a.s. for $t > 0$.

Our first two results are the following:

Theorem 1. Suppose F satisfies (2.2). There exists a constant C_Φ such that if T is any stopping time, then

$$E\Phi(F_T) \le C_\Phi E\Phi(T^{1/2}).$$

Theorem 2. Suppose G satisfies (2.3). There exists a constant C_Φ such that if T is any stopping time, then

$$E\Phi(T^{1/2}) \le C_\Phi E\Phi(G_T).$$

Before proving Theorems 1 and 2, we give some examples. The first example is $M_t^* = \sup_{s\le t} |B_s - B_0|$. The P^x distribution of M_t^* does not depend on x, and by scaling, we get (2.2) (i) and (2.3) (i). The subadditivity (2.2) (ii) is just the triangle inequality. Since $M_{t-s}^* \circ \theta_s = \sup_{s\le r\le t} |B_r - B_s| \le 2M_t^*$, we have (2.3) (ii). Thus M^* satisfies both (2.2) and (2.3), and observing that $P(B_0=0) = 1$, we recover from Theorems 1 and 2 the well-known Burkholder-Davis-Gundy inequalities.

A more interesting example is $L_t^* = \sup_x L(t,x)$. Because of the supremum in x, the P^x distribution of L_t^* does not depend on x. By scaling and the well-known fact that $0 < L_1^* < \infty$, a.s., we get (2.2) (i) and (2.3) (i). Since $L(t,x)$ is an additive functional,

$$(2.4) \qquad L(t,x) = L(s,x) + L(t-s,x) \circ \theta_s.$$

Taking suprema over x leads to (2.2) (ii). Since by (2.4),

$$L(t-s,x) \circ \theta_s \le L(t,x),$$

taking suprema over x again gives (2.3) (ii). Thus L^* satisfies both (2.2) and (2.3).

Two other examples satisfying both (2.2) and (2.3) that can be treated similarly are

$$C_t^B = \left[\sup_{0\le r<s\le t} \frac{|B_s - B_r|}{|s - r|^{1/2-\epsilon}} \right]^{\epsilon/2}$$

and

$$C_t^L = \left[\sup_{a\ne b} \frac{|L(t,a) - L(t,b)|}{|a - b|^{1/2-\epsilon}} \right]^{\frac{2}{1+2\epsilon}}.$$

To show C_t^B is continuous, one needs to use the fact that

$$\limsup_{\delta\downarrow 0,\ |s-r|\le\delta,\ r,s\in[0,t]} \frac{|B_s - B_r|}{|t - s|^{1/2-\epsilon/2}} = 0, \text{ a.s.,}$$

with a similar comment for C_t^L.

We now prove Theorems 1 and 2.

Proof of Theorem 1. Trivially we may assume $T < \infty$, a.s. Let $\beta > 1$, $\delta < 1$, and let $U = \inf\{t: F_t > \lambda\}$. Using the strong Markov property of Brownian motion at U,

$$\begin{aligned}
P(F_T > \beta\lambda, T^{1/2} \le \delta\lambda) &\le P(F_T - F_U \ge (\beta-1)\lambda, T \le \delta^2\lambda^2, U < T) \\
&\le P(F_{U+\delta^2\lambda^2} - F_U \ge (\beta-1)\lambda, U < T) \\
&\le P(F_{\delta^2\lambda^2} \circ \theta_U \ge (\beta-1)\lambda/K_1, U < T) \\
&= E[P(F_{\delta^2\lambda^2} \circ \theta_U \ge (\beta-1)\lambda/K_1 \mid \mathbf{F}_U);\ U < T] \\
&= E[P^{B_U}(F_{\delta^2\lambda^2} \ge (\beta-1)\lambda/K_1);\ U < T] \\
&\le \sup_x P^x(F_{\delta^2\lambda^2} \ge (\beta-1)\lambda/K_1)\ P(U < T) \\
&\le \sup_{x,\lambda} P^x(F_{\lambda^2} > \frac{\beta-1}{2K_1\delta}\lambda)\ P(F_T > \lambda).
\end{aligned}$$

By taking δ sufficiently small, using (2.2) (i), and appealing to Lemma 7.1 of [5], the proof is complete. □

Proof of Theorem 2. Suppose $\beta > 1$, $\delta < 1$. Using the Markov property at the fixed time λ^2, we have

$$\begin{aligned}
P(T^{1/2} > \beta\lambda, G_T \le \delta\lambda) &\le P(T > \beta^2\lambda^2, G_{T-\lambda^2} \circ \theta_{\lambda^2} \le K_2\delta\lambda) \\
&\le P(T > \lambda^2, G_{\beta^2\lambda^2-\lambda^2} \circ \theta_{\lambda^2} \le K_2\delta\lambda) \\
&= E[P(G_{(\beta^2-1)\lambda^2} \circ \theta_{\lambda^2} \le K_2\delta\lambda \mid \mathbf{F}_{\lambda^2});\ T > \lambda^2] \\
&= E[P^{B_{\lambda^2}}(G_{(\beta^2-1)\lambda^2} \le K_2\delta\lambda);\ T > \lambda^2] \\
&\le \sup_{x,\lambda} P^x(G_{(\beta^2-1)\lambda^2} < 2K_2\delta\lambda)\ P(T^{1/2} > \lambda).
\end{aligned}$$

Again, take δ sufficiently small and use (2.3)(i) and [5, Lemma 7.1] to complete the proof. □

3. Ratios of functionals

Our result here is

Theorem 3 *Suppose* $\alpha > 0$. *Suppose* F *satisfies* (2.2), G *satisfies* (2.3), *and moreover* G *is a continuous functional of* ω. *Then there exists* C_Φ *such that if* T *is any strictly positive stopping time,*

$$E\Phi\left[\frac{F_T^{\alpha+1}}{G_T^\alpha}\right] \leq C_\Phi E\Phi(G_T).$$

We make the obvious remark that if G_t satisfies (2.2) as well as (2.3), we can replace G_T on the right side of the above equation by $T^{1/2}$.

Proof. We start with

$$P\left[\frac{F_T^{\alpha+1}}{G_T^\alpha} > \beta\lambda, G_T \leq \delta\lambda\right] = \sum_{n=0}^{\infty} P(F_T^{\alpha+1} > \beta\lambda G_T^\alpha, \delta 2^{-(n+1)}\lambda < G_T \leq \delta 2^{-n}\lambda)$$

$$\leq \sum_{n=0}^{\infty} p_n,$$

where

$$p_n = P(F_T > \beta'\zeta 2^{-n\gamma}\lambda, G_T \leq \delta 2^{-n}\lambda),$$
$$\gamma = \alpha/(\alpha+1),$$
$$\beta' = \beta^{1/(\alpha+1)},$$

and $\zeta = \delta^\gamma 2^{-\gamma}$.

Let

(3.1) $$U_n = \inf\{t: F_t > 2^{-n\gamma}\zeta\lambda\},$$
$$V_n = \inf\{t: G_t > 2K_2\delta 2^{-n}\lambda\},$$
and $$W_n = U_n + V_n \circ \theta_{U_n} = \inf\{t > U_n: G_{t-U_n} \circ \theta_{U_n} > 2K_2\delta 2^{-n}\lambda\}.$$

Observe that by (2.3)(ii) we have $W_n \geq T$ on the set $(U_n \leq T, G_T \leq \delta 2^{-n}\lambda)$.

Then by the strong Markov property at U_n,

$$(3.2)\qquad p_n \leq P(F_T - F_{U_n} \geq (\beta'-1)\zeta 2^{-n\gamma}\lambda, U_n < T, G_T \leq \delta 2^{-n}\lambda)$$
$$\leq P(F_{W_n} - F_{U_n} \geq (\beta'-1)\zeta 2^{-n\gamma}\lambda, U_n < T)$$
$$\leq P(F_{V_n} \circ \theta_{U_n} \geq K_1^{-1}(\beta'-1)\zeta 2^{-n\gamma}\lambda, U_n < T)$$
$$= E[P^{B_{U_n}}(F_{V_n} \geq K_1^{-1}(\beta'-1)\zeta 2^{-n\gamma}\lambda); U_n < T]$$

For any x, any $r > 0$,

$$(3.3)\qquad P^x(F_{V_n} \geq K_1^{-1}(\beta'-1)\zeta 2^{-n\gamma}\lambda) \leq cE^x F_{V_n}^r 2^{rn\gamma}\lambda^{-r},$$

where here and in the remainder of the proof c denotes a constant whose value is unimportant and may change from place to place and which depends on β, α, δ, and r, but not λ or n. Using Theorems 1 and 2 with P replaced by P^x, the right side of (3.3) is

$$\leq cE^x V_n^{r/2} 2^{rn\gamma}\lambda^{-r} \leq cE^x G_{V_r}^r 2^{rn\gamma}\lambda^{-r}.$$

Since $G_{V_r} \leq 2K_2\delta 2^{-n}\lambda$, we then have

$$p_r \leq c2^{rn(\gamma-1)}P(U_n < T)$$

$$\leq c2^{rn(\gamma-1)}P(F_T > \zeta 2^{-n\gamma}\lambda).$$

Hence

$$(3.4)\qquad P\left[\frac{F_T^{\alpha+1}}{\beta G_T{}^{\alpha}} > \lambda\right] \leq P\left[\frac{F_T^{\alpha+1}}{G_T{}^{\alpha}} > \beta\lambda, G_T \leq \delta\lambda\right] + P(G_T > \delta\lambda)$$
$$\leq c\sum_{n=0}^{\infty} 2^{rn(\gamma-1)}P\left[\frac{2^{n\gamma}F_T}{\zeta} > \lambda\right] + P\left[\frac{G_T}{\delta} > \lambda\right].$$

We integrate (3.4) against $d\Phi(\lambda)$ and use integration by parts to get

$$E\Phi\left[\frac{F_T^{\alpha+1}}{\beta G_T}\right] \leq c\sum_{n=0}^{\infty} 2^{rn(\gamma-1)}\, E\Phi\left[\frac{2^{n\gamma}F_T}{\zeta}\right] + E\Phi\left[\frac{G_T}{\delta}\right]$$
$$\leq c\sum_{n=0}^{\infty} 2^{rn(\gamma-1)+np\gamma}\zeta^{-p}E\Phi(F_T) + \delta^{-p}E\Phi(G_T).$$

Since $\gamma-1 < 0$, the infinite series will be summable provided we choose r larger than $p\gamma/(1-\gamma)$. Another application of Theorems 1 and 2 to handle $E\Phi(F_T)$ completes the proof. □

4. Discontinuous functionals

To handle the results on upcrossings of [2], we need to consider discontinuous functionals.

Suppose H_t is a nondecreasing adapted functional of ω satisfying

(4.1)(i) (Uniform scaling near ∞) $\sup_{x,\lambda} P^x(H_{\lambda^2} > b\lambda) \to 0$ as $b \to \infty$.

(ii) There exists a continuous adapted nondecreasing functional F satisfying (2.2) such that

(a) (Bounded jumps) $\sup_{s\leq t} |\Delta H_s| \leq F_t$ for all s,t;

(b) (Partial subadditivity) $H_t - H_s \leq K_3 H_{t-s} \circ \theta_s + F_t$ for all s,t.

For such H we have

Theorem 4 Suppose H satisfies (4.1). There exists a constant C_Φ such that if T is any stopping time, then

$$E\Phi(H_T) \leq C_\Phi E\Phi(T^{1/2})$$

and

Theorem 5 Suppose H satisfies (4.1). Suppose G satisfies (2.3) and moreover is a continuous functional of ω. Suppose $\alpha > 0$. Then there exists a constant C_Φ such that for any strictly positive stopping time T

$$E\Phi\left[\frac{H_T^{\alpha+1}}{G_T^{\alpha}}\right] \leq C_\Phi E\Phi(G_T).$$

Before proceeding to the proofs, let us look at some examples. First consider $V_t = \sup_{a,\epsilon} \epsilon U_t(a,a+\epsilon)$, where $U_t(a,a+\epsilon)$ is the number of upcrossings of the interval $[a,a+\epsilon]$ by time t. The P^x distribution of V_t is independent of x because of the supremum in a, and the scaling in λ follows easily from that of the Brownian motion. Provided we know $P(V_1 < \infty) = 1$ (which we will show shortly), we then have (4.1)(i).

Let $F_t = 2M_t^*$ and observe that we cannot have an upcrossing before time t of size larger than $2M_t^*$. This gives (4.1)(iia). It is not hard to see that

$$U_t(a,a+\epsilon) \leq U_s(a,a+\epsilon) + U_{t-s}(a,a+\epsilon) \circ \theta_s + 1_{(2M_t^*>\epsilon)}.$$

Multiplying by ϵ and taking suprema over a and ϵ gives (4.1)(iib).

It remains to show $P(V_1 < \infty) = 1$. Let $\tau_r = \inf\{t: L_t^* > r\}$ and let $T_r(x) = \inf\{t: L(t,x) > r\}$. Let $\epsilon_n = 2^{-n}$. Fix M and let

$$W_n = \sup\{\epsilon_n U_{\tau_r}(a,a+\epsilon_n): |a| \leq M,\ a/\epsilon_n \text{ an integer}\}.$$

Since $L(\tau_r,x) \leq r$, then $T_r(x) \geq \tau_r$, and so $U_{\tau_r}(a,a+\epsilon_n) \leq U_{T_r(a)}(a,a+\epsilon_n)$. If N is the number of excursions at level a whose maxima exceed $a+\epsilon_n$ by time $T_r(a)$, then $U_{T_r(a)}(a,a+\epsilon_n) \leq N + 1$. By Ito's theory of excursions, N is a Poisson random variable, and the parameter is $r/2\epsilon_n$ (see [8]). By standard estimates for the tail of the Poisson distribution, if $\beta > 3r$,

$$P(N > \beta/\epsilon_n) \leq \exp(-cr/\epsilon_n),$$

where c is a constant whose value is unimportant. From this follows

$$P(W_n > \beta+1) \leq 2M\epsilon_n^{-1}\exp(-cr/\epsilon_n).$$

This is summable in n, and by Borel-Cantelli, $P(W_n > \beta+1 \text{ i.o.}) = 0$. Each $W_n < \infty$, a.s. by the continuity of Brownian paths, and so we conclude that $W = \sup_n W_n < \infty$, a.s.

Given a and ϵ, we can find n and x such that $a \leq x \leq x+\epsilon_n \leq a + \epsilon$, x is an integer multiple of ϵ_n, and $\epsilon/8 \le \epsilon_n \le \epsilon$. So

$$\epsilon U_{T_r}(a,a+\epsilon) \le 8\epsilon_n U_{T_r}(x,x+\epsilon_n).$$

Hence

$$\sup_{|a|\le M/2,\epsilon} \epsilon U_{T_r}(a,a+\epsilon) \le 8W < \infty, \text{ a.s.}$$

Finally, M and r are arbitrary; that $V_1 < \infty$, a.s. follows easily.

For a second example, consider $S_t = \sup_{\epsilon,a} \epsilon^{1/2} N_t(a,\epsilon)$, where $N_t(a,\epsilon)$ is the number of excursions at level a whose length exceeds ϵ and which are completed by time t. Let $F_t = t^{1/2}$. It is trivial that F satisfies (2.2). It is impossible to have completed an excursion of length longer than ϵ by time t if $\epsilon > t$, and so (4.1)(iia) is immediate. The argument for (4.1)(iib) is similar to the one for V_t, and by scaling, we will have (4.1)(i) as soon as we know $S_1 < \infty$, a.s.

Since $N_t(a,\epsilon) \le t/\epsilon$ for $\epsilon \le t$ and $= 0$ for $\epsilon > t$, it suffices to show $\limsup_{\epsilon\downarrow 0} \sup_a \epsilon^{1/2} N_t(a,\epsilon) < \infty$, a.s. But this follows from a result of Perkins [7].

We now prove Theorem 4.

<u>Proof of Theorem 4</u> Let $\beta > 3$. Let $U = \inf\{t: H_t > \lambda\}$. By (4.1)(iia), $H_U \le \lambda + F_U$. Then

$$\begin{aligned}
P(H_T > \beta\lambda, T^{1/2} \le \delta\lambda) &\le P(H_T > \beta\lambda, T \le \delta^2\lambda^2, F_T \le \lambda) + P(F_T > \lambda) \\
&\le P(H_T - H_U > (\beta-2)\lambda, U < T, F_T \le \lambda, T \le \delta^2\lambda^2) + P(F_T > \lambda) \\
&\le P(H_{\delta^2\lambda^2} \circ \theta_U > (\beta-3)\lambda/K_3, U < T) + P(F_T > \lambda) \\
&= E[P^{B_U}(H_{\delta^2\lambda^2} > (\beta-3)\lambda/K_3); U < T] + P(F_T > \lambda) \\
&\le \epsilon(\delta,\beta)\, P(H_T > \lambda) + P(F_T > \lambda),
\end{aligned}$$

where $\epsilon(\delta,\beta) = \sup_{x,\lambda} P^x(H_{\delta^2\lambda^2} > (\beta-3)\lambda/K_3)$.

Next,

$$(4.2) \qquad P\left[\frac{H_T}{\beta} > \lambda\right] \leq P(H_T > \beta\lambda, T^{1/2} \leq \delta\lambda) + P(T^{1/2} > \delta\lambda)$$
$$\leq \epsilon(\delta,\beta)\, P(H_T > \lambda) + P(F_T > \lambda) + P\left[\frac{T^{1/2}}{\delta} > \lambda\right].$$

Suppose for the moment that H_T is bounded. Integrating from 0 to ∞ with respect to $d\Phi(\lambda)$,

$$E\Phi\left[\frac{H_T}{\beta}\right] \leq \epsilon(\delta,\beta)\, E\Phi(H_T) + E\Phi(F_T) + E\Phi\left[\frac{T^{1/2}}{\delta}\right]$$

and so

$$(4.3) \qquad E\Phi(H_T) \leq \beta^p E\Phi\left[\frac{H_T}{\beta}\right] \leq \beta^p \epsilon(\delta,\beta) E\Phi(H_T) + \beta^p E\Phi(F_T) + \beta^p \delta^{-p} E\Phi(T^{1/2}).$$

Choose δ sufficiently small so that $\beta^p\epsilon(\delta,\beta) < 1/2$. Subtracting $\beta^p\epsilon(\delta,\beta)E\Phi(H_T)$ from both sides of (4.3), multiplying by $[1 - \beta^p\epsilon(\delta,\beta)]^{-1}$, and using Theorem 1 completes the proof when H_T is bounded.

If H_T is not bounded, note that (4.2) holding for H_T implies (4.2) holds for $H_T \wedge N$, for all $N > 0$. Arguing as above, we get

$$E\Phi(H_T \wedge N) \leq C_\Phi E\Phi(T^{1/2}),$$

C_Φ indepedent of N. Now let $N \to \infty$. □

Since the proof of Theorem 5 is very similar to that of Theorem 3, we omit the proof.

Note: B. Davis (in this volume) has independently discovered a simple proof of the main result of [3], and also an extension to the case of stable processes.

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Department of Mathematics
University of Washington
Seattle WA 98195.