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A SIMPLE PROOF OF A THEOREM OF BLACKWELL & DUBINS ON THE MAXIMUM OF A UNIFORMLY INTEGRABLE MARTINGALE

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For a random variable X with a finite mean, let X^* be a Hardy & Littlewood (1930) maximal random variable associated with X, which can be defined as follows:
Set $H_X(u) = E\big(f(U) \mid U \geq u\big) = \frac{1}{1-u} \int_u^1 f(t)dt, \quad 0 \leq u < 1,$
where f is the (essentially unique) nondecreasing function on $(0,1)$ whose Lebesgue-distribution is the same as the distribution of X, and U is uniformly distributed on $(0,1)$. Then $X^* = H_X(U)$.

Observe: If $x <$ ess.sup X and x_0 is the unique y for which $E(X|X \geq y) \leq x \leq E(X|X > y)$, then

$$P(X^* \geq x) = \frac{E(X - x_0)^+}{x - x_0}\,. \tag{1}$$

Note also that $x_0 < x$ in (1).

A (uniformly integrable) martingale (in discrete or continuous* time) whose last element is X, will be called a *martingale terminating with* X. As usual, if $P(Y \geq x) \leq P(Z \geq x)$ for all x, say that Z *stochastically dominates* Y.
This note presents a simple proof of the following
Theorem. (Blackwell & Dubins (1963))

X^* stochastically dominates the maximum of every martingale terminating with X.

The original proof of Blackwell & Dubins uses Doob's maximal inequality for martingales. Proofs via embedding in Brownian motion were given by Azéma & Yor in [1(b)] and, more recently by E. Perkins in [7] and D. van der Vecht in [8]. The present proof points out that the theorem is essentially a statement about martingale-pairs rather than martingale-processes. To isolate the main idea, it is convenient to break up the argument into two propositions, the first of which being the contribution of this note.

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* Here and henceforth (in continuous time) a cadlag version of a martingale is assumed.

Proposition 1.

If Z stochastically dominates every Y which qualifies to be the first member of a martingale-pair, whose second member is X, then Z also stochastically dominates the supremum of every martingale which terminates with X.

Remark. For a Y to qualify as above, it is necessary and sufficient that $E|Y-y| \leq E|X-y|$ for all y, or equivalently, that $EY = EX$ and $E(Y-y)^+ \leq E(X-y)^+$ for all y. It is only the necessity of this condition (which follows from Jensen's inequality) which is needed for our proof. To see however that the condition is also sufficient, the reader may consult, for example, Chacon & Walsh (1976).

Proof: For the proof of Proposition 1 it merely takes to realize that the distribution of the supremum of a cadlag process is determined by the distribution (one for each level x) of the state of the process at the instance of its first entrance into $[x,\infty)$. Formally, given a martingale terminating with X, let M be its supremum (over time) and, for each fixed x, let τ_x be its first entrance time into $[x,\infty)$. Introducing the natural convention $\tau_x = \infty$ on $\{M < x\}$ and $X_{\tau_x} = X$ on $\{\tau_x = \infty\}$, one obtains (recall the cadlag assumption)

(2) $$\{M > x\} \subset \{X_{\tau_x} \geq x\} \subset \{M \geq x\} ,$$

so that

(3) $$P(M \geq x) = P(M > x) = P(X_{\tau_x} \geq x) ,$$

for all x with $P(M = x) = 0$.

However, (X_{τ_x}, X) is a martingale-pair, hence by assumption, X_{τ_x} is stochastically dominated by Z, i.e., $P(X_{\tau_x} \geq y) \leq P(Z \geq y)$ for all y, in particular for $y = x$, i.e.,

(4) $$P(X_{\tau_x} \geq x) \leq P(Z \geq x) ,$$

which combined with (3) yields

(5) $$P(M \geq x) \leq P(Z \geq x) .$$

Since in (5), x is any continuity point of the distribution of M, stochastic domination of M by Z is established. The proof of the Blackwell-Dubins Theorem will now be complete, provided we can show that X^* qualifies to play the role of Z in Proposition 1. That this is indeed so, is the content of

Proposition 2 (Meilijson & Nadas (1979))

If $E(Y-y)^+ \leq E(X-y)^+$ for all y, then Y is stochastically dominated by X^*.

Proof: For the sake of completeness we reproduce the short argument of Meilijson & Nadas. Fix $x <$ ess.sup X and for the x_0 related to x as in (1), let $\varphi(t) = (t-x_0)^+$. Recall that $x > x_0$ and note that φ is strictly increasing on $[x_0, \infty)$. Now, calculate as follows:

$$\begin{aligned} P(Y \geq x) &= P(\varphi(Y) \geq \varphi(x)) \leq \text{(by Chebyshev)} \quad \frac{E\varphi(Y)}{\varphi(x)} \\ &= \frac{E(Y-x_0)^+}{x-x_0} \leq \text{(by assumption)} \quad \frac{E(X-x_0)^+}{x-x_0} \\ &= \text{(by (1))} \quad P(X^* \geq x) \ . \end{aligned}$$

Remark. Dubins & Gilat (1978) as well as Azéma & Yor (1978) construct, for each X, a martingale terminating with X, whose supremum is distributed like X^*, thus demonstrating that the Blackwell & Dubins stochastic upper bound is in fact an *attainable* least upper bound.

References

[1] J. Azéma & M. Yor,

a. Une solution simple au problème de Skorokhod.

b. Le problème de Skorokhod: complements.

Sem. Prob. XIII, LN 721, Springer (1978).

[2] D. Blackwell & L.E. Dubins, A converse to the dominated convergence theorem, *Illinois J. Math.* 7 (1963), 508-514.

[3] R.V. Chacon & J.B. Walsh, One dimensional potential embedding, *Sem. Prob.* X LN 511, Springer (1976).

[4] L.E. Dubins & D. Gilat, On the distribution of maxima of martingales, *Proc. AMS* 68 (1978), 337-338.

[5] G.H. Hardy & J.E. Littlewood, A maximal theorem with function theoretic applications, *Acta Math.* 54 (1930), 81-116.

[6] I. Meilijson & A. Nadas, Convex majorization with an application to the length of critical paths, *J. Appl. Prob.* 16 (1970), 671-677.

[7] E. Perkins, The Careteli-Davis solution to the H^1-embedding problem and an optimal embedding in BM, in *Sem. on Stochastic Processes*, Birkhauser (1985).

[8] D.P. van der Vecht, Inequalities for stopped Brownian motion, CWI Tract No. 21, Amsterdam, Holland (1986).