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SIMULTANEOUS BOUNDARY HITTING FOR A TWO POINT REFLECTING BROWNIAN MOTION

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For B_t a two-dimensional Brownian motion let X_t^1 and X_t^2 be two strong solutions of the Skorohod equation on the unit disc $D = \{Z, |Z| \le 1\}$, $dX_t = dB_t - X_t d\varphi_t$, where φ_t is an increasing process which grows only when X_t is on $\{Z; |Z| = 1\}$. Their starting points $X_0^\ell = x^\ell$, $\ell = 1,2$, are assumed to be unequal. In these equations φ_t^ℓ is an increasing process which grows only when X_t^ℓ. The associated local times are denoted by φ_t^ℓ. The authors (1988) have previously shown that the distance $z_t = \|X_t^1 - X_t^2\| /_2$ is non decreasing, tends to zero exponentially fast and is non zero for finite times t. The first two facts were established in the Ph. D. thesis of Weerasinghe (1987).

In this note, we show the following

Theorem : The points X_t^1 and X_t^2 visit ∂D simultaneously, almost-surely.

In fact, there is a three dimensional diffusion process ξ_t and a one dimensional non polar set C for ξ such that $\xi \in C$ exactly when both X_t^1 and X_t^2 are on ∂D. Thus local time of ξ on C gives the "local time" of (X_t^1, X_t^2) on $\partial D \times \partial D$. The process ξ involves a change of coordinates which exploits the rotation invariance of D. Namely, set $m_t = (X_t^1 + X_t^2)/_2$ and $i_t = \dfrac{X_t^2 - X_t^1}{2z_t}$, $j_t = i_t^\perp$, $x_t = \langle m_t, j_t\rangle$, $y_t = \langle m_t, i_t\rangle$.

The following equations were derived in our previous paper *(1)*

$$(1) \qquad dx_t = dW_t^1 + \left(\frac{x_t|y_t|}{2z_t} - \frac{x_t}{2}\right) d\varphi_t$$

$$(2) \qquad dy_t = dW_t^2 - \frac{y_t}{2}\, d\varphi_t - \left(\frac{x_t^2}{2z_t} + \frac{z_t}{2}\right)(d\varphi_t^2 - d\varphi_t^1)$$

$$(3) \qquad dz_t = -\frac{1}{2}(z_t + |y_t|)\, d\varphi_t$$

where (W_t^1, W_t^2) is two-dimensional Brownian motion and $\varphi_t = \varphi_t^1 + \varphi_t^2$.

From *(3)* it follows that $z_t = e^{-\frac{1}{2}\varphi_t}\left[z_o - \int_0^t |y_s| e^{\frac{1}{2}\varphi_s}\, d\varphi_s\right]$ and now

$\inf\{t > 0 : Z_t = 0\} \equiv T = \infty$ a.s. implies $z_o \geq \int_0^\infty |y_s| e^{\frac{1}{2}\varphi_s}\, d\varphi_s$ and thus

(4) $$\int_0^\infty |y_s|^\alpha \, d\varphi_s < \infty, \text{ for all } \alpha \geq 1.$$

Notice that from *(1)*, *(2)* and *(3)* it follows that $\xi_t = (x_t, y_t, z_t)$ is a diffusion process on $\mathcal{E} = \{(x,y,z) : z \geq 0,\ x^2 + (|y| + z)^2 \leq 1\}$. Also both X_t^1 and $X_t^2 \in \partial D$ when and only when $|y_t| = 0$ and $\xi_t \in \partial\mathcal{E}$, i.e. $x_t^2 + z_t^2 = 1$. Thus showing X_t^1 and X_t^2 visit ∂D simultaneously is equivalent to showing ξ_t hits the crease $C = \{(x,y,z) \in \partial\mathcal{E}, y = 0\}$.

Proof : It has been inspired by a paper of Varadhan and Williams where the idea is to select the correct harmonic function and argue from looking at Itô's expansion.

For this it is convenient to introduce two dimensional polar coordinates in each region $\mathcal{E}(z) = \{(x,y,z) : x^2 + (|y| + z)^2 \leq 1\}$, z being fixed, centered at $(\sqrt{1-z^2}, 0, z)$. Namely, set

$$\eta_t = |x_t| + 1 - \sqrt{1-z_t^2}\ ,$$

$$r_t \cos\theta_t = 1-\eta_t\ ,$$

$$r_t \sin\theta_t = y_t\ .$$

Now define $S = \inf\{t > 0 : r_t = 0\}$ and for $t < S$ set

$$u_t = u(x_t, y_t, z_t) = r_t^{\frac{1}{2}} \cos \tfrac{1}{2}\theta_t.$$

The reason for the choice of $\frac{1}{2}$ will become apparent in the course of the proof. Observe that when $r_t = 0$, ξ_t hits the crease C with positive x values. The situation however is symmetric. With this in mind we proceed with the Itô expansion for u_t which is expedited if one notices that $u_t = \operatorname{Re} w_t^{\frac{1}{2}}$, where $w_t = (1-\eta_t) + iy_t)$, is a harmonic function of (η_t, y_t). thus

(5) $$du_t = \operatorname{Re}\left[dw_t^{\frac{1}{2}}\right]$$

Examining *(6)*, one dominant term is obviously $-\frac{1}{2} r_t^{-\frac{1}{2}} |\sin \frac{1}{2} \theta_t| \frac{x_t^2}{2z_t} d\varphi_t$.
First observe $\int_0^t r_s^{-\frac{1}{2}} |\sin \frac{1}{2} \theta_s| \frac{x_t^2}{2z_t} d\varphi_t \longrightarrow \infty$ as $t \to \infty$. Indeed, by *(8)*; on the support of $d\varphi_s$, $r_s^{-\frac{1}{2}} |\sin \frac{1}{2} \theta_s| \frac{x_s^2}{2z_s} \geq \frac{1}{10\sqrt{2}} \frac{x_s^2}{2z_s} \geq \frac{1}{20} x_s^2$ (we assume $z_s \leq \frac{1}{10}$ which is true for all s large enough). Now $\int_0^\infty (|y_s| + z_s)^2 d\varphi_s < \infty$ (this uses *(4)* and $z_t \leq e^{-\frac{1}{2}\varphi_t} z_o$) and since $x_t + (|y_t| + z_t)^2 = 1$ on the support of $d\varphi_t$ one has $\int_0^\infty x_s^2 d\varphi_s = \infty$. (This is where we have used $T = \inf\{t > 0 : z_t = 0\} = \infty$ a.s.). Consequently, $\int_0^\infty r_s^{-\frac{1}{2}} |\sin \frac{1}{2} \theta_s| \frac{x_s^2}{2z_s} d\varphi_s = \infty$ on the set $\{S = \infty\}$. There are two $d\varphi_t$ terms in *(6)* with positive coefficients. These are

(10) $$\frac{1}{2} \cos \frac{1}{2} \theta_t \, (1-z_t^2)^{-\frac{1}{2}} z_t(|y_t| + z_t)$$

and

(11) $$- \cos \frac{1}{2} \theta_t \left[\frac{x_t|y_t|}{2z_t} - \frac{x_t}{2}\right] \cdot 1_{\{|y_t|<z_t\}}.$$

We now show these are dominated by

(12) $$- \frac{1}{2} |\sin \frac{1}{2} \theta_t|(|y_t| + z_t) - \frac{1}{2} |\sin \frac{1}{2} \theta_t| \frac{x^2}{2z_t}.$$

Before proving this, we notice the second term in *(12)* has a factor of $\frac{1}{2}$ which does not appear in *(6)*. This leaves $\frac{1}{2}$ of the term in *(6)* and thus *(6)* becomes

$$du_t = dM_t - c_t \, d\varphi_t - \frac{1}{2} |\sin \frac{1}{2} \theta_t|\frac{x_t^2}{2z_t} - d_t d\ell_o(x)_t$$

with c and d non negative. Thus

$$u_t = u_o + M_t - \int_0^t c_s d\varphi_s - \int_0^t d_s d\ell_o(x)_s - \frac{1}{2} \int_0^t |\sin \frac{1}{2} \theta_s|\frac{x_s^2}{2z_s} d\varphi_s$$

with the last integral tending to minus infinity as $t \to \infty$.
This gives the desired contradiction once it is establihed *(12)* dominates *(10)* + *(11)*. For this we assume $z_t \leq \frac{1}{20}$ for all t sufficiently large and we argue for large t.
First *(10)* may be dominated by the first term in *(12)* since from *(9)* and

$$= \frac{1}{2}\,\mathrm{Re}\left[\omega_t^{-\frac{1}{2}}\,d\omega_t\right]$$

$$= \frac{1}{2}\,r_t^{-\frac{1}{2}}(-\cos\tfrac{1}{2}\theta_t\,d\eta_t + \sin\tfrac{1}{2}\theta_t\,dy_t)$$

where the second order terms vanish due to the harmonicity of u. Thus, using the fact that $y_t \sin\frac{1}{2}\theta_t > 0$ on the support of $d\varphi_t$ and that $y_t(d\varphi_t^2 - d\varphi_t^1) = |y_t|d\varphi_t$ we have from *(5)*

(6)
$$du_t = dM_t + \frac{1}{2}\,r_t^{-\frac{1}{2}}\left[-\cos\tfrac{1}{2}\theta_t\left(\frac{|x_t||y_t|}{2z_t} - \frac{|x_t|}{2}\right)\right.$$
$$+ \frac{1}{2}\cos\tfrac{1}{2}\theta_t\,(1-z_t^2)^{-\frac{1}{2}}\,z_t(z_t + |y_t|) - \frac{1}{2}\sin\tfrac{1}{2}\theta_t\,y_t$$
$$\left.- |\sin\tfrac{1}{2}\theta_t|\left[\frac{X_t^2}{2z_t} + \frac{z_t}{2}\right]\right]d\varphi_t$$
$$- \frac{1}{2}\,r_t^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta_t\,d\ell_o(x)_t\ ,$$

with M_t on L^2 local martingale and $\ell_o(x)_t$ the local time of x at 0 up to time t.

The assumption $\{S = \infty\}$ on a set of positive probability will lead to a contradiction. On $\{S = \infty\}$ the bounded variation terms in the Itô expansion of u_t will be shown to tend to minus infinity. Since u_t is bounded this forces the local martingale M_t to converge to plus infinity which is the contradiction since M_t is a time change of Brownian motion. To make this precise, we first observe that z_t may be assumed as small as desired by selecting t large. Note the bound $z_t \le e^{-\frac{1}{2}\varphi_t} z_o$ from *(3)* and the asymptotic $\lim_{t\to\infty} \frac{\varphi_t}{t} = 2$. Next, the local time term for x_t at 0 has a negative coefficient so it suffices to show the integral of the $d\varphi_t$ terms from 0 to t tends to minus infinity as t tends to infinity. To this end we note the following hold on the support of $d\varphi_t$:

(7) $$\frac{\pi}{8} - \varepsilon(z_t) \le \frac{1}{2}\theta_t \le \frac{\pi}{4}\ , \quad \text{with } \varepsilon(z_t) \downarrow 0,$$

so that both

(8) $$\frac{1}{5} \le |\sin\tfrac{1}{2}\theta_t|$$

(9) $$\frac{1}{5} \le |\tan\tfrac{1}{2}\theta_t| \quad \text{for } z \text{ small enough.}$$

using $z_t \le \frac{1}{20}$,

$$\frac{1}{2}\cos\frac{1}{2}\theta_t(1-z_t^2)^{-\frac{1}{2}} z_t(|y_t| + z_t) \le \frac{1}{2}|\sin\frac{1}{2}\theta_t|(|y_t| + z_t).$$

Next we dominate *(11)* by the second term in *(12)*.
When $|y_t| \le z_t$ on the support $d\varphi_t$ it follows that $x_t^2 = 1 - (|y_t| + z_t)^2 \ge 1 - 4z_t^2 \ge .9$. Also, by *(9)* $|\cot\frac{1}{2}\theta_t| \le 5$ again on the support of $d\varphi_t$ so $|x_t| \ge 2|\cot\frac{1}{2}\theta_t|z_t$ and thus $\frac{1}{2}|\sin\frac{1}{2}\theta_t|\frac{x_t^2}{2z_t} \ge \cos\frac{1}{2}\theta_t \frac{|x_t|}{2}$ on ths support of $d\varphi_t \cap \{|y_t| \le z_t\}$ as desired. This completes the proof.

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