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THE BRANCHING PROCESS IN A BROWNIAN EXCURSION

by

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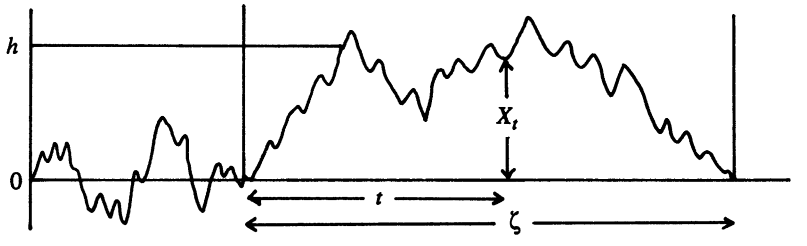
§1. Introduction.

This paper offers an alternative view of results in the preceding paper [NP] concerning the tree embedded in a Brownian excursion. Let BM denote Brownian motion on the line, BM_x a BM started at x. Fix $h > 0$, and let

$$X = (X_t, 0 \le t \le \zeta)$$

be governed by Itô's law for excursions of BM from zero conditioned to hit h. Here X may be presented as the portion of the path of a BM_0 after the last zero before the first hit of h, run till it returns to zero. See for instance [I], [R1].

Figure 1. Definition of X in terms of a BM_0.



For $x > 0$, let N_x^h be the number of excursions (or upcrossings) of X from x to $x+h$.

Theorem 1.1 *The process* $(N_x^h, x \ge 0)$ *is a continuous parameter birth and death process, starting from* $N_0^h = 1$, *with stationary transition intensities from* n *to* $n \pm 1$ *of* n/h.

In more detail, N_x^h is the number of branches at level x in a random tree, as in Figure 2, in which each branch dies at rate $1/h$, and splits at rate $1/h$, as it moves upward. The simple probabilistic structure of this tree is a key to the Markovian properties of Brownian local time. See [NP] and [LG] for further discussion, more careful definition of the tree, and references to earlier results in this vein.

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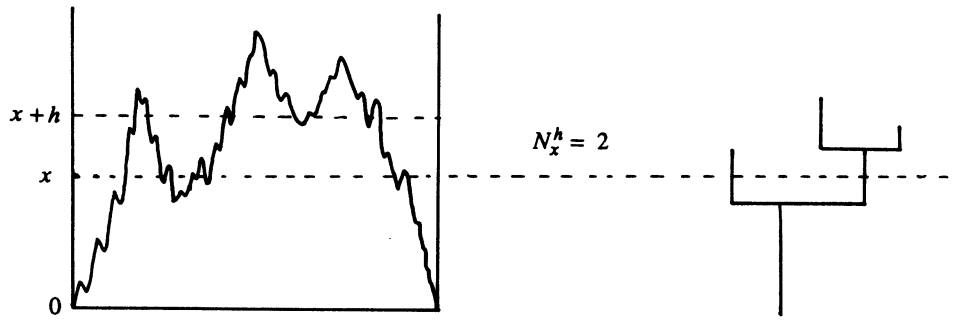
Figure 2. Tree associated with an excursion. Here the horizontal arms of the tree have no significance, except to order the vertical branches. Each vertical branch at level x corresponds to a different excursion from x to $x+h$.



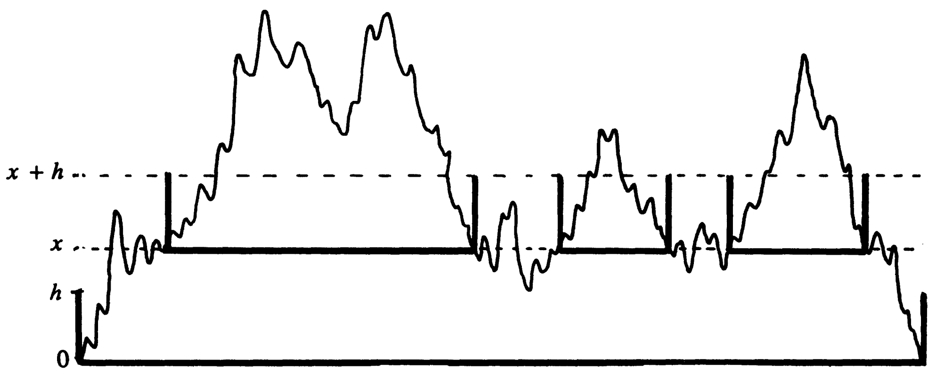
Section 2 of this paper sketches a proof of Theorem 1.1 and the Markovian property of the tree. This is very close to the development of Walsh [W], which provides all details of rigor for this section, as well as many interesting related results. Section 3 addresses the following question: At what moments on the original Brownian time scale does branching in the tree occur? This leads to a splitting of the Brownian excursion in the same vein as the path decompositions of Williams [W1] and [W2, Section 67, proved in R1]. See [LG] for a number of interesting variations of this theme for both random walks and Brownian motion.

§ 2. Sketch Proof of Theorem 1.1.

The number of downcrossings of the excursion X from $x + h$ to x is N_x^h. These downcrossings are all made after the time T_h when X first hits h. And after time T_h, the excursion X moves like a BM_h killed when it hits 0. Thus provided N_x^h is understood as a downcrossing count, Theorem 1.1 could just as well be formulated for X a BM_h with killing at 0 instead of a Brownian excursion from zero conditioned to reach h. Expressed this way, Theorem 1.1 is a corollary of calculations in [W, Sections 1 and 2]. But to understand the structure of the tree, it seems best to think in terms of complete excursions rather than upcrossings or downcrossings. So this section sketches the argument in terms of excursions.

Regard an excursion from x to $x+h$ as starting at x at the last hit of x before crossing up to $x+h$, and finishing at the first hit of x on the way down, as in Figure 3.

Figure 3. Here $N_x^h = 3$, and the 3 excursions from x to $x + h$ are shown in 3 small boxes inside the big box containing the original excursion X from 0 to h and back.



Notice that (no matter whether x is larger or smaller than h)

$$\begin{aligned} P(N_x^h = 0) &= P(X \textit{ never hits } x+h) \\ &= P(BM_h \textit{ hits } 0 \textit{ before } x+h) \\ &= \frac{x}{x+h} = q\text{, say.} \end{aligned}$$

And given that the excursion X reaches $x+h$, the excursion is certain to drop back to x. So given $N_x^h > 0$, there are a geometric (p) number of excursions from x to $x+h$ before hitting 0, where

$$\begin{aligned} p &= P(BM_x \textit{ hits } 0 \textit{ before } x+h) \\ &= \frac{h}{x+h} = 1-q. \end{aligned}$$

Thus

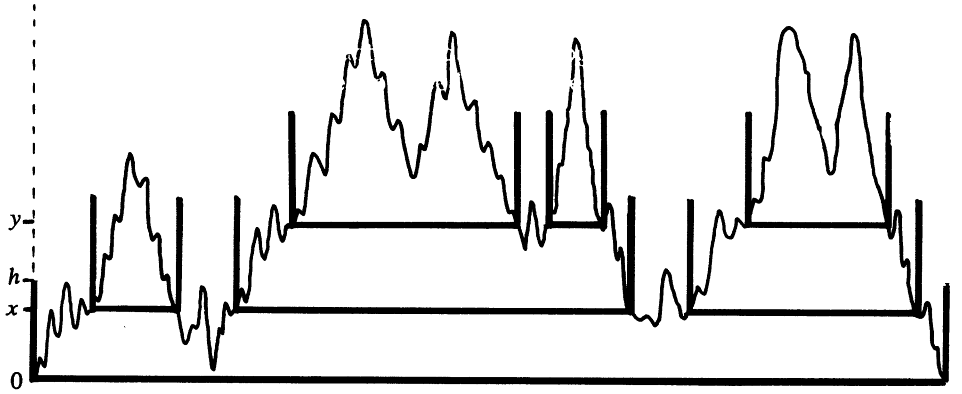
$$N_x^h = \begin{cases} 0 \textit{ with probability } q = \dfrac{x}{x+h} = \dfrac{x/h}{x/h+1} \\ n \textit{ with probability } p \cdot q^{n-1}p,\ n \geq 1. \end{cases} \tag{2.1}$$

According to Feller [F, Problem XVII.10.11], this is the distribution at "time" x of the number of individuals in a simple birth and death process as in Theorem 1.1. Here "time" means level, and the individuals are excursions. Given $N_x^h = n$, there are n excursions from x to $x+h$. When these excursions are shifted in space and time to start at (0, 0), they are independent with the same law as the original excursion X (conditioned to reach h), by straightforward calculations. Now the crucial observation is the following:

Thus these excursions $y \to y+h$ can be found inside the boxes of $x \to x+h$ excursions, and, after shifting the $x \to x+h$ boxes back to zero, using the same recipe as if looking for excursions from $y-x$ to $y-x+h$ in the original box. This idea of "excursions within excursions", illustrated in the next diagram by boxes within boxes, gives the basic homogeneous branching property, that for $x<y$

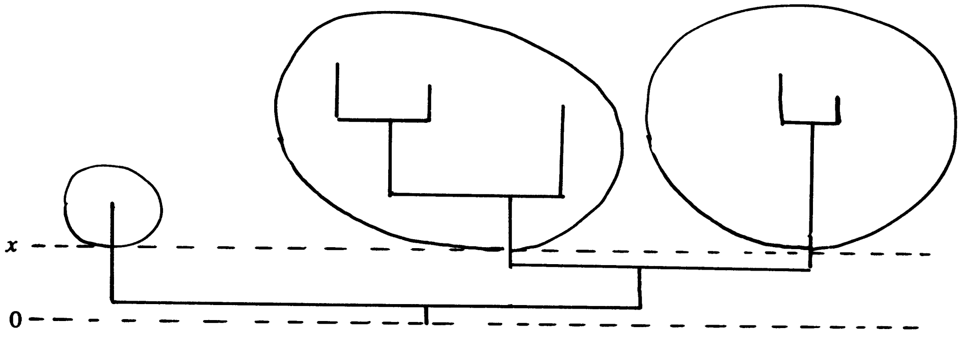
(2.2) $$N_y^h = \text{sum of } N_x^h \text{ independent copies of } N_{y-x}^h.$$

Figure 4. Illustration for (2.2). Here $N_x^h = 3$. These 3 excursions from x to $x+h$ contribute 0, 2, and 1 excursions respectively from y to $y+h$.



Formulae (2.1) and (2.2) show that the joint distribution of N_x^h and N_y^h for any $x<y$ is as asserted by Theorem 1.1. Pushed harder, an argument along these lines gives the homogeneous branching property of the tree, and implies that $(N_x^h, x \geq 0)$ is a homogeneous Markov branching process, as identified in Theorem 1.1.

Figure 5. Reproductive property of the tree. Each branch of the tree above x, as circled below is an independent copy of the whole tree.



§ 3. The Path Decomposition at the Lowest Branch.

Here is another way to characterise the law of the Markovian random tree appearing here:

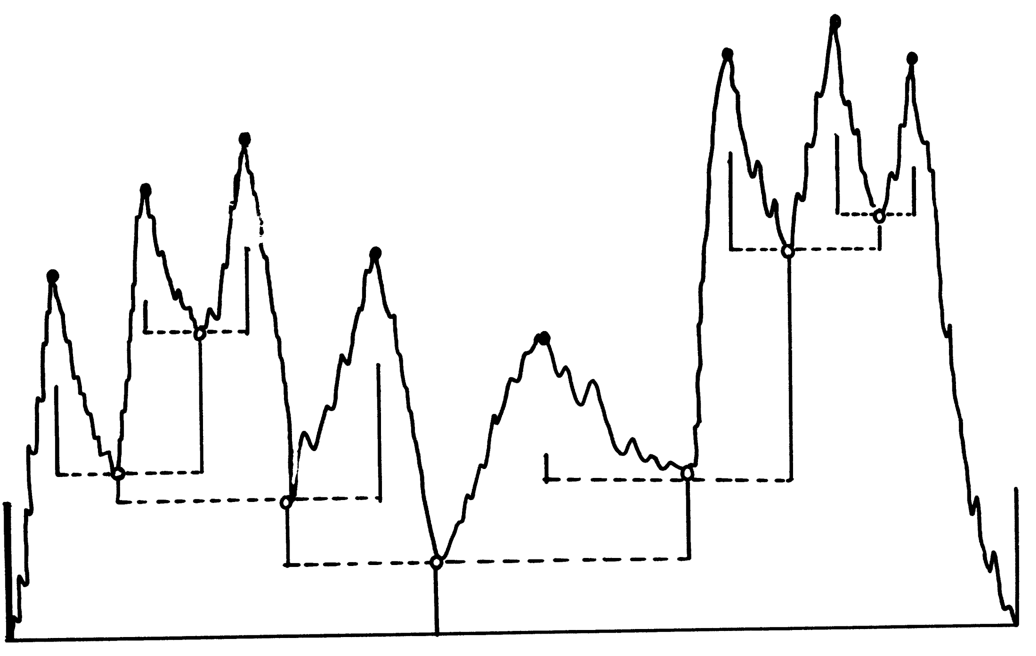
(i) the trunk has height α with exponential $(2/h)$ distribution.

(ii) the tree either dies at height α, or branches in two at height α, with equal probabilities, independently of α.

(iii) given that the tree branches at α, the two parts of the tree above α are independent copies of the original tree.

These statements can be understood directly in terms of the Brownian excursion X. To do so, it helps to associate a point on the timescale of X with each "subexcursion" represented in the tree. Strictly speaking, these timepoints are not part of the structure of the Markovian tree, rather a convenient way to arrange its branches. But one fairly natural assignment of times to branches of the tree is illustrated by Figure 6, then discussed in more detail.

Figure 6. Growing the tree beneath an excursion. Points marked o are h-minima, points marked ● are h-maxima, as defined below, following [NP]. Beneath these points along the path are vertical intervals representing branches of the tree. The number of these intervals passing through level x is N_x^h. Horizontal dotted lines indicate geneology of the excursions. Each split occurs at a point on the path that is an h-minimum. Each death of a branch occurs at the time of an h-maximum, at level h below the path.



Let *hmax* denote the set of times of *h–maxima,* meaning local maxima in the path of X that are higher than h, *hmin* the set of times of *h–minima,* local minima that are deeper than h:

$$hmax = \{t : \textit{there exist } s < t \text{ and } u > t \textit{ with } X_s = X_u = X_t - h \textit{ and } X_v \le X_t \text{ for } v \in [s,u]\}$$

$$= \{hmax_1, hmax_2, \cdots, hmax_K\} \textit{ say};$$

$$hmin = \{t : \textit{there exist } s < t \text{ and } u > t \textit{ with } X_s = X_u = X_t + h \text{ and } X_v \ge X_t \text{ for } v \in [s,u]\}$$

$$= \{hmin_1, hmin_2, \cdots, hmin_{K-1}\} \textit{ say},$$

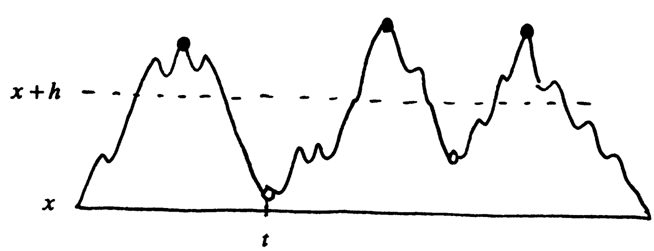
where $K = \#\{hmax\} \ge 1$. If $K = 1$ the set *hmin* is empty, and say X is *h–sterile.* If $K \ge 1$, say X is *h–fertile.* The times in *hmax* and *hmin* are assumed to be in increasing order, and they must alternate:

$$hmax_1 < hmin_1 < hmax_2 < \cdots hmin_{K-1} < hmax_K .$$

Consider now for $x > 0$ an excursion from x to $x+h$ and back. Associate with this excursion a single time t during the excursion, as follows:

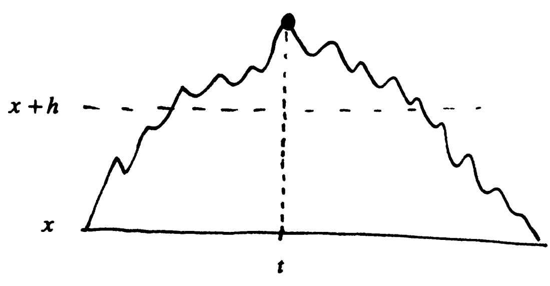
(i) If the excursion contains one or more points in *hmin*, let t be the time at which the trajectory is lowest over all these times in *hmin*:

Figure 7. Illustration of Case (i).



(ii) If the excursion contains no points in *hmin*, let t be the time of its maximum, which is the unique point in *hmax*:

Figure 8. Illustration of Case (ii).



For each fixed h, the set of all points (t, x) in the plane associated in this way with some excursion from x to $x+h$ is then a finite union of vertical intervals, one interval of levels x above each

In particular, the height α of the trunk of the tree is identified as

$$\alpha = \begin{cases} X(hmax_1) - h & \textit{if excursion } X \textit{ is } h\textit{-sterile}, \\ X(hmin_*) & \textit{if excursion } X \textit{ is } h\textit{-fertile}, \end{cases}$$

where $hmin_*$ attains the minimum of X over $hmin$. The basic properties (i), (ii), (iii) of the tree now follow from the following:

Theorem 3.1. (Decomposition of Brownian excursion at its deepest h-minimum). *Let X be a Brownian excursion conditioned to reach $h > 0$, let $\alpha_0 = X(hmax_1) - h$, α as above. Then*

(i) α_0 *has exponential distribution with rate* $1/h$, *and* $(\alpha_0, \alpha) \overset{d}{=} (\alpha_0, \alpha_0 \wedge \alpha_2)$ *where* α_2 *is independent of* α_0 *with the same exponential distribution.*

(ii) *The event* $(\alpha_0 > \alpha)$, *which is the event that X has a minimum deeper than h, is independent of* α, *with probability*

$$P(\alpha_0 > \alpha) = P(\alpha_0 > \alpha_0 \wedge \alpha_2) = P(\alpha_0 > \alpha_2) = \tfrac{1}{2},$$

(iii) *Conditional on this event, let* $\rho_\alpha = hmin_*$ *denote the time at which the deepest minimum is attained, so* $X(\rho_\alpha) = \alpha$. *Let*

$$\sigma_\alpha = \sup\{t < \rho_\alpha : X_t = \alpha\}; \qquad \tau_\alpha = \inf\{t > \rho_\alpha : X_t = \alpha\}.$$

Then $(X(\sigma_\alpha + s) - \alpha,\ 0 \le s \le \rho_\alpha - \sigma_\alpha)$ *and* $(X(\rho_\alpha + u) - \alpha,\ 0 \le u \le \tau_\alpha - \rho_\alpha)$

are two independent copies of X, independent also of α.

Figure 9. Illustration for Theorem 3.1 in case X has an h-minimum.

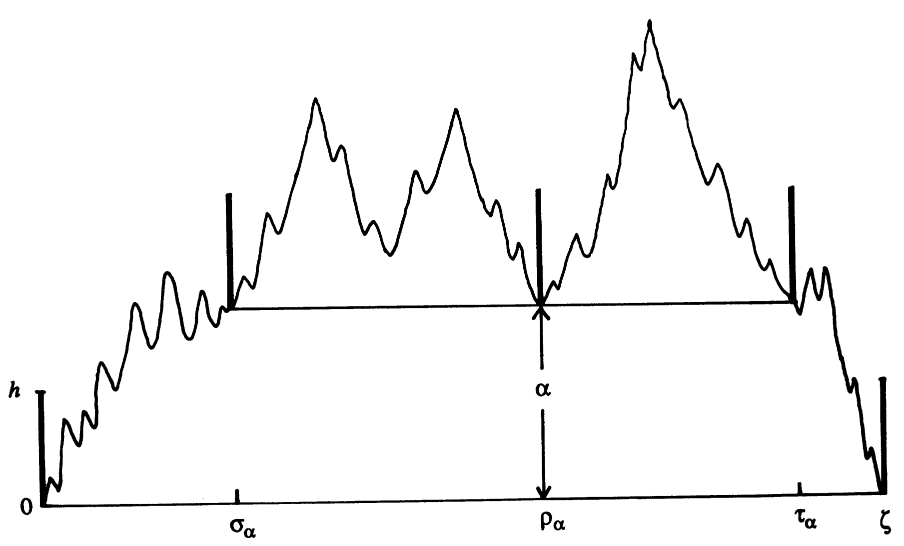
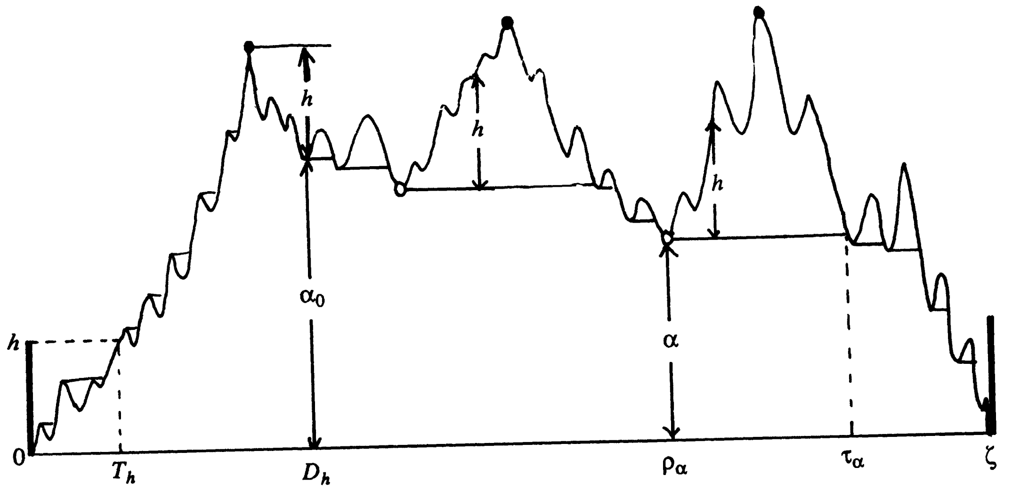


Figure 10. Illustration of definitions for the proof.



Proof. Let $T_h = \inf\{t : X_t = h\}$; $D_h = \inf\{t : \bar{X}_t - X_t = h\}$ where $\bar{X}_t = \sup_{s \le t} X_s$. For a random time $T \le \zeta$, let $X^{(T)}$ be the process $X^{(T)}(s) = X(T+s)$ $0 \le s \le \zeta - T$. Because $X^{(T_h)}$ is a BM^h run till it hits 0, and $D_h - T_h$ is the first time this process drops h below its previous maximum, it is easy to see that

$$\alpha_0 = X(D_h) = h max_1 - h$$

has exponential $(1/h)$ distribution. Now $X^{(D_h)}$ is a BM starting with this distribution and run till it hits 0. Look at excursions of this process above its previous minimum. The original excursion X has no h minimum if none of these excursions above the minimum process after time D_h rises more than h above its starting point. Then $\alpha = \alpha_0$ by definition. Otherwise, the deepest min of X is deeper than h, at some level $\alpha < \alpha_0$. Then the last excursion to rise by at least h from its starting point on the minimum process starts at level α at time ρ_α, and ends at time τ_α, as shown in Figure 10. When indexed by their starting level $0 \le x \le \alpha_0$, these excursions rising by at least h appear as a Poisson point process governed by the Brownian excursion law for positive excursions, stopped at an independent level α_0. The rate of excursions which rise by h is $1/h$, so

$$\alpha = \alpha_0 \wedge \inf\{x : x < \alpha_0 \text{ and } N^*(x) = 1, \}$$

where $(N^*(x), 0 \le x \le \alpha_0)$ is a Poisson process with rate $1/h$ killed at an independent exponential $(1/h)$ random time α_0. That is to say

$$(\alpha_0, \alpha) \overset{d}{=} (\alpha_0, \alpha_0 \wedge \alpha_2)$$

This proves (i), and (ii) follows. Given that $\alpha<\alpha_0$, the excursion from ρ_α to τ_α is distributed as the first excursion reaching h in an Itô point process of Brownian excursions, that is according to the original law of X, independently of α and of all other points in the process of excursions above the post-D_h minimum, and independently of the pre-D_h process. Since the whole pre-ρ_α process can be recovered from these objects independent of the excursion from ρ_α to τ_α, this excursion is independent in particular of the excursion of X above level α between σ_α and ρ_α. Time reversal shows that this excursion must have the same law as the excursion for ρ_α to τ_α. This gives (iii). □

Remarks.

(i) Details of the above argument can be filled in by consulting Greenwood and Pitman [GP] or Rogers [R2], where the same technique is used to derive path decompositions from excursion theory.

(ii) The argument shows also that the two copies of X are independent also of the pre-ρ_α and post-τ_α parts of X. By the time reversal of Williams [W1], $(X(t), 0 \le t \le \sigma_\alpha)$ is a $BES_0(3)$ run till it last hits α, conditioned never to drop more than h below its past maximum. This is the reverse in law of $X(\tau_\alpha+t)$, $0 \le t \le \zeta-\tau_\alpha)$, which is a BM started at α, run till it hits 0, conditioned never to rise more than h above its minimum.

(iii) The decomposition at the deepest h-minimum described above is different from the one which is obtained by considering the minimum between the first hitting and the last exit time of h. Indeed, the latter leads to a different branching tree in the Brownian excursion. (This remark is due to J.-F. Le Gall).

Acknowledgements.

We thank J.-F. Le Gall for several stimulating conversations, and for remarks on a first draft of this paper. Also, Jim Pitman thanks the Laboratoire de Probabilités for hospitality in the fall of 1985.

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