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A REMARK ON THE CLASS OF MARTINGALES WITH BOUNDED QUADRATIC VARIATION

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Let (Ω, F, P) be a fixed probability space with a filtration (F_t) satisfying the usual conditions, and consider the class H_∞ of all martingales M adapted to this filtration such that $\langle M\rangle_\infty \in L_\infty$.

The aim of this short note is to prove the following.

PROPOSITION 1. Suppose the existence of a predictable mobile time T such that $P(T>0)>0$. Then there exists a bounded continuous martingale which does not belong to the closure $\overline{H}_\infty$ in BMO.

The definition of a mobile time is given in [1]: that is, it is a stopping time T such that for some continuous local martingale X we have $\langle X\rangle_t < \langle X\rangle_T$ on $\{t<T\}$. Recall that a uniformly integrable martingale M is said to be in the class BMO if

$$\|M\|_{BMO} = \sup_T \|E[(M_\infty - M_{T-})^2 \mid F_T]^{1/2}\|_\infty < \infty ,$$

where the supremum is taken over all stopping times T.

In order to prove Proposition 1, we need the next three lemmas.

LEMMA1. For $M \in BMO$, let $b(M)$ be the supremum of the set of b for which

$$\sup_T \|E[\exp\{b^2(\langle M\rangle_\infty - \langle M\rangle_{T-})\} \mid F_T]\|_\infty < \infty .$$

Then we have

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$$b(M) \geqq \frac{1}{\sqrt{2}\, d(M,H_\infty)} ,$$

where $d(M,H_\infty)$ is the distance in BMO from M to H_∞.

PROOF. Recall that, if $||X||_{BMO} < 1$, then

$$E[\exp(\langle X\rangle_\infty - \langle X\rangle_T) | F_T] \leqq (1-||X||^2_{BMO})^{-1}$$

for every stopping time T. This is well known as the John-Nirenberg theorem. Let now $0<b< 1/\{\sqrt{2}\, d(M,H_\infty)\}$. Then we have $2b^2||M-N||^2_{BMO} < 1$ for some $N\in H_\infty$. Since $\langle N\rangle_\infty \leqq C$ for some constant $C>0$, we find for $s<t$

$$\langle M\rangle_t - \langle M\rangle_{s-} \leqq 2(\langle M-N\rangle_t - \langle M-N\rangle_{s-}) + 2C.$$

Then from the John-Nirenberg theorem it follows that

$$E[\exp\{b^2(\langle M\rangle_\infty - \langle M\rangle_{T-})\} | F_T] \leqq e^{2b^2C} E[\exp\{2b^2(\langle M-N\rangle_\infty - \langle M-N\rangle_{T-})\} | F_T]$$

$$\leqq e^{2b^2C}(1-2b^2||M-N||^2_{BMO})^{-1},$$

which implies that $b \leqq b(M)$. Thus the lemma is proved.

LEMMA 2. Let A be an increasing process such that $A_t < A_\infty$ for every finite t. Then there exists a positive continuous increasing process U such that $\int_0^\infty U_s dA_s = \infty$ a.s.

For the proof, see Lemma 2 in [1].

LEMMA 3. Suppose the existence of a predictable mobile time $T>0$ a.s. Then there exists a continuous local martingale L satisfying $\langle L\rangle_\infty = \infty$ a.s.

PROOF. By the definition of a mobile time, for some continuous local martingale X we have $\langle X\rangle_t < \langle X\rangle_T$ on $\{t<T\}$, and, T being predictable, there is a sequence (T_n) of stopping times such that $T_n \uparrow T$ a.s. and $T_n<T$ for every n. Let now $g_n:[n-1,n[\to[0,\infty[$ be an increasing homeomorphic function, and set

$$\tau_t = \max[\,T_{n-1}\,,\min\{T_n,\,g_n(t)\}\,] \quad (t\varepsilon[n-1,n[).$$

Then $(\tau_t)_{0\leqq t<\infty}$ is a continuous change of time such that $\tau_0=0$, $\tau_k=T_k$ $(k=1,2,\dots)$ and further $\tau_t<T$ for every finite t. So, the process Y defined by $Y_t=X_{\tau_t}$ $(0\leqq t<\infty)$ is a continuous local martingale over (F_{τ_t}), and we have for every finite t

$$\langle Y\rangle_t = \langle X\rangle_{\tau_t} < \langle X\rangle_T = \langle Y\rangle_\infty \,.$$

Thus from Lemma 2 it follows that $\int_0^\infty U_s d\langle Y\rangle_s = \infty$ a.s. for some positive continuous increasing process $U=(U_t,F_{\tau_t})$.

Next, let $\sigma_t = \inf\{s : \tau_s>t\}$ and $V_t=U_{\sigma_t}$. As is easily verified, each σ_t is a stopping time with respect to (F_{τ_t}), so that V_t is $F_{\tau_{\sigma_t}}$-measurable. However, since $\tau_{\sigma_t}\leqq t$ by the definition of σ_t, we find that V_t is in fact F_t-measurable. Thus the stochastic integral $L_t=\int_0^t\sqrt{V_s}\,dX_s$ is well-defined. Since $L_{\tau_t}=\int_0^t\sqrt{V_{\tau_s}}\,dY_s$, we find

$$\langle L\rangle_\infty \geqq \langle L\rangle_{\tau_\infty} = \int_0^\infty V_{\tau_s}\,d\langle Y\rangle_s = \int_0^\infty U_{\sigma_{\tau_s}}\,d\langle Y\rangle_s \,.$$

Noticing that U is increasing and $\sigma_{\tau_s} \geqq s$, we have in conclusion

$$\langle L\rangle_\infty \geqq \int_0^\infty U_s \, d\langle Y\rangle_s = \infty \text{ a.s.}$$

This completes the proof.

PROOF OF PROPOSITION 1. Let T be a predictable mobile time such that P(T>0)>0. We may assume that T>0 a.s, because there is no question on {T=0}. Then by Lemma 3 there exists a continuous local martingale L such that $\langle L\rangle_\infty = \infty$ a.s. Let $\theta_t = \inf\{s : \langle L\rangle_s > t\}$ and $W_t = L_{\theta_t}$. As is well known, the process $W=(W_t, F_{\theta_t})$ is a one dimensional Brownian motion. Next, let $\sigma = \inf\{t : |W_t| = 1\}$. Note that $\exp(\pi^2\sigma/8)$ is not integrable. It is easy to see that θ_σ is a stopping time with respect to (F_t), and so the process $M_t = L_{t\wedge\theta_\sigma}$ is a continuous local martingale over (F_t). Recalling that L is constant on the stochastic interval $[[t, \theta_{\langle L\rangle_t}]]$, we find

$$M_t = L_{\theta_{\langle L\rangle_t} \wedge \theta_\sigma} = W_{\langle L\rangle_t \wedge \sigma},$$

from which it follows that $|M| \leqq 1$. On the other hand, we have $\langle M\rangle_\infty = \langle L\rangle_{\theta_\sigma} = \sigma$, so that $\exp(\pi^2\langle M\rangle_\infty/8)$ is not integrable. Then $b(M) \leqq \pi/\sqrt{8}$ by the definition of b(M) and so we have $d(M, H_\infty) \geqq 2/\pi$ by Lemma 1. This completes the proof.

We can also verify, under the same assumption as in Proposition 1, that $H_\infty \setminus L_\infty \neq \phi$.

As a corollary to Proposition 1, we shall remark that a change of law gives sometimes rise to a morbid phenomenon. For that, let M be a local martingale such that the solution Z of the equation $Z_t = 1 + \int_0^t Z_{s-} dM_s$ is a uniformly integrable positive martingale, and let $d\hat{P} = Z_\infty dP$. Then for every continuous local martingale X the process $\hat{X} = \langle X,M\rangle - X$ is a continuous local martingale with respect to $d\hat{P}$ such that $\langle\hat{X}\rangle = \langle X\rangle$ under either probability measure (see [3]). This is better known as transformation of drift or the Girsanov transformation. Note that, if $M \in BMO$ and $\Delta M \geqq -1+\delta$ for some δ with $0<\delta\leqq 1$, then the mapping : $X \to \hat{X}$ is an isomorphism of BMO onto BMO($\hat{P}$) (see [2]). However, we obtain the following interesting result.

PROPOSITION 2. Suppose the existence of a predictable mobile time T such that $P(T>0)>0$. Then there is a probability measure $\hat{P}$ equivalent to P such that $\hat{X} \notin H_1(\hat{P})$ for some bounded continuous martingale X.

PROOF. By Proposition 1 there is a bounded continuous martingale X which does not belong to $\bar{H}_\infty$. As a matter of course, $\langle X\rangle_\infty^{1/2}$ is not bounded. Since the dual of L_1 is L_∞, there exists a random variable $W>0$ a.s., $E[W]=1$, such that $E[W\langle X\rangle_\infty^{1/2}] = \infty$. Then, letting $d\hat{P} = WdP$, the conclusion follows immediately.

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