BHASKARAN RAJEEV

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On Semi-Martingales Associated with Crossings

B. RAJEEV, Indian Statistical Institute

Introduction. Let $(X_t)_{t\geq 0}$ be a Brownian motion, $X_0 = x$ almost surely, $x < a < b$. Let σ_t be the last exit time of X before t from $(a,b)^c$, defined in sec. 1.1. We note that when $b = \infty$, $X_t - X_{\sigma_t} = (X_t - a)^+$ and by Tanaka's formula it follows that $X_t - X_{\sigma_t}$ and hence X_{σ_t}, are semi-martingales. It is easy to see from Theorem 1 of [6] that when $b < \infty$, $|X_t - X_{\sigma_t}|$ is a semi-martingale given by

$$(b-a)c(t) + |X_t - X_{\sigma_t}| = \int_0^t I_{(a,b)}(X_s)\Theta(s)dX_s + \frac{1}{2}(L(t,a)+L(t,b))$$

where $c(t)$ is the numbers of crossings of (a,b) in time t, $\Theta(s,w)$ is 1 during an upcrossing and -1 during a downcrossing and $L(t,.)$ is the local time of X.

In the case of a continuous semi-martingale $(X_t, \mathcal{F}_t)$, where $\mathcal{F}_t$ is the underlying filtration and σ_t as above, it is an immediate consequence of Tanaka's formula that $(X_{\sigma_t}, \mathcal{F}_t)$, $(X_t - X_{\sigma_t}, \mathcal{F}_t)$ are semi-martingales (Theorem 2.1). In this case, time changing by σ_t does not change the underlying filtration. In this paper, as our main result we determine the martingale and bounded variation parts of $|X_t - X_{\sigma_t}|$ (Theorem 4.1). In sec. 5, we state a few applications of this result. These include Levy's crossing theorem, an asymptotic relationship between local times and crossings of Brownian motion and a probabilistic approximation of the remainder term in the 2nd order Taylor expansion of a function.

1. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$ a filtration on it satisfying usual conditions. For a continuous adopted process $(X_t)_{t\geq 0}$ and $a < b$, the upcrossing intervals $(\sigma_{2k}, \sigma_{2k+1}]$, $k = 0,1,2,\ldots$, are defined by $\sigma_0 = \inf\{s\geq 0, X_s \leq a\}$, $\sigma_{2k} = \inf\{s > \sigma_{2k-1}, X_s \leq a\}$ and $\sigma_{2k+1} = \inf\{s > \sigma_{2k} : X_s \geq b\}$. As usual the infimum over the empty set is infinity. The downcrossing intervals $(\tau_{2k}, \tau_{2k+1}]$, $k = 0,1,\ldots$ are similarly defined. Let $\theta^u(s) = \sum_{k=0}^{\infty} I_{(\sigma_{2k},\sigma_{2k+1}]}(s)$; $\theta^d(s) = \sum_{k=0}^{\infty} I_{(\tau_{2k},\tau_{2k+1}]}(s)$ and $\theta(s) = \theta^u(s) - \theta^d(s)$. The number of upcrossings in time t, denoted by $U(t)$ is defined as $U(t) = \max\{k : \sigma_{2k+1} \leq t\}$. The number of downcrossings is similarly defined. $C(t) = U(t)+D(t)$ is the total number of crossings. Let $\tau = \inf\{s>0 : X_s \notin (a,b)\}$.

$$\text{Let } \sigma_t = \begin{cases} t & t \leq \tau \\ \max & < s < t : X_s \; t(a,b)^c \quad , t > \tau \end{cases}$$

σ_t is in general not a stop time, but is however $\mathcal{F}_t$ measurable. Consequently X_{σ_t} is $\mathcal{F}_t$ measurable.

2. The Semi-Martingale $X_t - X_{\sigma_t}$

From now on we fix a continuous $\mathcal{F}_t$ - semi-martingale $X_t = X_0 + M_t + V_t$ and $a < b$. Let $L(t,x,w)$ be a jointly (t,x,w) measurable version of the local time of X which is continuous in t and right continuous in x. For the existence of such versions see [10]. Let $Y_t = X_t - X_{\sigma_t}$ and $Z_t = X_{\sigma_t}$.

Theorem 2.1. The process Y_t is an $\mathcal{F}_t$ semi-martingale and we have

$$Y_t = \int_{\tau_t}^{t} I_{(a,b]}(X_s)dX_s + \frac{1}{2}(L(t,a) - L(t,b)) - (b-a)(U(t)-D(t)) \quad (1)$$

Proof. The proof is immediate from Tanaka's formula and the following pathwise identity :

$$(X_{\tau_t} - X_0) + (b-a)[U(t)-D(t)] + (X_t - X_{\sigma_t})$$
$$= (X_t-a)^+ - (X_0-a)^+ - (X_t-b)^+ + (X_0-b)^+ \qquad (2)$$

Remarks.

2.2 It is immediate from Theorem 2.1 that X_{σ_t} is also a semi-martingale whose components can be got by subtracting (X_t) from both sides of eqn. (1).

2.3 The sum of the jumps of Y in time t — $\sum_{s\leq t} \Delta Y_s$ — is precisely $(b-a)[D(t)-U(t)]$. Since $|U(t) - D(t)| \leq 1$ this implies that Y (and hence Z) is a special semimartingale. Further the representation (1) of Y_t is unique (see [9]). The jump times of these processes are precisely the times of crossings of (a,b) by X and $|\Delta Y_s| = b-a$ or 0.

2.4 Equation (2) and hence Theorem 2.1 are still valid for a semi-martingale (X_t) with $\sum_{s\leq t} |\Delta X_s| < \infty \;\forall\; t$, almost surely. Now $(b-a)[U(t)-D(t)]$ is replaced by $-\sum_{s\leq t} \Delta Y_s$ and $(X_t-a)^+$, $(X_t-b)^+$ are replaced by $(X_t-a)^+ - \sum_{s\leq t} \Delta(X_s-a)^+$, $(X_t-b)^+ - \sum_{s\leq t} \Delta(X_s-b)^+$ respectively.

3. Local times of $X_t - X_{\sigma_t}$

We now determine the local times of Y in terms of that of X. We note that the process lives in [0,b-a) during an upcrossing of (a,b) and in (-(b-a),0] during a downcrossing. Also $Y_t = 0$ whenever $X_t = a$ or b . Let I(t,x) denote the local time of the Y process.

Lemma 3.1

(i) For $x \in [0,b-a)$,

$$(Y_t-x)^+ = \int_{\tau_t}^{t} I_{(a,b]}(X_s)I_{(x,\infty)}(Y_{s-})dX_s - (b-a-x)U(t) + \frac{1}{2}I(t,x) \qquad (3)$$

(ii) For $x \in (-(b-a),0]$

$$(Y_t-x)^- = -\int_{\tau_t}^{t} I_{(a,b]}(X_s)I_{(-\infty,x]}(Y_{s-})dX_s - (b-a+x)D(t) + \frac{1}{2}I(t,x)$$

$$+ \frac{1}{2}\left(\int_0^t I_{(-\infty,x]}(Y_{s-})L(ds,b) - \int_0^t I_{(-\infty,x]}(Y_{s-})L(ds,a)\right) \qquad (4)$$

Remark 3.2 Observe that in case $x < 0$, the 2nd term on the RHS of (4) is zero whereas when $x = 0$ it is $\frac{1}{2}(L(t,b)-L(t,a))$.

Proof. Tanaka's formula (see [4]) applied to Y at the point $x \in [0,b-a)$ gives

$$(Y_t-x)^+ = (Y_0-x)^+ + \int_0^t I_{(x,\infty)}(Y_{s-})dY_s$$

$$+ \sum_{0<s\le t} I_{(x,\infty)}(Y_{s-})(Y_s - x)^-$$

$$+ \sum_{0<s\le t} I_{(-\infty,x]}(Y_{s-})(Y_s-x)^+ + \frac{1}{2}I(t,x)$$

$$= I_0 + I_1 + I_2 + I_3 + \frac{1}{2}I(t,x) .$$

Since $Y_0 \equiv 0$, $I_0 \equiv 0$. Using eqn. (2) for Y_t and noting that the measures $L(ds,a)$, $L(ds,b)$, $D(ds)$ have no support on the set $s : Y_{s-} > x$ we get

$$I_1(t) = \int_{\tau_t}^{t} I_{(a,b]}(X_s)I_{(x,\infty)}(Y_{s-})dX_s - (b-a)U(t) .$$

Since the jumps of Y occur at the crossing times σ_{2k+1}, τ_{2k+1} it is easy to see that almost surely for $x \in [0,b-a)$, $I_2(t) = x\,U(t)$, $I_3(t) \equiv 0$. This proves the first part of the lemma. The proof of (4) is similar using the Tanaka formula for $(Y_t-x)^-$.

The following theorem gives I in terms of L.

Theorem 3.3

(i) For $x \in (0,b-a)$, almost surely,

$$I(t,x) = \int_0^t \theta^u(s)L(ds,a+x) \tag{5}$$

(ii) For $x \in (-(b-a),0)$, almost surely,

$$I(t,x) = \int_0^t \theta^d(s)L(ds,b+x) \tag{6}$$

(iii) For $x = 0$, almost surely,

$$I(t,0) = L(t,a) \tag{7}$$

Proof.

(i) Let $x \in (0,b-a)$. Fix $k \geq 0$. Let $Y_1(t) = (Y_t - x)^+$ $Y_2(t) = (X_t - (a+x))^+$. We note that, $\forall\, t \in (\sigma_{2k},\sigma_{2k+1})$

$$\int_0^t I_{(\sigma_{2k},\sigma_{2k+1}]}(s)dY_1(s) = \int_0^t I_{(\sigma_{2k},\sigma_{2k+1}]}(s)dY_2(s) \tag{8}$$

By Tanaka's formula,

$$\int_0^t I_{(\sigma_{2k},\sigma_{2k+1}]}dY_2(s) = \int_0^t I_{(\sigma_{2k},\sigma_{2k+1}]}(s)I_{(a,b]}(X_s)I_{(x,\infty)}(Y_{s-})dX_s$$
$$+ \frac{1}{2}\,(L(t\wedge \sigma_{2k+1},a+x)-L(t\wedge\sigma_{2k},a+x)) .$$

By eqn. (3), $\forall\, t \in (\sigma_{2k},\sigma_{2k+1})$

$$\int_0^t I_{(\sigma_{2k},\sigma_{2k+1}]}dY_1(s) = \int_0^t I_{(\sigma_{2k},\sigma_{2k+1}]}(s)I_{(a,b]}(X_s)I_{(x,\infty)}(Y_{s-})dX_s$$
$$+ \frac{1}{2}\,(I\,(t\wedge \sigma_{2k+1},x)-I(t\wedge \sigma_{2k},x))$$

eqn. (8) now implies that $\forall\, t \geq 0$,

$$I(t\wedge \sigma_{2k+1},x) - I(t\wedge \sigma_{2k},x) = L(t\wedge \sigma_{2k+1},a+x) - L(t\wedge \sigma_{2k},a+x)$$

since $I(ds,x)$ is supported on the upcrossing intervals, the proof of (i) is complete.

(ii) Let $x \in (-(b-a),0)$. Then $Y_1(t) = (Y_t - x)^-$ and

$Y_2(t) = (X_t-(b+x))^-$ agree on the downcrossing intervals. Applying Tanaka's formula for Y_1 and eqn. (4) to Y_2 the proof is completed as in (i) above.

(iii) Let $x = 0$. Proceeding as in case (i) we show that

$$\int_0^t \theta^u(s)I(ds,0) = \int_0^t \theta^u(s)L(ds,a) = L(t,a).$$

To complete the proof we show that $\int_0^t \theta^d(s)I(ds,0) = 0$. To see this we fix k and as in case (ii), compare the expressions for $(X_t-b)^-$ and $(Y_t)^-$ for $t \in (\tau_{2k},\tau_{2k+1})$ given by Tanaka's formula and eqn. (4) respectively. Using Remark 3.2 we see that

$$L(t\wedge\tau_{2k+1},b)-L(t\wedge\tau_{2k},b) + (I(t\wedge\tau_{2k+1},0) - I(t\wedge\tau_{2k},0))$$

$$= L(t\wedge\tau_{2k+1},b) - L(t\wedge\tau_{2k},b)$$

whence $I(t\wedge\tau_{2k+1},0) - I(t\wedge\tau_{2k},0) = 0$.

<u>Remarks</u>.

3.4 We recall from [10] that for the semi-martingale (X_t) with $-\sum_{s\leq t}\Delta X_s + X_t = X_0 + M_t + V_t$, where M and V are the continuous martingale and bounded variation parts respectively, the jumps of the local time $L(t,x)$ is given by the formula : almost surely,

$$L(t,x) - L(t,x-) = \int_0^t I_{\{X_s = x\}} dV_s \tag{9}$$

Using (9) it is easy to see that for $x \in (0,b-a)$, $I(t,x)$ is continuous at x if $L(t,.)$ is continuous at $a+x$. The case $x \in (-(b-a),0)$ is similar. When $x = 0$, it is easy to see that $I(t,0-) = L(t,b-) \neq L(t,a)$.

3.5 Let $\bar{I}(t,x)$ denote the local time process of $Z_t = X_{\sigma_t}$. The martingale, bounded variation part and the jumps of Z_t are easily calculated from eqn. (1). By using Tanaka's formula it is easily verified that $\bar{I}(t,x) = L(t,x)$,

$\forall\ x \in (-\infty,a) \cup [b,\infty)$, $\bar{I}(t,a) = 0$ and $\bar{I}(t,a) = L(\tau \wedge t,x)$, $\forall\ x \in (a,b)$.

4. The Semi-Martingale $|X_t - X_{\sigma_t}|$

We now determine the continuous martingale and the continuous bounded variation parts of $|X_t - X_{\sigma_t}|$. We note that the sum of the jumps upto time t is $-(b-a)c(t)$.

Theorem 4.1 For $a < b$, we have almost surely,

$$(b-a)c(t) + |X_t - X_{\sigma_t}| = \int_0^t \theta(s,w) I_{(a,b)}(X_s)dX_s + \tfrac{1}{2}(L(t,a)+L(t,b-)) \quad (10)$$

Proof. Lemma 3.1 and Theorem 3.3 together give

$$\begin{aligned} |X_t - X_{\sigma_t}| &= (X_t - X_{\sigma_t})^+ + (X_t - X_{\sigma_t})^- \\ &= \int_{\tau_t}^t (I_{(0,\infty)}(Y_{s-}) - I_{(-\infty,0]}(Y_{s-}))I_{(a,b]}(X_s)dX_s \\ &\quad - (b-a)c(t) + \tfrac{1}{2}(L(t,a) + L(t,b)) \\ &= \int_0^t \theta(s) I_{(a,b)}(X_s)dX_s - (b-a)c(t) \\ &\quad + \tfrac{1}{2}(L(t,a) + L(t,b-)) \end{aligned}$$

where in the last equality we have used eqn. (9).

Remark 4.2 We refer to [6] for an analogous result on crossings of closed intervals by a continuous martingale.

5. Applications

We now give some applications of the previous results. We mention only the results and refer the proofs to [5], [6] and [7].

Firstly we note that letting $a \uparrow b$ in Theorem 4.1 eqn. (10) yields Levy's crossing theorem. We note that if $\varepsilon_1 \le \varepsilon \le \varepsilon_2$ then $\varepsilon_1 C_{\varepsilon_2}(t) \le \varepsilon C_\varepsilon(t) \le \varepsilon_2 C_{\varepsilon_1}(t)$ where

$C_\varepsilon(t)$ = number of crossings of $(b-\varepsilon,b)$ in time $t = C((b-\varepsilon,b),t)$. Hence sufficient to let $a \uparrow b$ along a sequence. This is done via the Borel-Cantelli lemma and an estimate due to Yor (Theorem 1, [10]). The following theorem (Levy's (down) crossing theorem) was first proved in the case of a continuous semi-martingale in El Karoui [3] where the discontinuous case is also discussed.

<u>Theorem</u> 5.1 Let (X_t) be a continuous semi-martingale. Then

(a) almost surely, $\underset{a\uparrow b}{\mathrm{Lt}}\ (b-a)C((a,b),t) = L(t,b-)$

$$\underset{b\downarrow a}{\mathrm{Lt}}\ (b-a)C((a,b),t) = L(t,a)$$

(b) If further $(X_t) \in H^p$, $p \geq 1$ then the above limits hold in H^p.

Next let (X_t) be a Brownian motion. We now state a result somewhat related to Theorem 5.1 above and whose proof can be found in [6], [7]. The crossing theorem say, that $(b-a)C(t) \sim L(t,a)$ as $b \downarrow a$, the parameter t being fixed. It is an interesting fact that the same is true when we let $t \to \infty$. We have the following theorem.

<u>Theorem</u> 5.2 Let (X_t) be a Brownian motion and $a < b$. Then almost surely,

$$\underset{t\to\infty}{\mathrm{Lt}}\ \frac{L(t,a)}{C((a,b),t)} = \underset{t\to\infty}{\mathrm{Lt}}\ \frac{E\ L(t,a)}{E\ C((a,b),t)} = (b-a)$$

<u>Remark</u> 5.3 The proof of the 2nd equality is immediate from Theorem 4.1 and Theorem 2.1

<u>Corollary</u> 5.4 Let $a < b$, $d < e$. Then almost surely,

$$\underset{t\to\infty}{\mathrm{Lt}}\ \frac{C((a,b),t)}{C((d,e),t)} = \underset{t\to\infty}{\mathrm{Lt}}\ \frac{E\ C((a,b),t)}{E\ C((d,e),t)} = \frac{b-a}{e-d}$$

We continue with a Brownian motion (X_t). The following result gives the average sojourn time in (a,b) per crossing.

Theorem 5.3 If (X_t) is a Brownian motion and $a < b$, then almost surely,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{\int_0^t I_{(a,b)}(X_s)ds}{C((a,b),t)} = \underset{t \to \infty}{\mathrm{Lt}} \frac{E\int_0^t I_{(a,b)}(X_s)ds}{E\,C((a,b),t)} = (b-a)^2$$

We refer to [8] for a proof of this result. The 2nd equality is an immediate consequence of Theorem 1, [5] which is also proved in [11]. We refer to [1] for a more general result in the context of Hunt processes and to [2] for related results involving recurrent diffusions. The following is a different generalization of Theorem 5.3 and can be thought off as a random approximation to the remainder term in a 2nd order Taylor expansion for a C^2-function. For the proof of this result see [6], [7].

Theorem 5.4 Let (X_t) be a Brownian motion, $a < b$, and f a C^2-function. Then almost surely,

$$\underset{t \to \infty}{\mathrm{Lt}} \frac{\int_0^t f''(|X_s-X_{\sigma_s}|_-)I_{(a,b)}(X_s)ds}{C((a,b),t)}$$

$$= \underset{t \to \infty}{\mathrm{Lt}} \frac{E\int_0^t f''(|X_s-X_{\sigma_s}|_-)I_{(a,b)}(X_s)ds}{E\,C((a,b),t)}$$

$$= f(b-a)-f(0)-f'(0)(b-a).$$

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