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GENERALIZED HARMONIC OSCILLATORS IN QUANTUM PROBABILITY

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Introduction.

By a *generalized harmonic oscillator* we mean a pair (H, X) of selfadjoint operators in a complex separable Hilbert space $\mathcal{H}$ satisfying

$$[H, [H, X]]u = c^2 Xu \quad \text{for all} \quad u \in \mathcal{D} \tag{1.1}$$

where $c^2 \geq 0$ is a constant and $\mathcal{D}$ is a dense linear manifold in $\mathcal{H}$. When H is fixed we say that X is *harmonic* with respect to H in the domain $\mathcal{D}$. Such a definition is motivated by the fact that in Heisenberg's picture of quantum dynamics with energy operator H the rate of change (or velocity) and the acceleration of the observable X are determined by the operators $i[H, X]$ and $-[H, [H, X]]$ respectively and (1.1) expresses the relation that in every pure state $u \in \mathcal{D}$ the mean acceleration of X is proportional to the mean value of X, the constant of proportionality being $-c^2 \leq 0$. In the present exposition we shall discuss several examples of generalized harmonic oscillators and establish the following : given any symmetric probability distribution μ on the real line satisfying the property that polynomials are dense in $L^2(\mu)$ there exists a generalized harmonic oscillator (H, X) and a unit vector u in a Hilbert space such that $Hu = 0$, the spectrum of H is contained in $\{0, 1, 2, \cdots\}$ and the probability distribution of X in the pure state u is μ. We shall also indicate situations when an arbitrary observable may be expressed as a superposition of harmonic observables with respect to a selfadjoint operator having pure point spectrum. Finally examples of quantum martingales are constructed in a boson Fock space for which the observable at time t is harmonic with respect to the conservation operator $\Lambda(t)$ for every t. These include fermion brownian motion, Azéma martingales and also martingales for which the distribution at time t in the vacuum state is a properly scaled Wigner distribution.

Examples of generalized harmonic oscillators.

We shall now present a few concrete examples of generalized harmonic oscillators and examine their properties.

Example 2.1. Let $\mathcal{H} = \mathbb{C}^2$ with the orthonormal basis $\{e_0, e_1\}$ where $e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Define

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\sigma_i, i = 1,2,3$ are the well known Pauli spin matrices. Then $[\sigma_3, [\sigma_3, \sigma_i]] = 4\sigma_i$ if $i = 1,2$ and $= 0$ otherwise. Thus σ_i is harmonic with respect to σ_3 for each i and any observable σ in $\mathcal{H}$ can be expressed as $\sigma = \sum_i x_i \sigma_i$ where x_i is a real scalar for each i. In the pure state e_0, σ_0 and σ_3 have degenerate distribution at 1 whereas σ_1 and σ_2 have Bernoulli distribution with equal probability for the values 1 and -1.

Example 2.2. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_0, e_1, e_2, \ldots\}$. We adopt the convention that $e_n = 0$ whenever $n \geq \dim\mathcal{H}$. For any $u, v \in \mathcal{H}$ define the operator $|u><v|$ in Dirac's notation so that

$$|u><v|w = <v, w> u \quad \text{for all } w \text{ in } \mathcal{H}.$$

Let

$$N = \sum_j j|e_j><e_j|\,, \qquad L = \sum_j |e_j><e_{j+1}|\,. \tag{2.1}$$

Denote by $\mathcal{D}$ the linear manifold generated by $e_0, e_1, \ldots$. Then N is essentially selfadjoint on $\mathcal{D}$, $[N, L] = -L$ and $L^k = \sum_j |e_j><e_{j+k}|$. In particular, $L^k = 0$ for $k \geq \dim\mathcal{H}$. (Since $Le_0 = 0$, $Le_j = e_{j-1}$ for $1 \leq j < \dim\mathcal{H}$ we may call L and L^* the ***standard annihilation*** and ***creation*** operators respectively. N may be called the ***number operator***.) We have the relations

$$L^*L = 1 - |e_0><e_0|\,, \; LL^* = 1 - |e_{n-1}><e_{n-1}| \quad \text{where } n = \dim\mathcal{H}, \tag{2.2}$$
$$N = \sum_{j\geq 1} L^{*j} L^j.$$

For any bounded complex valued function f on $\{0,1,2,\ldots,\}$ define the bounded selfadjoint operator

$$X = f(N)L^k + L^{*k}\bar{f}(N)\,, \qquad k \geq 0\,. \tag{2.3}$$

Then

$$[N, [N, X]]\,u = k^2 X u\,, \qquad u \in \mathcal{D}\,,$$

so that X is harmonic with respect to N on $\mathcal{D}$. When $\dim\mathcal{H} = \infty$, $f(j) = 1/2$ for all j and $k = 1$, $X = (L + L^*)/2$ is a harmonic observable with respect to N on $\mathcal{D}$ having the ***standard Wigner distribution*** with density function $\frac{2}{\pi}(1 - x^2)^{1/2}$ in the interval $[-1, 1]$ in the pure state e_0. This is easily shown by proving that $<e_0, X^n e_0> = 0$ if n is odd and $= 2^{-2k}(k+1)^{-1}\binom{2k}{k}$ when $n = 2k$ through a routine computation.

The boson annihilation operator a can be expressed as $a = (N+1)^{1/2}L$. Then

$$X = (N+1)^{1/2}L + L^*(N+1)^{1/2} \tag{2.4}$$

can be closed to an unbounded selfadjoint operator with $\mathcal{D}$ as a core and

$$[N, [N, X]]\,u = Xu\,, \qquad u \in \mathcal{D}\,.$$

This is covered by (2.3) by putting $k = 1$, $\dim\mathcal{H} = \infty$ and allowing the unbounded function f with $f(j) = (j+1)^{1/2}$ for all j. In the pure state e_0, X has the standard normal distribution. In the pure state e_k the density function of X is $(2\pi)^{-1/2}q_k(x)^2 e^{-x^2/2}$ where q_k is a suitably normalised k-th degree Hermite polynomial.

Now consider an arbitrary operator X whose domain includes $\mathcal{D}$ and express it as $X = \sum_{i,j} x_{ij}|e_i><e_j|$ where $\sum_i |x_{ij}|^2 < \infty$ for each j. Define the functions

$$f_0(i) = x_{ii}\ ,\ f_k(i) = x_{ii+k}\ ,\ \tilde{f}_k(i) = x_{i+ki}$$

for all $i \geq 0$ where $x_{ii+k} = x_{i+ki} = 0$ whenever $i + k \geq \dim\mathcal{H}$. Then we have

$$Xu = f_0(N)u + \sum_{k\geq 1}\{f_k(N)L^k + L^{*k}\tilde{f}_k(N)\}u\ , \quad u \in \mathcal{D}\ , \tag{2.5}$$

$$\begin{aligned} &e^{itN}Xe^{-itN}u = \\ &f_0(N)u + \sum_{k\geq 1}\{e^{-itk}f_k(N)L^k + e^{itk}L^{*k}\tilde{f}_k(N)\}u\ , \quad u \in \mathcal{D}\ . \end{aligned} \tag{2.6}$$

In particular, (2.6) implies that $f_0, f_k, \tilde{f}_k$ are determined by the identities :

$$\begin{aligned} f_0(N)u &= \frac{1}{2\pi}\int_0^{2\pi} e^{itN}Xe^{-itN}u\,dt\ , \\ f_k(N)L^k u &= \frac{1}{2\pi}\int_0^{2\pi} e^{it(k+N)}Xe^{-itN}u\,dt\ , \\ L^{*k}\tilde{f}_k(N)u &= \frac{1}{2\pi}\int_0^{2\pi} e^{it(N-k)}Xe^{-itN}u\,dt\ . \end{aligned}$$

for all $u \in \mathcal{D}$, where the right hand side integrals are in the strong sense. If X is symmetric on $\mathcal{D}$ then $\tilde{f}_k = \bar{f}_k$ and f_0 is real. If X is bounded

$$\max(\|f_0(N)\|, \|f_k(N)\|, \|\tilde{f}_k(N)\|) \leq \|X\| \quad \text{for all } k\,.$$

If X is Hilbert-Schmidt we have the "Plancherel identity"

$$\operatorname{Tr} X^*X = \operatorname{Tr} |f_0|^2(N) + \sum_{k\geq 1} \operatorname{Tr} L^kL^{*k}(|f_k|^2 + |\tilde{f}_k|^2)(N) \tag{2.7}$$

where $L^kL^{*k} = 1$ if $\dim\mathcal{H} = \infty$.

When X is a bounded selfadjoint operator $\tilde{f}_k = \bar{f}_k$ in (2.5) and $X_k = f_k(N)L^k + L^{*k}\bar{f}_k(N)$ is a bounded selfadjoint operator. Thus (2.5) may be interpreted as follows : every bounded observable is a superposition of bounded observables harmonic with respect to N.

Example 2.3. Let H be any selfadjoint operator in a complex separable Hilbert space $\mathcal{H}$ with pure point spectrum S. Then S is a finite or countable subset of $\mathbb{R}$. Denote by G the countable additive group generated by S and endowed with the discrete topology. Let

$\widehat{G}$ be its compact character group with the normalized Haar measure. For any bounded operator X on $\mathcal{H}$ and $\lambda \in G$ define the bounded operator

$$X_\lambda = \int_{\widehat{G}} \chi(H)\overline{\chi}(\lambda) X \overline{\chi}(H)\, d\chi \,. \tag{2.8}$$

Let $\mu, \nu \in S$ and $u, v \in \mathcal{H}$ be such that $Hu = \mu u, Hv = \nu v$. Then

$$< u, X_\lambda v > = \{ \int_{\widehat{G}} \chi(\mu - \nu - \lambda)\, d\chi \} < u, Xv > .$$

Thus

$$\begin{aligned} < u, X_\lambda v > &= < u, Xv > \quad \text{if } \lambda = \mu - \nu, \\ &= 0 \quad \text{if } \lambda \neq \mu - \nu. \end{aligned}$$

In other words, for any nonzero bounded operator X there exists a $\lambda \in S - S = \{\mu - \nu \,|\, \mu, \nu \in S\}$ such that $X_\lambda \neq 0$ and on the linear manifold $\mathcal{D}$ generated by all the eigenvectors of H

$$\begin{gathered} [H, [H, X_\lambda]] = \lambda^2 X_\lambda \,, \\ Xu = X_0 u + \sum_{\substack{\lambda > 0 \\ \lambda \in S - S}} (X_{-\lambda} + X_\lambda) u \,, \qquad u \in \mathcal{D} \,. \end{gathered}$$

If X is selfadjoint $(X_\lambda)^* = X_{-\lambda}$ and X is a "superposition" of bounded harmonic observables X_0 and $\{X_{-\lambda} + X_\lambda \,,\, \lambda \in S - S \,,\, \lambda > 0\}$ with respect to H. Whenever X, Y are Hilbert–Schmidt operators we have the analogue of (2.7) :

$$\operatorname{Tr} X^* Y = \operatorname{Tr} X_0^* Y_0 + \sum_{\lambda \in S - S \,,\, \lambda > 0} \operatorname{Tr} X_\lambda^* Y_\lambda \,.$$

and $X = X_0 + \sum_{\lambda \in S - S \,, \lambda > 0} (X_\lambda + X_{-\lambda})$ converges in Hilbert-Schmidt norm.

Example 2.4. In contrast to the preceding examples where the energy operator H had pure point spectrum we may consider $\mathcal{H} = L^2(\mathbb{R})$, $H = p$, $X = \cos \alpha q$, $\alpha \in \mathbb{R}$ where p, q is a canonical Schrödinger pair satisfying $[q, p] = i$. Then $[p, [p, \cos \alpha q]] = \alpha^2 \cos \alpha q$ on the domain of smooth functions with compact support. Similarly $[p, [p, \sin \alpha q]] = \alpha^2 \sin \alpha q$. More generally one can construct examples of harmonic observables of the form $X = f(p) e^{i\alpha q} + e^{-i\alpha q} \overline{f}(p)$ where f is a sufficiently regular complex valued fucntion and hope to describe an arbitrary observable as a "continuous superposition" of such harmonic observables.

Harmonic observables with a prescribed distribution

Adopting the notations of Example 2.2 consider a selfadjoint operator Z in $\mathcal{H}$ whose restriction to the linear manifold $\mathcal{D}$ generated by an orthonormal basis has the form

$$Z = f(N) + g(N) L + L^* \overline{g}(N) \tag{3.1}$$

where f and g are functions defined on the set $\{0,1,2,\ldots\}$, f is real and $|g(j)| > 0$ for all j. Define the projections

$$P_k = \sum_{j=0}^{k} |e_j><e_j| , \qquad 0 \leq k < \dim\mathcal{H}$$

with range $\mathcal{H}_k$ equal to the linear span of $\{e_0, e_1, \ldots, e_k\}$ and the operators

$$A_{k+1} = P_k Z P_k \restriction_{\mathcal{H}_k} . \tag{3.2}$$

Consider the polynomials

$$p_0(x) = 1 , \quad p_k(x) = \det(x - A_k) , \quad 1 \leq k < \dim\mathcal{H} . \tag{3.3}$$

Inspired by the theory of orthogonal polynomials as expounded in [1] we shall establish the following theorem.

THEOREM 3.1. *The sequence $\{p_k,\ 0 \leq k < \dim\mathcal{H}\}$ defined by (3.3) is also the sequence of monic orthogonal polynomials of the distribution of the observable Z in the pure state e_0.*

We reduce the proof to two elementary lemmas.

LEMMA 3.2. *The sequence $\{p_k\}$ obeys the following recurrence relations :*

$$p_0(x) = 1 , \quad p_1(x) = x - f(0) ,$$
$$p_k(x) = (x - f(k-1))p_{k-1}(x) - |g(k-2)|^2 p_{k-2}(x) \quad \text{if } 2 \leq k < \dim\mathcal{H}.$$

PROOF. In the orthonormal basis $\{e_0, e_1, \ldots e_{k-1}\}$ of the subspace $\mathcal{H}_{k-1}$ the operator A_k has the tridiagonal matrix representation

$$A_k = \begin{pmatrix} f(0) & g(0) & 0 & 0 & \cdots & 0 \\ \bar{g}(0) & f(1) & g(1) & 0 & \cdots & 0 \\ 0 & \bar{g}(1) & f(2) & g(2) & \cdots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots & g(k-2) \\ 0 & 0 & \ldots & \ldots & \overline{g(k-2)} & f(k-1) \end{pmatrix} \tag{3.4}$$

Expanding the determinant of $x - A_k$ by the last row we obtain the required relations immediately. ∎

LEMMA 3.3. *The polynomials $\{p_k\}$ satisfy the following :*

$$p_k(Z)\, e_0 = h(k)\, e_k \quad \text{for all } 0 \leq k < \dim\mathcal{H} \tag{3.5}$$

where

$$h(0) = 1 , \quad \overline{h(k)} = g(0)g(1)\cdots g(k-1) \quad \text{if } 1 \leq k < \dim\mathcal{H}. \tag{3.6}$$

PROOF. We have trivially

$$p_0(Z)\, e_0 = e_0 , \quad p_1(Z)\, e_0 = (Z - f(0))\, e_0 = \overline{g(0)}\, e_1 = h(1)\, e_1 .$$

For $k \geq 2$ we use induction. By Lemma 3.2, induction hypothesis and (3.1) we have

$$\begin{aligned} p_k(Z)e_0 &= (Z - f(k-1))p_{k-1}(Z)e_0 - |g(k-2)|^2 p_{k-2}(Z)e_0 \\ &= (Z - f(k-1))h(k-1)e_{k-1} - |g(k-2)|^2 h(k-2)e_{k-2} \\ &= g(k-2)h(k-1)e_{k-2} + \overline{g(k-1)}h(k-1)e_k - |g(k-2)|^2 h(k-2)e_{k-2} \\ &= \overline{g(k-1)}h(k-1)e_k = h(k)e_k . \end{aligned}$$

■

PROOF OF THEOREM 3.1. Let μ_0 be the probability distribution of Z in the pure state e_0. Lemma 3.2 implies that the coefficients of the polynomials p_k are all real and by Lemma 3.3

$$\begin{aligned} \int p_k(x)p_\ell(x)d\mu_0(x) &= < e_0, ,p_k(Z)p_\ell(Z)e_0 > \\ & < p_k(Z)e_0, p_\ell(Z)e_0 > \\ &= < h(k)e_k, h(\ell)e_\ell > = |h(k)|^2 \delta_{k\ell} . \end{aligned}$$

Since $|h(k)| > 0$ for all $0 \leq k < \dim\mathcal{H}$ the required result follows. ■

COROLLARY 3.4. *Define the polynomials* $\{q_k, 0 \leq k < \dim\mathcal{H}\}$ *by*

$$q_k(x) = |h(k)|^{-1} p_k(x)$$

where p_k *and* $h(k)$ *are defined by (3.3) and (3.6). Let* μ_k *be the probability distribution of* Z *in the pure state* e_k *for each* $0 \leq k < \dim\mathcal{H}$. *Then* $\mu_k \ll \mu_0$ *and*

$$\frac{d\mu_k}{d\mu_0}(x) = q_k(x)^2 .$$

If dim $\mathcal{H} = \infty$ *and* μ_0 *is determined uniquely by its moments then the distribution of the observable* A_k *defined by (3.2) in the pure state* e_0 *has its support in the set of zeros of* q_k *and converges weakly to* μ_0 *as* $k \to \infty$.

Proof. For any real t we have from Lemma 3.3

$$\begin{aligned} < e_k, e^{itZ}e_k > &= < q_k(Z)e_0, e^{itZ}q_k(Z)e_0 > \\ &= < e_0, e^{itZ}q_k(Z)^2 e_0 > = \int e^{itx}q_k(x)^2\, d\mu_0 . \end{aligned}$$

This proves the first part. To prove the second part first observe that p_k is the characteristic polynomial of A_k for each k. Since $q_0, q_1, \ldots, q_{k-1}$ are the orthonormal polynomials for the distribution of A_k in the pure state e_0 as well as the first k orthonormal polynomials for the distribution μ_0 it follows that their first k moments are same. ■

Remark 1. Suppose in (3.1) we drop the hypothesis $|g(j)| > 0$ for all j. We can still define the sequence $\{p_k\}$ by (3.3) and obtain the recurrence relations of Lemma 3.2. From the proof of Theorem 3.1 and (3.6) we obtain

$$\int |p_k(x)|^2 d\mu_0(x) = |h(k)|^2 = \prod_{j=0}^{k-1} |g(j)|^2 .$$

If $k = \min\{j : g(j) = 0\}$ then it follows that the polynomials $1, x, x^2, \ldots, x^{k-1}$ are linearly independent and $\{p_0, p_1, \ldots, p_{k-1}\}$ is an orthogonal basis of $L^2(\mu_0)$.

Remark 2. Let $\dim \mathcal{H} = n < \infty$ and let the observable Z be defined by (3.1) with $g(j) > 0$ for all $0 \leq j \leq n-1$. Translating the recurrence relations of Lemma 3.2 in terms of the normalized polynomials $\{q_k\}$ in Corollary 3.4 we obtain

$$(Z - x) \sum_{j=0}^{n-1} q_j(x) e_j = g(n-1) q_n(x) e_{n-1} .$$

If $x_0, x_1, \ldots, x_{n-1}$ is an enumeration of the zeros of q_n then

$$\tilde{e}_j = \frac{\sum_{r=0}^{n-1} q_r(x_j) e_r}{\left(\sum_{r=0}^{n-1} q_r(x_j)^2\right)^{1/2}}, \qquad 0 \leq j \leq n-1$$

is a unit eigenvector of Z for the eigenvalue x_j. In particular, the observable Z assumes the values $x_0, x_1, \ldots, x_{n-1}$ with respective probabilities

$$p_{ij} = \frac{q_i(x_j)^2}{\sum_{r=0}^{n-1} q_r(x_j)^2}, \qquad 0 \leq j \leq n-1$$

in the pure state e_i for each $i = 0, 1, 2, \ldots, n-1$. It is to be noted that $((p_{ij}))$ is a doubly stochastic matrix. In this remark we have used the fact that the roots $x_0, x_1, \ldots, x_{n-1}$ of $q_n(x)$ are distinct.

Remark 3. From the table of orthogonal polynomials as presented in [1] it is possible to determine the functions f and g in (3.1) so that the corresponding observable Z has some of the well known probability distributions. We shall present a few examples :

(1) Let $f(j) = 0$, $g(j) = \frac{1}{2}$ for all j so that $Z = (L + L^*)/2$. As remarked in Example 2.2, Z has standard Wigner distribution in the pure state e_0 whenever $\dim \mathcal{H} = \infty$. Then its density function in the pure state e_k is $\frac{2}{\pi} q_k(x)^2 (1 - x^2)^{1/2}$ in the interval $[-1, 1]$ where $\{q_k, k \geq 0\}$ is the sequence of Chebyshev's polynomials of the second kind :

$$q_k(x) = \frac{\sin((k+1) \operatorname{Arc} \cos x)}{\sin(\operatorname{Arc} \cos x)} .$$

Suppose $\dim \mathcal{H} = n$. By the discussion in Remark 2 and the fact that the zeros of q_n are $\{\cos \frac{j+1}{n+1} \pi,\ j = 0, 1, \ldots, n-1\}$ it follows that the observable Z assumes the value $\cos \frac{j+1}{n+1} \pi$ with probability

$$p_{ij} = \frac{1 - \cos \frac{2(i+1)(j+1)}{n+1} \pi}{n+1}, \qquad j = 0, 1, 2, \ldots, n-1$$

in the pure state e_i for each $i = 0,1,2,\ldots,n-1$. It is curious that $((p_{ij}))$ is a symmetric matrix.

(2) Let $\dim\mathcal{H} = \infty$, $f(j) = 0$, $g(0) = 2^{-1/2}$, $g(j) = 1/2$ for all $j \geq 1$. Then the distribution of Z in the pure state e_0 is the symmetric Arcsin law with density function $\pi^{-1}(1-x^2)^{-1/2}$ in the interval $(-1,1)$. In the pure state e_k, Z has the density function $\pi^{-1}q_k(x)^2(1-x^2)^{-1/2}$ where $\{q_k,\ k \geq 0\}$ is the sequence of Chebyshev's polynomials of the first kind :

$$q_0 = 1\ ,\ q_k(x) = \sqrt{2}\cos(k \operatorname{Arc}\cos x)\ , \quad k \geq 1\ ,\ |x| \leq 1\ .$$

(3) Let $\dim\mathcal{H} = \infty$, $f(j) = 2j+\alpha$, $g(j) = \{(j+1)(j+\alpha)\}^{1/2}$ for all j where $\alpha > 0$ is a constant. Then Z has Gamma distribution in the interval $(0,\infty)$ with density function $\Gamma(\alpha)^{-1}x^{\alpha-1}e^{-x}$. In the pure state e_k the density function of Z is $\Gamma(\alpha)^{-1}q_k(x)^2x^{\alpha-1}e^{-x}$ in $(0,\infty)$ where $\{q_k,\ k \geq 0\}$ are the Laguerre polynomials :

$$q_k(x) = \left\{\frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma(\alpha+k+1)}\right\}^{1/2} L_k^\alpha(x)\ , \qquad k \geq 1$$

where $\{L_k^\alpha,\ k \geq 0\}$ is determined by the generating fucntion

$$\sum_{k=0}^{\infty} L_k^\alpha(x)\, w^k = (1-w)^{-\alpha} \exp\Big(-\frac{xw}{1-w}\Big)\ .$$

The binomial, Poisson, normal and Beta distributions along with their Krawtchuk, Charlier, Hermite and Jacobi orthogonal polynomials can be similarly realized through the observable Z in (3.1) by an appropriate choice of f and g using the extensive table in [1].

THEOREM 3.5. *Let μ be a probability distribution on the real line with moments of all order and satisfying the condition that the linear manifold $\mathcal{D}$ of all polynomials is dense in $L^2(\mu)$. Suppose Z is the selfadjoint operator of multiplication by x, i.e., $(Zu)(x) = xu(x)$ with maximal domain. Then there exist an orthonormal basis $\{q_k,\ 0 \leq k < \dim L^2(\mu)\}$ of polynomials and real sequences $\{f(k),\ g(k) \mid 0 \leq k < \dim L^2(\mu)\}$ satisfying the following :*

(i) *$q_0 = 1$, $\deg q_k = k$, $g(k) > 0$ for every k ;*

(ii) *$Zu = \{f(N) + g(N)L + L^*g(N)\}u$ for all $u \in \mathcal{D}$,*

where N and L are the number and standard annihilation operators defined by

$$N = \sum_k k\,|q_k><q_k|\ ,\ L = \sum_{k=0}^{\dim L^2(\mu)-1} |q_k><q_{k+1}|$$

respectively.

Proof. First observe that the sequence $\{x^k, 0 \leq k < \dim L^2(\mu)\}$ is linearly independent in $L^2(\mu)$. By applying the Gram-Schmidt process on this sequence construct an orthonormal basis $\{q_k,\ 0 \leq k < \dim L^2(\mu)\}$ of real polynomials with $q_0 = 1$. Since $xq_k(x)$ is in the linear span of $q_0, q_1, \ldots, q_{k+1}$ it follows that $< q_m, Zq_k > = 0$ for $m \geq k+2$. For $k \geq m+2$ we have $< q_m, Zq_k > = < q_k, Zq_m > = 0$. Combining both we conclude

$$< q_m, Zq_k > = 0 \quad \text{if} \quad |m-k| \geq 2$$

In other words, in the orthonormal basis $\{q_k\}$ the matrix of Z has the tri-diagonal form

$$\begin{pmatrix} f(0) & g(0) & 0 & 0 & 0 & 0 & \ldots \\ g(0) & f(1) & g(1) & 0 & 0 & 0 & \ldots \\ 0 & g(1) & f(2) & g(2) & 0 & 0 & \ldots \\ 0 & 0 & g(2) & f(3) & g(3) & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

where $f(k) = < q_k, Zq_k >$, $g(k) = < q_k, Zq_{k+1} >$ for each k. By Remark 1 after the proof of corollary 3.4, $g(k) \neq 0$ for $k < \dim L^2(\mu)$. If $g(k) < 0$ change q_{k+1} to $-q_{k+1}$. This can be done successively to ensure that $g(k) > 0$ for all $0 \leq k < \dim L^2(\mu)$. This shows that Z satisfies the required properties. ∎

COROLLARY 3.6. *In Theorem 3.5 suppose that μ is a symmetric probability distribution. Then the function f can be chosen to be identically 0.*

PROOF. The symmetry of μ implies that the odd moments of μ vanish. Thus $L^2(\mu) = S_+ \oplus S_-$ where S_+ and S_- are respectively the closed subspaces spanned by $\{x^{2j}, j = 0, 1, 2, \ldots\}$ and $\{x^{2j+1}, j = 0, 1, 2, \ldots\}$. Thus the polynomials q_k of Theorem 3.5 satisfy : $q_k \in S_\pm$ according as k is even or odd. In particular, $xq_k(x)$ is orthogonal to $q_k(x)$ in $L^2(\mu)$ and hence $f(k) = < q_k, Zq_k > = 0$ for all k. ∎

COROLLARY 3.7. *Let μ be any symmetric probability distribution on the real line with moments of all order. Suppose that the set of all polynomials is dense in $L^2(\mu)$. Then there exists an orthonormal basis $\{e_0, e_1, e_2, \ldots\}$ and a selfadjoint operator X in $L^2(\mu)$ satisfying the following :*

(i) *e_j belongs to the domain of X for each j ;*

(ii) *X is harmonic with respect to the number operator $N = \sum_j j\,|e_j><e_j|$ in the linear manifold $\mathcal{D}$ generated by $\{e_0, e_1, \ldots\}$;*

(iii) *The distribution of X in the pure state e_0 is μ.*

PROOF. This is immediate from Theorem 3.5, Corollary 3.6 and the relation

$$[N, [N, g(N)L + L^* g(N)]] = g(N)L + L^* g(N)$$

on the domain $\mathcal{D}$, where $L = \sum_j |e_j><e_{j+1}|$. ∎

Examples of processes satisfying harmonic property

We begin with a heuristic argument. Consider two commuting selfadjoint operators H, K in a Hilbert space h. Let $\lambda(H), \lambda(K)$ be their respective differential second quantizations in the boson Fock space $\mathcal{H} = \Gamma(h)$ defined by $\Gamma(e^{itH}) = e^{it\lambda(H)}$ for all $t \in \mathbb{R}$. For any

$u \in h$ let $a(u), a^\dagger(u)$ be the associated annihilation and creation operators in $\mathcal{H}$. Define $X = \lambda(K)a(u) + a^\dagger(u)\lambda(K)$. If $Hu = cu$ for some scalar c we have the commutation relations : $[\lambda(H), X] = -c\lambda(K)a(u) + ca^\dagger(u)\lambda(K)$ and $[\lambda(H), [\lambda(H), X]] = c^2 X$. By imposing suitable domain restrictions on H and K it is possible to construct many examples of generalised harmonic oscillators of the form $(\lambda(H), X)$. A slightly modified form of this construction reveals the harmonic property of many processes in Fock space with respect to the conservation process. We follow the methods of quantum stochastic calculus as described in [4], [5], [7]. Consider $\mathcal{H} = \Gamma(L^2(\mathbb{R}_+))$, the boson Fock space over $L^2(\mathbb{R}_+)$ and the creation, conservation and annihilation processes $A^\dagger, \Lambda$ and A respectively. Let $\{X(t),\, t \geq 0\}$ be an adapted family of selfadjoint operators satisfying

$$dX = E\,dA + E^\dagger dA^\dagger$$

where $(E,\, E^\dagger)$ is a pair of adapted processes adjoint to each other on a suitable domain of exponential vectors and satisfying $[\Lambda(t),\, E(t)] = 0$. From quantum Ito's formula it follows that $[\Lambda(t),\, [\Lambda(t), X(t)]] = X(t)$ modulo domain considerations. As a consequence of this heuristic discussion we have the following examples.

Example 4.1. Let $F, F^\dagger$ be the fermion annihilation and creation processes in $\Gamma(L^2(\mathbb{R}_+))$ satisfying

$$dF = J\,dA\,, \quad dF^\dagger = J\,dA^\dagger$$

where J is the reflection process [2]. Then $(\Lambda(t), F(t) + F^\dagger(t))$ is a generalised harmonic oscillator in the linear manifold generated by all exponential vectors.

Example 4.2. Let $-1 \leq c < 1$ be any constant. Following [6] consider the Azéma martingale $\{X_c(t)\}$ obeying the stochastic differential equation

$$dX_c = (c-1)\,X_c\,d\Lambda + dA + dA^\dagger\,, \quad X_c(0) = 0\,.$$

This is not a process of the form described earlier but it once again follows from quantum Ito's formula that

$$[\Lambda(t)\,, [\Lambda(t), X_c(t)]] = X_c(t)$$

on a dense linear manifold generated by vectors of the exponential type.

Using Maassen's kernel formalism [3], [4] in Guichardet's version of Fock space we shall now prove a lemma and use it to exhibit some examples of quantum martingales with the harmonic property. To this end consider a standard, totally finite and non-atomic measure space $(S, \mathcal{F}, m)$ and the associated Guichardet space $\Gamma_S = \{\sigma : \sigma \subset S, \#\sigma < \infty\}$ with its symmetric measure constructed from m, integration with respect to which being indicated by $d\sigma$ and $\#\sigma$ denoting cardinality of σ. Consider the annihilation, conservation and creation operators in $L^2(\Gamma_S)$ defined by

$$\{a(u)f\}(\sigma) = \int \bar{u}(s) f(\sigma \cup s)\, dm(s)\,,$$
$$\{\lambda(\varphi)f\}(\sigma) = \{\sum_{s\in\sigma} \varphi(s)\}\, f(\sigma)\,,$$
$$\{a^\dagger(u)f\}(\sigma) = \sum_{s\in\sigma} u(s) f(\sigma \setminus s)$$

where $u \in L^2(m)$, $\varphi \in L_\infty(m)$.

LEMMA 4.3. *Let $u \in L^2(m)$ and let ψ be a function on the set $\{0,1,2,\ldots\}$ satisfying $\sup_n n^{1/2}|\psi(n)| < \infty$. Then the closure of the operator $a^\dagger(u)\psi(\lambda(1))$ is bounded.*

PROOF. Put $B = a^\dagger(u)\psi(\lambda(1))$. Then

$$\begin{aligned}(Bf)(\sigma) &= \sum_{s\in\sigma} u(s) f(\sigma\setminus s)\psi(\#(\sigma\setminus s)) \\ &= \psi(\#\sigma - 1) \sum_{s\in\sigma} u(s) f(\sigma\setminus s) \quad \text{if } \sigma \neq \varphi, \\ &= 0 \quad \text{if } \sigma = \varphi.\end{aligned}$$

By the sum-integral formula for integration with respect to $d\sigma$ we have

$$\begin{aligned}\|Bf\|^2 &= \int_{\sigma\neq\varphi} \sum_{s\in\sigma} |\psi(\#\sigma-1)|^2 |u(s)|^2 |f(\sigma\setminus s)|^2\, d\sigma \\ &\quad + \int_{\#\sigma\geq 2} |\psi(\#\sigma-1)|^2 \sum_{s_1,s_2\in\sigma,\, s_1\neq s_2} \bar{u}(s_1)\bar{f}(\sigma\setminus s_1) u(s_2) f(\sigma\setminus s_2)\, d\sigma \\ &= \int |\psi(\#\sigma)|^2 |u(s)|^2 |f(\sigma)|^2\, d\sigma\, dm(s) \\ &\quad + \int |\psi(\#\sigma+1)|^2 \bar{u}(s_1)\bar{f}(\sigma\cup s_2) u(s_2) f(\sigma\cup s_1)\, d\sigma\, dm(s_1)\, dm(s_2)\end{aligned} \tag{4.1}$$

The second term on the right hand side of (4.1) is equal to

$$\begin{aligned}&\int \Big| \int \bar{u}(s) f(\sigma\cup s)\, dm(s)\Big|^2 |\psi(\#\sigma+1)|^2\, d\sigma \\ &\leq \Big(\int |u(s)|^2\, dm(s)\Big) \int |f(\sigma\cup s)|^2 |\psi(\#\sigma+1)|^2\, d\sigma\, dm(s) \\ &= \|u\|^2 \int \sum_{s\in\delta} |f(\delta)|^2 |\psi(\#\delta)|^2\, d\delta \\ &= \|u\|^2 \int \#\delta\, |\psi(\#\delta)|^2 |f(\delta)|^2\, d\delta\,.\end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2) we have for any f in the domain of B

$$\|Bf\|^2 \leq \{\sup_n (n+1)|\psi(n)|^2\}\, \|u\|^2\, \|f\|^2\,. \qquad \blacksquare$$

COROLLARY 4.4. *Let ψ be any function on $\{0,1,2,\ldots\}$ such that*

$$\sup_n n^{1/2}\,|\psi(n)\,| < \infty .$$

Define

$$X(t) = \{\overline{\psi}(\Lambda(t))\,A(t) + A^{\dagger}(t)\,\psi(\Lambda(t))\,\}^{\sim}$$

in $\Gamma(L^2(\mathbb{R}_+))$ where $A^{\dagger}$, Λ, A are the boson creation, conservation and annihilation processes and $\sim$ denotes closure. Then $\{X(t), t \geq 0\}$ is a quantum martingale of bounded selfadjoint operators in the standard Fock filtration. Furthermore $(\Lambda(t),\, X(t)\,)$ is a generalised harmonic oscillator on the linear manifold generated by exponential vectors for every t.

PROOF. By considering $X(t)$ as an operator in $L^2(\Gamma_S)$ where S is the interval $[0,t]$ with Lebesgue measure and putting $u(s)\equiv 1$ it follows from Lemma 4.3 that

$$||X(t)\,|| \leq \sqrt{t}\,\sup_n\,(n+1)^{1/2}\,|\psi(n)\,| \quad \text{for all} \quad t \geq 0\,.$$

Since $(A^{\dagger}$, Λ, $A)$ is the correct Wick ordering so that $dA^{\dagger}d\Lambda = dA^{\dagger}dA = d\Lambda dA = 0$ it follows that $(X(t))_{t\geq 0}$ obeys the martingale property. ■

Remark. In Corollary 4.4 the probability distribution of $t^{-1/2}X(t)$ in the vacuum state is also the probability distribution of the bounded operator

$$Z = \bar{\psi}(N+1)(N+1)^{1/2}\,L + L^{*}\psi(N+1)(N+1)^{1/2}$$

in the pure state e_0 where N, L, e_0 are as in Example 2.2 with the dimension of the underlying Hilbert space $\mathcal{H}$ being infinite. In particular,

$$X(t) = \frac{1}{2}\{\,(\Lambda(t)+1)^{-1/2}A(t) \;+\; A^{\dagger}(t)(\Lambda(t)+1)^{-1/2}\,\}^{\sim}$$

is a quantum martingale of bounded selfadjoint operators where $t^{-1/2}X(t)$ has standard Wigner distribution in $[-1,1]$ for all $t > 0$. Indeed, this is immediate by putting $L(t) = \{(\Lambda(t)+1)^{-1/2}A(t)\}^{\sim}$, $L^{*}(t) = A^{\dagger}(t)(\Lambda(t)+1)^{-1/2}$ and observing that $L_tL_t^{*} = t$, $[\Lambda(t), L(t)] = -L(t)$ for all t. In this construction $X(t)$ is a noncommutative martingale. It will be interesting to construct a commutative version.

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